Momentum of Light in a Dielectric Medium

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We review different expressions that have been proposed for the stress tensor and for the linear momentum of light in dielectric media, focusing on the Abraham and Minkowski forms. Analyses of simple models and consideration of available experimental results support the interpretation of the Abraham momentum as the kinetic momentum of the field, while the Minkowski momentum is the recoil momentum of absorbing or emitting guest atoms in a host dielectric. Momentum conservation requires consideration not only of the momentum of the field and of recoiling guest atoms, but also of the momentum the field imparts to the medium. Different model assumptions with respect to electrostriction and the dipole force lead to different expressions for this momentum. We summarize recent work on the definition of the canonical momentum for the field in a dielectric medium.

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1. Introduction

In a vacuum, the energy and linear momentum per unit volume carried by an electromagnetic wave are, respectively,

$$u = \frac{1}{2}(\epsilon_0 \mathbf{E}^2 + \mu_0 \mathbf{H}^2), \tag{1}$$

$$\mathbf{g} = \frac{1}{c^2}\mathbf{E} \times \mathbf{H} = \mathbf{D} \times \mathbf{B}, \tag{2}$$

in the usual notation for the fields and the speed of light (c). For a monochromatic plane wave,

$$\mathbf{E}(\mathbf{r},t) = \hat{\mathbf{x}} E_0 \cos\omega(t - z/c), \quad \mathbf{H}(\mathbf{r},t) = \hat{\mathbf{y}}\sqrt{\frac{\epsilon_0}{\mu_0}} E_0 \cos\omega(t - z/c), \tag{3}$$

these energy and momentum densities are

$$u = \frac{1}{2}\epsilon_0 E_0^2 \quad \text{and} \quad \mathbf{g} = \frac{1}{2c}\hat{\mathbf{z}}\epsilon_0 E_0^2 \tag{4}$$

when we replace $\cos^2 \omega(t - z/c)$ by its average, $1/2$.

We can express these quantities in terms of photons by writing $u = q\hbar\omega/V$, where q is the average number of photons in a volume V. This implies from Eq. (4) that we should replace E_0^2 with $2q\hbar\omega/\epsilon_0 V$ and therefore $\mathbf{g}$ with $q\hbar\omega/cV$, consistent with a single photon in vacuum having a linear momentum $p = \hbar\omega/c$ and with the requirement of special relativity that the energy E and linear momentum p of a particle with zero rest mass satisfy $E = pc$.

We can extend these considerations heuristically to a wave with

$$\mathbf{E}(\mathbf{r},t) = \hat{\mathbf{x}} E_0 \cos\omega(t - nz/c), \quad \mathbf{H}(\mathbf{r},t) = \hat{\mathbf{y}}\sqrt{\frac{\epsilon}{\mu_0}} E_0 \cos\omega(t - nz/c) \tag{5}$$

in the case of a dielectric medium with refractive index $n = \sqrt{\epsilon/\epsilon_0}$ at the frequency ω. For utmost simplicity we assume for now that both absorption and dispersion are negligible at frequency ω, so that n may be taken to be real and $d\epsilon/d\omega$ to be 0. We will also assume, throughout this review, that $\mu \cong \mu_0$, generally an excellent approxi-

mation at optical frequencies. Then the cycle-averaged energy density is

$$u = \frac{1}{2}\epsilon E_0^2. \tag{6}$$

However, the form of the momentum density depends on which form of Eq. (2) we use. If we use the $(1/c^2)\mathbf{E}\times\mathbf{H}$ form, we obtain the (cycle-averaged) momentum density

$$\mathbf{g} = \frac{1}{2c^2}\hat{\mathbf{z}}\sqrt{\frac{\epsilon}{\mu_0}}E_0^2. \tag{7}$$

