

# Influence of atmospheric turbulence on the propagation of quantum states of light using plane-wave encoding

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**Abstract:** We consider the possibility of performing quantum key distribution (QKD) by encoding information onto individual photons using plane-wave basis states. We compare the results of this calculation to those obtained by earlier workers, who considered encoding using OAM-carrying vortex modes of the field. We find theoretically that plane-wave encoding is less strongly influenced by atmospheric turbulence than is OAM encoding, with potentially important implications for free-space quantum key distribution.

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**OCIS codes:** (270.5568) Quantum cryptography; (010.1330) Atmospheric turbulence.

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There has recently been great interest in encoding information onto the transverse degree of freedom of individual photons [1] for applications such as optical communication and especially for quantum key distribution (QKD) [2]. The transverse degree of freedom of the photon is associated with an infinite-dimensional Hilbert space, and thus there is in principle no limit to how much information can be encoded in this manner onto an individual photon. One motivation for impressing many bits of information onto each photon is to increase the data transmission rate of a QKD system. Another more subtle motivation is to exploit the increased security of a QKD system that is afforded by the use of a larger state space [3,4].

Much of the prior work on exploiting the transverse degree of freedom of the photon for optical communication has made use of states that carry orbital angular momentum (OAM) [5], such as the Laguerre-Gauss (LG) states [1,6,7]. Single-photon states that carry OAM are readily synthesized in the laboratory [8], and the quantum optical properties of these states have been studied by many workers [9].

A prototypical free-space quantum communication system is shown in Fig. 1. Here information is impressed onto individual photons by means of some quantum-state generator, the photons are transmitted through a free-space channel, and they are then collected and analyzed by a receiver. A potential problem with any such communication system is that atmospheric turbulence can lead to the loss of quantum coherence [10], which in the context of a QKD protocol would lead to the received photon being detected in a quantum state other than the one in which it was launched. Earlier theoretical studies [11–18] of QKD based on the use of OAM-carrying beams have shown that such a loss of information is likely to be important under realistic scenarios and have produced explicit predictions for the expected error rates. Encoding in the polarization degree of freedom leads to a communication link that is less susceptible to the influence of turbulence [19], but of course this protocol limits the communication efficiency to one bit per photon.

For practical reasons, it is important to know if some protocols for encoding information onto the transverse degree of freedom of the photon are more robust than others in the presence of atmospheric turbulence. If one could identify such states, one could use this knowledge to develop free-space QKD protocols with increased reliability.

In our earlier study [14], an analysis was presented of an encoding scheme based on the use of pure vortex beams, that is, beams having the form  $V(r, \phi) = A_0 \exp(i\ell\phi)$ , where  $r$  and  $\phi$  are radial and azimuthal coordinates, respectively, and  $\ell$  is the OAM quantum number. It was found that these beams showed essentially the same sensitivity to turbulence as do LG beams, thereby suggesting that the specific form of the transverse field distribution used to encode information is not important in determining the robustness of the communication protocol.

In the present article, we explore the possibility of encoding information through the use of plane-wave states of light [20, 21]. We choose to explore this form of encoding for two different reasons. One is that a system based on such an encoding scheme should prove easier to implement in the laboratory than one based on the use of OAM states. The other reason is that we want to examine the conceptual question of whether some states are more robust than others, and one might well expect that encoding by impressing phase information onto a cartesian coordinate would be significantly different from encoding onto an azimuthal coordinate.

Specifically, we consider the situation in which the propagation direction of each of the plane-wave modes lies in the  $x$ - $z$  plane, and these modes are launched through a square aperture of side  $L$ . We represent each of the plane wave modes in the paraxial approximation as

$$E_n = A \exp(ikz + imqx). \quad (1)$$

Here  $m$  is any positive or negative integer that identifies a particular mode and  $q$  is some characteristic transverse wave vector component. We choose the value of  $q$  to ensure that the modes are orthogonal over the transmitting aperture. To do so, we consider the overlap integral

$$O_{mn} = \int_{-L/2}^{L/2} E_m E_n^* dx, \quad (2)$$

which, through use of Eq. (1), becomes

$$O_{mn} = |A|^2 \int_{-L/2}^{L/2} e^{iq(m-n)x} dx. \quad (3)$$

For  $m = n$  this integral is simply evaluated to give  $O_{nn} = |A|^2 L$ , whereas more generally we obtain

