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● *Technical Note*

COULD LINEAR HYSTERESIS CONTRIBUTE TO SHEAR WAVE LOSSES IN TISSUES?

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Abstract—For nearly 100 y in the study of cyclical motion in materials, a particular phenomenon called "linear hysteresis" or "ideal hysteretic damping" has been widely observed. More recently in the field of shear wave elastography, the basic mechanisms underlying shear wave losses in soft tissues are in question. Could linear hysteresis play a role? An underlying theoretical question must be answered: Is there a real and causal physical model that is capable of producing linear hysteresis over a band of shear wave frequencies used in diagnostic imaging schemes? One model that can approximately produce classic linear hysteresis behavior, by examining a generalized Maxwell model with a specific power law relaxation spectrum, is described here. This provides a theoretical plausibility for the phenomenon as a candidate for models of tissue behavior. (E-mail: kevin.parker@rochester.edu)   © 2015 World Federation for Ultrasound in Medicine & Biology.

Key Words: Shear waves, Elastography, Biomaterials, Biomechanics, Visco-elastic models, Hysteresis.

INTRODUCTION

New imaging approaches have made it possible to infer tissue properties from shear wave measurements at low frequencies (40–1000 Hz). This has led to renewed interest in the low-frequency visco-elastic properties of normal and diseased tissues (Asbach et al. 2008; Barry et al. 2012, 2014a; Carstensen and Parker 2014; Catheline et al. 2004; Chen et al. 2013a, 2013b; Deffieux et al. 2009; Kruse et al. 2000; Salameh et al. 2007). We are in the early stages of understanding shear wave propagation in soft tissues such as the liver, and a recent review paper asked if a linear hysteresis mechanism could contribute to the observed lossy behavior (Carstensen and Parker 2014). For nearly a century, in a variety of situations from the movement of soil to waves in metals, it has been recognized that the energy dissipation during cyclical motion can increase as the first power of frequency, over an extended frequency range (Kimball and Lovell 1927; Mason and McSkimin 1947; Wegel and Walther 1935). That implies that the losses per cycle are constant over many octaves, and this behavior has been called *linear hysteresis*, or "ideal hysteretic damping" (De Silva 2007), a hypothetical loss element or process that creates a constant phase lag between stress and strain over all frequencies (Crandall 1997; Mason 1950; Theodorsen and Garrick 1940). In the early 20th century, linear hysteresis effects were thought to be prominent across a diverse range of materials and conditions. Kimball and Lovell (1927) at GE Laboratories reported that hysteresis was found "over a considerable frequency range" and "for a number of solids of very different physical properties." Later, Mason stated that "the component proportional to frequency is the same as observed for most metals and solid materials at low frequencies, and indicates the presence of an elastic hysteresis" (Mason and McSkimin 1947). Fung (1981:chs 7, 8) considered relaxation models that approximate linear hysteresis for viscous biomaterials. The issue of linear hysteresis is of continuing importance in a diverse set of areas, including earthquake motion and damping of structures (Makris and Zhang 2000; Nakamura 2007) and, possibly, in shear wave propagation in biomedical tissues (Carstensen and Parker 2014). We emphasize that in this Technical Note we are not referring to the generic loading/unloading hysteresis that is exhibited by all lossy materials, but are specifically considering the special frequency-independent behavior called "linear hysteresis" and other closely related names (Caughey 1962; Crandall 1997; De Silva 2007; Inaudi and Kelly 1995; Mason 1950; Muravskii 2004).

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Despite the simplicity of the classic, idealized frequency domain description of linear hysteresis, it has been difficult to find a practical, real, causal time domain impulse response that produces hysteresis, and this problem has been the subject of numerous articles over the past decade (Nakamura 2007).

The Kramers–Kronig relationship links and constrains the relationship between the real and imaginary parts of a transfer function in the frequency domain, based on the constraint that the impulse response of a material is a real and causal function (Nachman et al. 1990; Nasholm and Holm 2011; Szabo 1995; Szabo and Wu 2000). Nevertheless, the most straightforward description of a constant phase shift in the frequency domain is simply a transfer function with constant real and imaginary parts, as given by Mason (1950). However, if formulated to be consistent with a real impulse response, the corresponding impulse response is an acausal $1/t$ function (valid for both positive and negative time t), and this well-known transform pair resembles the Hilbert transform (Bracewell 1965; Crandall 1963, 1970). Because physical objects respond in a causal manner, the simple Mason formulation with constant real and imaginary modulus is not realistic.

In Parker (2014b), we re-examined the fundamental requirements for linear hysteresis and causality and indicated that there is a diverse set of continuous, real, causal analytic functions that provide linear hysteresis behavior over a range of observable frequencies, but within a set of constraints that permit only an approximation to the classic formulation of constants. What remains to be seen is how a physical model can approach the requirements of hysteresis and, then, if these physical mechanisms actually exist in soft tissues such as the liver. The first of these requirements is established in this Technical Note.