Using the same little heuristic argument as above to express this momentum in terms of photons, we write u as $q\hbar\omega/V$, E_0^2 as $2q\hbar\omega/\epsilon V$, and therefore $\mathbf{g}$ as

$$\mathbf{g} = \frac{q}{V}\frac{\hbar\omega}{nc}\hat{\mathbf{z}}, \tag{8}$$

implying that the momentum of a photon in a dielectric medium is

$$p_A = \frac{1}{n}\frac{\hbar\omega}{c}. \tag{9}$$

It is often asserted, however, that the momentum of a photon in a dielectric medium is $\hbar$ times the wave vector:

$$p_M = \hbar|\mathbf{k}| = n\frac{\hbar\omega}{c}. \tag{10}$$

In fact, this form follows directly from the $\mathbf{B}\times\mathbf{D}$ form of Eq. (2). Here p_A is the "Abraham" expression for the photon momentum in a dielectric medium, whereas p_M is the "Minkowski" expression. There is a long-standing question as to which expression is correct, and this review is primarily concerned with this question.

The theory of light momentum has an interesting history even when restricted to the propagation of light in a vacuum or nearly a vacuum. Maxwell deduced from his electromagnetic theory that a light wave has a momentum density given by Eq. (2), i.e., a pressure $P=I/c$, where $I=cu$ is the intensity. Adolfo Bartoli concluded independently that the second law of thermodynamics requires this same expression for radiation pressure. If the intensity of sunlight at the Earth's surface is assumed to be $1.4\ \mathrm{kW/m^2}$, it follows that $P=4.7\times10^{-6}\ \mathrm{N/m^2}$, not far from Maxwell's estimate of $8.82\times10^{-8}\ \mathrm{lb/ft^2}$. According to Maxwell [1], "A flat body exposed to sunlight would experience this pressure on its illuminated side, and would therefore be repelled from the side on which the light falls. It is probable that a much greater energy of radiation might be obtained by means of the concentrated rays of the electric lamp. Such rays falling on a thin metallic disk, delicately suspended in a vacuum, might perhaps produce an observable mechanical effect." Shortly thereafter Crookes, initially unaware of the earlier theoretical work, began an extensive series of experiments on "attraction and repulsion" due to radiation. He observed that hot objects repelled a pith ball, and since the density of surrounding air was such as to rule out convection as the cause of the repulsion, he cautiously suggested that the effect was due to "a repulsive action of radiation" [2]. Maxwell, acting as a referee of several of Crookes's papers [3], initially regarded Crookes's data

as a confirmation of his theory. The fluid dynamicist Reynolds was apparently the first to understand that the force must be a more complicated consequence of heating than radiation pressure [4]. Indeed in further experiments following the construction of his eponymous radiometer, Crookes found that it was the blackened sides of the vanes of the radiometer and not the silvered ones that were seemingly being pushed away from the source of radiation, whereas radiation pressure would have the opposite effect. Maxwell and Reynolds both subsequently published detailed analyses of the Crookes radiometer based on Reynolds's ideas. Over the next half-century hundreds of papers on the subject were published, and to this day the theory of radiometric forces remains a subject of research [5–7].

The first direct experimental demonstration of radiation pressure was evidently made by Lebedev [8], who inferred the pressure from the deflection caused by the reflection of light off a small mirror hanging from a torsion fiber in an evacuated glass jar. Similar experiments were performed at the same time by Nichols and Hull [9,10], who were able to more accurately subtract out the force on the mirror due to ambient gas effects rather than radiation pressure; they concluded that "The Maxwell–Bartoli theory is thus quantitatively confirmed within the probable errors of observation." It is worth emphasizing how weak were the forces measured by these pioneers. Nichols and Hull, for example, measured forces of the order of 7×10^{-5} dynes on a mirror of diameter $\sim$13 mm, corresponding approximately to the radiation pressure of sunlight calculated by Maxwell. The forces they measured are comparable with the gravitational force between two 3 kg masses separated by a meter. Gerlach and Golsen later confirmed the Maxwell–Bartoli theory to an accuracy of 2% [11].