$$O_{mn} = |A|^2 \frac{2 \sin[q(m-n)L/2]}{q(m-n)}. \quad (4)$$

Note that if we choose  $q$  to be equal to  $2\pi/L$ , this equation becomes

$$O_{mn} = |A|^2 \frac{\sin[\pi(m-n)]}{\pi(m-n)/L}, \quad (5)$$

which vanishes for any non-zero value of the integer  $m - n$ . For this value of  $q$ , the plane wave modes are orthogonal over the transmitting aperture. We note that this result has the simple interpretation that for  $q = 2\pi/L$  the angle between adjacent modes  $\alpha = q/k$  is just equal to  $\lambda/L$  where  $\lambda$  is the optical wavelength, which is the approximate angular spread of each such mode. Thus, these modes are just barely resolved in the far field.

To proceed, we assume that the field launched by the transmitter can be represented as

$$A(x, y) = A_0 W(x/L) W(y/L) e^{ilqx}, \quad (6)$$

where  $A_0$  is the (spatially uniform) field amplitude,  $W(\xi)$  is the aperture function defined so that  $W(\xi) = 1$  for  $|\eta| \leq 1$  and zero otherwise, and  $l$  is the mode index of the launched field.

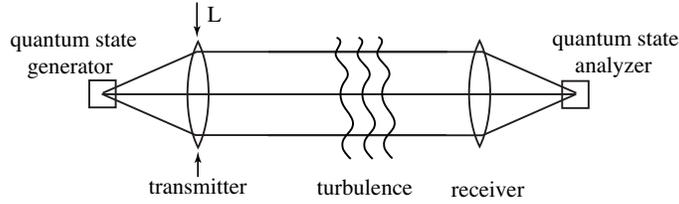


Fig. 1. Schematic diagram of a free-space quantum communication link.

We further assume that the transmitted beam remains sufficiently well collimated that the field at the receiver aperture can be well described by

$$V(x, y) = A_0 W(x/L) W(y/L) e^{iLqx} e^{i\phi(x, y)}, \quad (7)$$

where  $\phi(x, y)$  represents the turbulence-induced distortion of the wavefront and where we have omitted the overall phase factor  $\exp(ikz)$ . Here we are describing the influence of turbulence in the phase-screen approximation, which is valid when the turbulence is not too strong. We are also assuming that the apertures sizes and separations are chosen so that essentially all of the light leaving the transmitter is intercepted by the receiver. This condition is readily achieved. The necessary condition is that the Fresnel number ( $F = L^2/\lambda Z_0$ , where  $Z_0$  is the distance from the transmitter to the receiver) of the communication link be much greater than unity. For a wavelength  $\lambda$  of  $1 \mu\text{m}$  and a separation of  $Z_0$  of 10 km, a telescope aperture of only  $L = 0.1 \text{ m}$  is needed to obtain a Fresnel number of unity.

We next consider explicitly the nature of the mode scrambling that is induced by the atmospheric turbulence. The specific motivation for the present calculation is in determining how the integrity of QKD is compromised by the presence of atmospheric turbulence. We are thus interested in determining how the quantum state of an individual photon is modified by atmospheric turbulence. Nonetheless, the calculation can be performed at an entirely classical level by determining how the modes of the transmitted light become scrambled by atmospheric turbulence. The connection between the quantum properties and the classical analyses can be understood from the point of view that a “photon” is one unit of excitation of a given mode of the optical field. To proceed, we expand the quantity  $\exp[i\phi(x, y)]$  in a Fourier series in the  $x$  direction only as

$$e^{i\phi(x, y)} = \sum_{m=-\infty}^{\infty} g_m(y) e^{imqx}, \quad (8)$$

where the expansion coefficients  $g_m(y)$  are given by

$$g_m(y) = \frac{1}{L} \int_{-L/2}^{L/2} dx e^{i\phi(x, y)} e^{-imqx}. \quad (9)$$