THEORY

Necessary requirements for linear hysteresis

Under wave propagation, the requirements for linear hysteresis are stringent because we require the attenuation (the imaginary part of the wavenumber) to increase linearly with frequency. This constrains the material properties. For example, in a sinusoidal steady-state plane shear wave propagation in an isotropic elastic material, the general relationship is

$$T(\omega) = \mu S(\omega) \qquad (1)$$

where $T(\omega)$ and $S(\omega)$ are the shear stress and strain at frequency ω, respectively; μ is the shear modulus; and the shear wave speed is $c_s = \sqrt{\frac{\mu}{\rho}}$, where ρ is the density (Carstensen and Parker 2014). For many biomaterials

where Poisson's ratio υ approaches the incompressible limit $\upsilon \to 0.5$, the shear modulus can be approximated by $\mu \cong E/3$, where E is the Young's modulus, commonly referred to as the "stiffness" (Parker et al. 2011). In a lossy material, μ or E can be described as a complex quantity, for example, let $\mu(\omega) = K(\omega) + jH(\omega)$; then the complex wave number is

$$k = \frac{\omega}{c_s} = \beta - j\alpha = \frac{\omega}{\sqrt{\frac{K(\omega)+jH(\omega)}{\rho}}} \qquad (2)$$

Here, k is the wavenumber with real (β) and imaginary (α) parts (Blackstock 2000). The attenuation coefficient α of a propagating wave will therefore be a function of frequency depending on $K(\omega)$ and $H(\omega)$. Expanding on the real and imaginary parts of eqn (2), we have (Carstensen and Parker 2014)

$$\beta = \omega\sqrt{\frac{\rho}{K^2+H^2}}\left[\frac{1}{2}\left(1+\frac{1}{\sqrt{1+\frac{H^2}{K^2}}}\right)\right]^{\frac{1}{2}} \qquad (3)$$

the shear wave speed

$$c_s = \sqrt{\frac{K^2+H^2}{\rho}}\left[\frac{1}{2}\left(1+\frac{1}{\sqrt{1+\frac{H^2}{K^2}}}\right)\right]^{-\frac{1}{2}} \qquad (4)$$

and the absorption coefficient

$$\alpha = \omega\sqrt{\frac{\rho}{K^2+H^2}}\left[\frac{1}{2}\left(1-\frac{1}{\sqrt{1+\frac{H^2}{K^2}}}\right)\right]^{\frac{1}{2}} = \frac{\omega}{c_s}\sqrt{\frac{1-\frac{1}{\sqrt{1+\frac{H^2}{K^2}}}}{1+\frac{1}{\sqrt{1+\frac{H^2}{K^2}}}}} \qquad (5)$$

Considering eqns (3)–(5), we see that if $K^2(\omega) + H^2(\omega) = $ constant, and if $K(\omega)/H(\omega) = $ constant, then c_s will be independent of frequency while α will be linearly proportional to frequency. This behavior has been traditionally associated with waves in a linear or ideal hysteretic material since Mason (Mason 1950; Mason and McSkimin 1947). However, this behavior can only be observed in a passive medium if both $K(\omega)$ and $H(\omega)$ are approximately constant over some extended frequency range. We call this the "strict hysteresis" criterion (Parker 2014b) and have already noted that achieving the strict criterion, $|H(\omega)| = H_0$ and $|K(\omega)| = K_0$, over all frequencies is not possible with real, causal functions. Furthermore, none of the simple models such as the Kelvin, Maxwell or Zener model

● *Erratum*

ERRATUM TO 'COULD LINEAR HYSTERESIS CONTRIBUTE TO SHEAR WAVE LOSSES IN TISSUES? [ULTRASOUND MED BIOL 41 (2015) 1100–1104]

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The authors regret that the above paper was printed with errors in equations (3), (4), and (5). These are corrected in the following equations:

$$\beta = \omega \sqrt{\frac{\rho}{\sqrt{K^2 + H^2}}} \left[\frac{1}{2} \left(1 + \frac{1}{\sqrt{1 + \frac{H^2}{K^2}}} \right) \right]^{\frac{1}{2}} \tag{3}$$

$$c_s = \sqrt{\frac{\sqrt{K^2 + H^2}}{\rho}} \left[\frac{1}{2} \left(1 + \frac{1}{\sqrt{1 + \frac{H^2}{K^2}}} \right) \right]^{-\frac{1}{2}} \tag{4}$$

$$\alpha = \omega \sqrt{\frac{\rho}{\sqrt{K^2 + H^2}}} \left[\frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + \frac{H^2}{K^2}}} \right) \right]^{\frac{1}{2}} = \frac{\omega}{c_s} \sqrt{\frac{1 - \frac{1}{\sqrt{1 + \frac{H^2}{K^2}}}}{1 + \frac{1}{\sqrt{1 + \frac{H^2}{K^2}}}}}, \tag{5}$$

The authors would like to apologise for any inconvenience caused.

have a constant imaginary component $K(\omega)$, so a different model must be established.