The list of the many people who have contributed to the theory of the Crookes radiometer includes Einstein [12]. Far more important, of course, was his work on the photon concept. In his celebrated 1905 paper on the photoelectric effect, Einstein postulated that a photon of radiation of frequency ω has an energy $\hbar\omega$, and, based on the formula $E=pc$ noted earlier, one might think that he would also have postulated a photon momentum $p=\hbar\omega/c$. But neither Einstein nor anyone else associated this momentum with a photon until 1916; Pais [13] has called attention to this "remarkable fact that it took the father of special relativity theory twelve years to write down the relation [$p=\hbar\omega/c$] side by side with [$E=\hbar\omega$]." What Einstein did in 1916 was to show that the recoil momentum $\hbar\omega/c$ that accompanies the absorption and emission of radiation by atoms is consistent with the Planck spectrum of thermal equilibrium radiation. He considered this result to be more important than his derivation of the Planck spectrum based on his A and B coefficients, because "a theory [of thermal radiation] can only be regarded as justified when it is able to show that the impulses transmitted by the radiation field to matter lead to motions that are in accord with the theory of heat" [14,15].

Frisch [16] verified "Einstein's recoil radiation" in experiments on the deflection of atomic beams by radiation, and more accurate experiments were later performed with laser radiation [17]. There is of course plenty of evidence that photons have momentum $\hbar\omega/c$, and the exchange of this momentum between atoms and light is the basis for laser cooling and trapping, among other things.

But what about the momentum of light in a dielectric medium? In the experiments cited thus far it was not possible to observe any difference between the momentum of light in a vacuum and in a material medium because the refractive index was so close to unity. As discussed below, however, there have been other

types of experiment in which this difference has been observed and which have a direct bearing on the Abraham–Minkowski controversy. We begin in the following section by reviewing different stress tensors and momentum densities that have been proposed for electromagnetic radiation in material media, restricting our considerations to dielectric media and field frequencies at which absorption is negligible. We devote most of our attention, throughout this review, to the Abraham and Minkowski formulations. The field momentum in the Abraham theory is widely regarded as being the correct one, while at the same time the Minkowski momentum correctly describes, for example, the observed recoil of objects embedded in dielectrics; this circumstance explains in part why the Abraham and Minkowski theories have been the most widely favored ones, and why different authors have advocated the use of one over the other. In Section 3 we discuss two basic and instructive examples of the momentum exchange between light and matter: (i) the recoil and displacement of a transparent dielectric block when light passes through it, and (ii) the Doppler effect in the absorption or emission of light by an atom. These examples are intended to highlight the significance of the Abraham and Minkowski momenta, respectively. Section 4 is a brief overview of experiments on momentum exchange involving light propagating in a dielectric medium, and in Section 5 we analyze a few examples of such momentum exchange. In Section 6 we review the concepts of canonical and kinetic momentum and their relevance to the Abraham–Minkowski controversy, and in the final section we summarize our conclusions with respect to it.

There is no shortage of reviews of this topic; see, for instance, Refs. [18–25]. In fact the literature on the comparison of the Abraham and Minkowski stress tensors is sufficiently large that we can claim familiarity with only a portion of it. This review is intended for readers with little or no previous exposure to the subject. We will not be concerned with formal symmetry and transformation properties of the fully relativistic energy-momentum tensor, and the 3×3 Maxwell stress tensor appropriate to considerations of electromagnetic momentum will be introduced and discussed mainly in connection with recoil forces in dielectric media.

2. Different Forms of the Stress Tensor

The macroscopic Maxwell equations are

$$\nabla\cdot\mathbf{D}=\rho,\quad \nabla\cdot\mathbf{B}=0,$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\quad \nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}, \tag{11}$$

in the standard notation. The force on a volume V of the material medium may be obtained by consideration of the integral over this volume of the Lorentz force density $\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}$. It follows straightforwardly from Maxwell's equations that this integral, equal to the rate of change of the "mechanical" momentum $\mathbf{P}_m$ of the material in the volume V, is