Of course, we could have expanded  $\exp[i\phi(x, y)]$  in Fourier series in both the  $x$  and  $y$  directions, but doing so is not needed and in fact would complicate the ensuing analysis. It is crucial to note that we are *not* assuming that the atmospheric properties are uniform in the  $y$  direction. Rather, we are arguing that  $y$ -dependent inhomogeneities will not lead to beam deflections in the  $x$  direction, the direction in which information is encoded. Similarly, we expand the received field  $V(x, y)$  in a Fourier series as

$$V(x, y) = \sum_{n=-\infty}^{\infty} V_n(y) e^{inqx}, \quad (10)$$

where each Fourier component  $V_n(y)$  is given by

$$V_n(y) = \frac{1}{L} \int_{-L/2}^{L/2} dx V(x, y) e^{-inqx}. \quad (11)$$

Eqs. (7) and (8) are now substituted into Eq. (11) which becomes

$$V_n(y) = A_0 W(y/L) \sum_{m=-\infty}^{\infty} g_m(y) \int_{-L/2}^{L/2} dx e^{i(l+m-n)qx}. \quad (12)$$

As established above in connection with Eq. (5), the integral in this expression is equal to  $L$  for  $n - l - m = 0$  and vanishes otherwise. Using this result, the summation in Eq. (12) can be performed directly to give

$$V_n(y) = A_0 W(y/L) g_{n-l}(y). \quad (13)$$

This result illustrates the manner in which the Fourier components  $g_{n-l}(r)$  associated with atmospheric turbulence are coupled to the plane-wave states of the received field. Specifically, if mode  $l$  is launched, mode  $n$  will be received with a probability amplitude that is proportional to the quantity  $g_{n-l}(y)$ , which represents the amplitude of the spectrum of atmospheric fluctuations at spatial frequency  $(n-l)q$ .

Under many practical situations, one is interested primarily in determining the power contained in each plane-wave component of the received field. The total power collected by the receiver is given by

$$P = \frac{1}{2} \epsilon_0 c \int dx dy V^*(\mathbf{r}) V(\mathbf{r}) = \frac{1}{2} \epsilon_0 c |A_0|^2 L^2. \quad (14)$$

where in obtaining the last form we have used the field given by Eq. (7). This power is distributed among the various (orthogonal) plane-wave modes of the received field according to

$$P = \sum_{\Delta=-\infty}^{\infty} P_{\Delta} \quad (15)$$

where

$$P_{\Delta} = \frac{1}{2} \epsilon_0 c |A_0|^2 L \int_{-L/2}^{L/2} dy g_{\Delta}^*(y) g_{\Delta}(y) \quad (16)$$

and where we have made use of Eq. (13) and have defined  $\Delta$  as  $\Delta = n - l$ .

It is also useful to define the fraction  $s_{\Delta} = P_{\Delta}/P$  of the received power contained in each plane-wave mode. This quantity is given by

$$s_{\Delta} = \frac{1}{L} \int_{-L/2}^{L/2} dy g_{\Delta}^*(y) g_{\Delta}(y). \quad (17)$$

For any statistical realization of the atmospheric turbulence,  $s_{\Delta}$  gives the probability that the quantum number  $n$  of the received photon departs from that  $l$  of the transmitted photon by the amount  $\Delta = n - l$ .

The result presented in Eq. (17) is valid for any realization of atmospheric turbulence. Usually we are interested in the ensemble average of this quantity, which is given by

$$\langle s_{\Delta} \rangle = \frac{1}{L} \int_{-L/2}^{L/2} dy \langle g_{\Delta}^*(y) g_{\Delta}(y) \rangle \quad (18)$$

where the angle brackets  $\langle \dots \rangle$ , represent an ensemble average over the turbulence statistics.