A physical model for linear hysteresis

In the simple Maxwell model of a series spring E and dashpot η, the stress relaxation curve $\sigma_{SR}(t)$ is a simple exponential decay (Fung 1981:ch 2). If the applied strain $\varepsilon(t) = \varepsilon_0 \text{UnitStep}(t)$, then

$$\sigma_{SR}(t) = \varepsilon_0 E e^{-t/\tau} \quad \text{for } t \geq 0 \tag{6}$$

where the time constant $\tau = \eta/E$. It should be noted that the single-time-constant exponential decay is not capable of matching soft tissue stress relaxation responses, nor the frequency responses (Carstensen and Parker 2014).

Now assume there are multiple relaxation components within the tissue. In this case, if each component contributes to the stress relaxation at its respective time constant τ_n, then the simplest model for this looks like a parallel set of Maxwell elements (Fig. 1). This configuration of multiple parallel elements and an optional single spring element is the generalized Maxwell–Weichert model (Fung 1981:ch 2; Parker 2014a). Generally, we can write the stress relaxation solution for N Maxwell elements as

$$\sigma_{SR}(t) = \sum_N A_N e^{\frac{-t}{\tau_N}} \tag{7}$$

where A_N are the relative strengths of the components with characteristic relaxation time constant τ_N. In the limit, as we allow a continuous distribution of time constants τ, the summation becomes an integral, and $A(\tau)$ is the relaxation spectrum, which can be either discrete or continuous, depending on the particular medium under study (Fung 1981:ch 2; Lakes 1999). Given a material's $A(\tau)$, we can write

$$\sigma_{SR}(t) = \int_0^\infty A(\tau) e^{\frac{-t}{\tau}} d\tau \tag{8}$$

Now consider a specific power law distribution:

$$A(\tau) = A_0 \tau^{-(1+\varepsilon)}, \quad 0 < \varepsilon \ll 1 \tag{9}$$
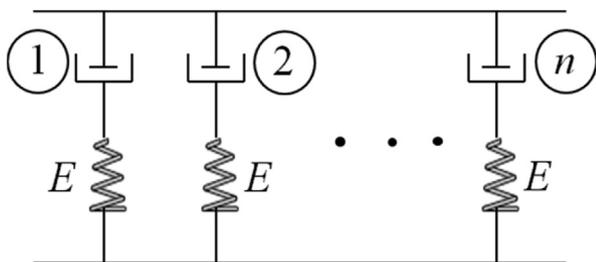
Fig. 1. Parallel elements forming the generalized Maxwell–Weichert model.

One rationale for introducing this function is that the power law distribution is frequently found to describe fractal systems in nature and biology (West et al. 1999). Specifically, power law distributions have been observed in metrics related to branching vasculature, including normal and pathologic circulatory systems (Gazit et al. 1997; Risser et al. 2007). However, here we restrict the power law to be slightly greater than unity.

Now, substituting eqn (9) into eqn (8) and solving yield

$$\int_0^\infty \left(\frac{A_0}{\tau^{(1+\varepsilon)}}\right) e^{\frac{-t}{\tau}} d\tau = A_0 t^{-\varepsilon} \Gamma(\varepsilon) \quad \text{for} \quad t > 0 \tag{10}$$

where $\Gamma(\ )$ is the gamma function. Taking the derivative to find the impulse $\sigma_i(t)$ response yields

$$\frac{\sigma_i(t)}{A_0} = t^{-\varepsilon} \delta(t) \Gamma(\varepsilon) - \varepsilon t^{(-1-\varepsilon)} \Gamma(\varepsilon) \theta(t) \tag{11}$$

where $\delta(t)$ is the Dirac delta and $\theta(t)$ is the Heaviside theta function. Taking the Fourier transform yields

$$-\varepsilon(2\pi)^\varepsilon \text{abs}(\omega)^\varepsilon \Gamma(-\varepsilon) \Gamma(\varepsilon) \left[\cos\left(\frac{\varepsilon\pi}{2}\right) + i\text{Sign}(\omega)\sin\left(\frac{\varepsilon\pi}{2}\right)\right] \tag{12}$$

This describes the frequency response of the complex shear modulus of the material that is characterized by a relaxation spectrum $A(\tau)$ of a power law just greater than unity, as given in eqn (9).

RESULTS AND DISCUSSION

Under this model, the magnitude of the shear modulus increases as ω^ε, where it is assumed that $\varepsilon \ll 1$. Because of the square root relation in eqns (2)–(4), dispersion of shear wave speed will be even smaller, proportional to $\omega^{\varepsilon/2}$, which results in a very slight dispersion over many decades. The loss tangent, determined by the ratio of imaginary to real parts of the transfer function, is a constant over all frequencies and is proportional to $\varepsilon(\pi/2)$. The shear wave absorption coefficient will be nearly linear with frequency, proportional to $\omega^{1-\varepsilon/2}$. As a specific example, let $A_0 = 1$ and $\varepsilon = 0.01$. The real and imaginary components are plotted on log–log scales in Figure 2 from $\omega = 1$ to 2,000 rad/s. The increases in magnitudes are within 5% over this range, which would be experimentally seen as negligible shear wave velocity dispersion (eqn 4) and nearly linear frequency dependence of attenuation (eqn 5). These are the characteristic properties of classic linear hysteresis. It has also been found that the high-frequency approximation to the fractional Caputo wave equation (and the Kelvin–Voigt fractional derivative model) can produce a nearly linear-with-frequency attenuation
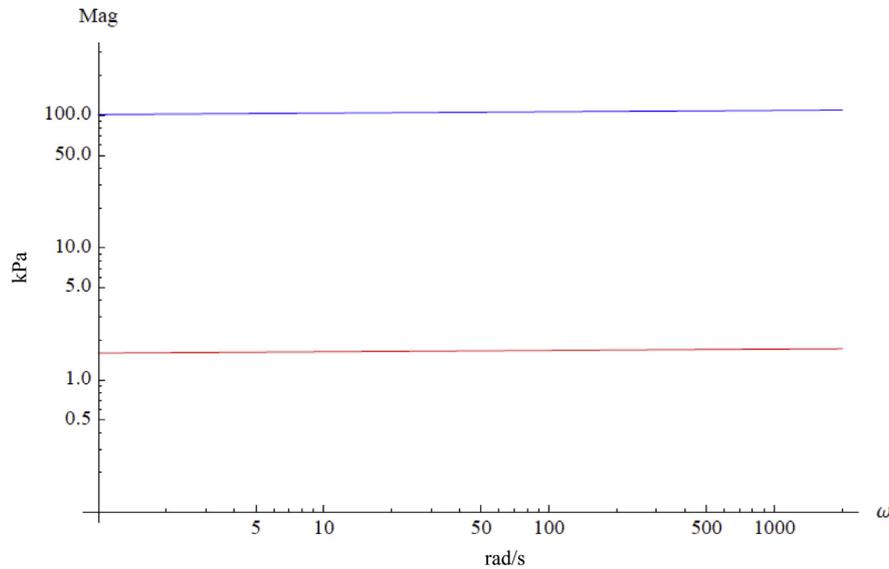
Fig. 2. Real (upper line) and imaginary (lower line) components of complex shear modulus from a generalized Maxwell model with power law relaxation spectrum of the form $1/\tau^{1+\varepsilon}$. Vertical axis (log–log) magnitude is in kilopascals; horizontal axis ω is in radians per second. For the power law parameter $\varepsilon = 0.01$, the shear moduli are nearly constant with frequency over any decade of measurement.

proportional to $|\omega|^{1+\varepsilon}$ (Holm and Sinkus 2010:sect IV.C). This establishes a corollary path using fractional mathematical operators.

From the medical ultrasound literature, including shear wave measurements in tissue, some of the lowest tissue dispersion values have been reported from *ex vivo* measurements on livers of leptin-deficient strains of mice and rats (Barry et al. 2012, 2014a, 2014b). In these lean, young groups, the dispersion was below 0.1 m/s per 100 Hz in the range 100–300 Hz shear wave frequency. For example, a measured liver shear speed of 3.2 m/s at 100 Hz increases to 3.3 m/s at 260 Hz shear wave in the lean group of Barry et al. (2014b). If fit to a power law, this corresponds to an $\varepsilon/2$ of approximately 0.03 or an ε of 0.06. However, independent measurements of attenuation and the complex modulus were not made, so further research is required to evaluate the suitability of this model of linear hysteresis.

CONCLUSIONS

The derivations indicate that there is at least one physical model of tissue capable of exhibiting (approximately) the features of classic linear hysteresis. A generalized Maxwell model with a simple stress–strain relaxation spectrum that follows a power law distribution slightly greater than unity will exhibit nearly constant wave speed, a constant loss tangent and linearly increasing attenuation across many decades of frequency. This establishes a conventional physical model capable of

illustrating linear hysteresis. Whether or not any tissue does, in fact, exhibit this behavior and the underlying mechanisms are left for further research.

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