To proceed, Eq. (9) is used to express Eq. (18) in terms of the random phase associated with atmospheric turbulence. One obtains

$$\begin{aligned}\langle s_{\Delta} \rangle &= \frac{1}{L} \int_{-L/2}^{L/2} dy \left\langle \frac{1}{L} \int_{-L/2}^{L/2} dx_1 e^{-i\phi(x_1,y)} e^{i\Delta q x_1} \frac{1}{L} \int_{-L/2}^{L/2} dx_2 e^{i\phi(x_2,y)} e^{-i\Delta q x_2} \right\rangle \\ &= \frac{1}{L^3} \int_{-L/2}^{L/2} dy \int_{-L/2}^{L/2} dx_1 \int_{-L/2}^{L/2} dx_2 \left\langle e^{-i\phi(x_1,y)} e^{i\phi(x_2,y)} \right\rangle e^{i\Delta q(x_1-x_2)}\end{aligned}\quad (19)$$

The analysis proceeds using standard methods [14, 22, 23]. Since the random aberrations introduced by atmospheric turbulence are normal random variables, the ensemble average present in Eq. (19) can be expressed as

$$\left\langle e^{-i[\phi(x_1,y) - \phi(x_2,y)]} \right\rangle = e^{-\frac{1}{2} \langle [\phi(x_1,y) - \phi(x_2,y)]^2 \rangle}.\quad (20)$$

The quantity  $\langle [\phi(x_1,y) - \phi(x_2,y)]^2 \rangle$  is known as the phase structure function. It can be evaluated [22] by means of Kolmogorov turbulence theory to give the result

$$\langle [\phi(x_1,y) - \phi(x_2,y)]^2 \rangle = 6.88 \left| \frac{x_1 - x_2}{r_0} \right|^{5/3},\quad (21)$$

where  $r_0$  is Fried's coherence diameter, which is a measure of the transverse distance scale over which refractive index correlations remain correlated. When Eqs. (20) and (21) are introduced into Eq. (19), the resulting integral simplifies dramatically. The result becomes

$$\langle s_{\Delta} \rangle = \frac{1}{L^3} \int \int \int dy dx_1 dx_2 e^{i\Delta q(x_1-x_2)} e^{-3.44(|x_2-x_1|/r_0)^{5/3}}\quad (22)$$

We evaluate this expression as follows. The integral over  $y$  can be performed directly. We also make a change of integration variables to  $\zeta = (x_1 + x_2)/2L$  and  $\eta = (x_1 - x_2)/2L$ . Eq. (22) thus becomes

$$\langle s_{\Delta} \rangle = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\eta \left( \int_{-\frac{1}{2}+|\eta|}^{\frac{1}{2}-|\eta|} d\zeta \right) e^{-3.44(L/r_0)^{5/3} |\eta|^{5/3}} e^{2i\Delta L q \eta}\quad (23)$$

which can be expressed as

$$\langle s_{\Delta} \rangle = 8 \int_0^{\frac{1}{2}} d\eta \left( \frac{1}{2} - \eta \right) e^{-3.44(\eta L/r_0)^{5/3}} \cos(4\pi\Delta\eta).\quad (24)$$

We note that  $\langle s_{\Delta} \rangle$  clearly depends only on  $L/r_0$ .

This expression can be evaluated numerically under general conditions. Some of the results of this numerical evaluation are shown in Fig. 2. The solid curve labeled  $\Delta = 0$  represents the fraction of the power that remains in the the launched mode after propagation through atmospheric turbulence. The other solid curves indicate the modes into which this power is transferred. For comparison, the dashed curve represents the results for the OAM case studied in reference [14]. We see that plane-wave encoding is more robust to the influence of turbulence than is OAM encoding. The origin of the difference between the two situation can be traced to the form of Eq. (24) and the analogous equation (Eq. 14) of reference [14] for the OAM case. Specifically, the integrands of these two expressions are essentially the same, but the integral in the plane-wave case (expressed in physical units) extends from  $-L/2$  to  $L/2$ , whereas the integral for the OAM case extends over an angular range of  $2\pi$  radians or a distance range of  $\pi$  times the diameter of the OAM mode. For an OAM mode that just fills the aperture of

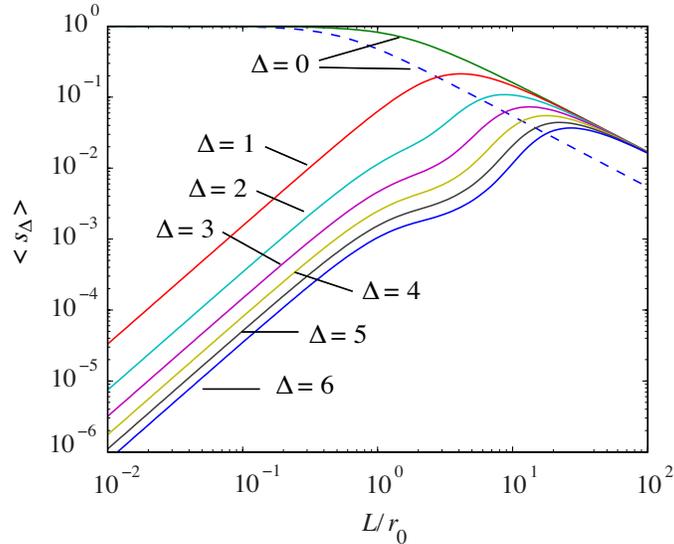


Fig. 2. The quantity  $\langle s_\Delta \rangle$  plotted against the strength of the atmospheric turbulence as quantified by the ratio of the linear size  $L$  of the telescope aperture to the Fried parameter  $r_0$  for several values of  $\Delta$ .  $\langle s_\Delta \rangle$  is the ensemble average of the fraction of the received power that is found to be in plane-wave mode  $n = \ell + \Delta$ , assuming that the transmitted beam was in plane-wave mode  $\ell$ . Solid lines give the predictions based on a numerical evaluation of the integral in Eq. (24). The dashed curve refers to the OAM case treated in reference [14]. The quantity  $L$  represents the diameter of the circular aperture for the OAM case and the length of each side of the square aperture for the plane wave case. Note that the plane wave encoding is more robust (by about as much as a factor of three) than the OAM encoding.

the transmitter telescope, this will lead to about a factor-of- $\pi$  increase in the robustness of the plane-wave modes as compared to the OAM modes, as is seen in the numerical results of Fig. 2. In fact, the two curves labeled  $\Delta = 0$  in this figure have essentially identical shapes, with the curves being offset horizontally from one another by a value of  $\pi$ . Stated somewhat differently, these results indicate that plane-wave modes are corrupted by fluctuations that occur along the  $x$  direction in the range  $-L/2$  to  $L/2$ , whereas OAM modes are corrupted by fluctuations that occur anywhere along the circumference of the OAM mode.

In addition to the numerical results presented in Fig. 2, we have found analytic expressions for the integral of Eq. (24) valid in the limits of very small and very large receiver aperture. For small aperture (i.e. for  $L/r_0 \rightarrow 0$ ) we can expand the exponent in Eq. (24) in a power series in  $(L/r_0)^{5/3}$  and retain only the lowest-order correction term. For  $\Delta = 0$  we integrate the resulting equation directly to obtain

$$\langle s_0 \rangle = 1 - 0.22 (L/r_0)^{5/3}. \quad (25)$$

For  $\Delta \neq 0$ , we find that to good approximation

$$\langle s_\Delta \rangle = \frac{0.0674}{\Delta^2} \left( \frac{L}{r_0} \right)^{5/3}. \quad (26)$$

For the opposite limit of very large receiver aperture, we obtain

$$\langle s_\Delta \rangle = 1.70 (r_0/L) \quad (27)$$

We have also found a highly accurate (at most 1.6% error) approximate expression for  $\langle s_0 \rangle$

valid over the entire domain of  $L/r_0$ , namely

$$\langle s_0 \rangle = \left[ 1 + (0.659L/r_0)^2 \right]^{-1/2}. \quad (28)$$

In any real implementation of a QKD system, it is necessary to be able to transmit in at least two different mutually orthogonal bases. One of the bases can be the plane-wave states considered in this paper. The other can be some linear combination of these states. To assess the performance of the entire system, a calculation of the sort presented here would need to be performed for each one of the additional bases. Such an analysis lies outside of the scope of this article. However, we can make some general comments. Certain linear combination of plane waves are highly localized in space. We expect these states to be less affected by turbulence than the more extended plane-wave states.

In summary, we have presented a calculation that quantifies the rate at which quantum information encoded on the plane-wave states of individual photons is lost as a result of propagation through atmospheric turbulence. These results are summarized by the simple relation of Eq. (28). By comparison with the results of previous workers, we find that plane-wave encoding is less quickly degraded by about a factor of three than is encoding in the OAM states of individual photons. These results should prove useful in the design of practical free-space quantum communication systems.

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