$$\frac{d\mathbf{P}_m}{dt} \equiv \int_V \frac{d\mathbf{p}_m}{dt} dV = \int_V [\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}] dV$$
$$= \int_V \left[(\nabla \cdot \mathbf{D})\mathbf{E} + (\nabla \times \mathbf{H}) \times \mathbf{B} - \frac{\partial \mathbf{D}}{\partial t} \times \mathbf{B} \right] dV$$
$$= \int_V \left[(\nabla \cdot \mathbf{D})\mathbf{E} + (\nabla \times \mathbf{H}) \times \mathbf{B} + \mathbf{D} \times \frac{\partial \mathbf{B}}{\partial t} - \frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B}) \right] dV$$
$$= \int_V [(\nabla \cdot \mathbf{D})\mathbf{E} + (\nabla \cdot \mathbf{H})\mathbf{B} - \mathbf{D} \times (\nabla \times \mathbf{E}) - \mathbf{B} \times (\nabla \times \mathbf{H})] dV$$
$$- \frac{d}{dt} \int_V (\mathbf{D} \times \mathbf{B}) dV, \tag{12}$$

and therefore that

$$\int_V \left[\frac{d}{dt}(\mathbf{p}_m + \mathbf{D} \times \mathbf{B}) \right] dV = \int_V [(\nabla \cdot \mathbf{D})\mathbf{E} + (\nabla \cdot \mathbf{H})\mathbf{B} - \mathbf{D} \times (\nabla \times \mathbf{E})$$
$$- \mathbf{B} \times (\nabla \times \mathbf{H})] dV. \tag{13}$$

We have added $0 = (\nabla \cdot \mathbf{H})\mathbf{B}$ to the right-hand sides in order to put these equations in a form more symmetrical in the electric and magnetic fields. If we assume a linear (but not necessarily isotropic) relation between $\mathbf{D}$ and $\mathbf{E}$, it also follows from Maxwell's equations that we can write the ith Cartesian component of a force density as

$$\frac{d}{dt}(\mathbf{p}_m + \mathbf{D} \times \mathbf{B})_i = \sum_{j=1}^{3} \frac{\partial T_{ij}^M}{\partial x_j}, \tag{14}$$

where the stress tensor has components

$$T_{ij}^M = E_i D_j + H_i B_j - \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})\delta_{ij} \quad (i,j = 1,2,3). \tag{15}$$

2.1. Minkowski and Abraham

If one accepts the validity of the macroscopic Maxwell equations, there can be no objections to Eqs. (14) and (15). These equations suggest that the total momentum density is the mechanical momentum density $\mathbf{p}_m$ plus

$$\mathbf{g}_M = \mathbf{D} \times \mathbf{B}, \tag{16}$$

in terms of which Eq. (14) is

$$\left(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} + \frac{\partial \mathbf{g}_M}{\partial t} \right)_i = \sum_{j=1}^{3} \frac{\partial T_{ij}^M}{\partial x_j}. \tag{17}$$

Equation (16) defines the Minkowski momentum density, and T_{ij}^M is the Minkowski form of the stress tensor.

Let us also recall another relation—Poynting's theorem—that follows from the macroscopic Maxwell equations:

$$\nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}. \tag{18}$$

The Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ gives the flux of electromagnetic energy. The right-hand side is the rate of change of the total energy, that of the field plus that of the material medium, but *only energy attributable to the field actually propagates out of any given volume element of the medium*. (This assumes that we can ignore any elastic forces in the material that can cause "mechanical" energy to be transported.) In other words, $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ gives the energy flux of the field in the medium as well as in free space. From this assumption, and the relation $\mathbf{g}=\mathbf{S}/c^2$ of special relativity theory for the momentum density associated with any process, electromagnetic or otherwise, by which energy is transported with a flux $\mathbf{S}$, we are led to assign to the field a momentum density

$$\mathbf{g}_A = \mathbf{E} \times \mathbf{H}/c^2. \tag{19}$$

This defines the Abraham momentum density [Eq. (2)]. Using $\mathbf{D}\times\mathbf{B}=(1/c^2)\times(1+n^2-1)\mathbf{E}\times\mathbf{H}$ for an effectively nondispersive and isotropic linear medium with refractive index n, we can write Eq. (14) as

$$\left(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} + \mathbf{f}^A + \frac{\partial \mathbf{g}_A}{\partial t}\right)_i = \sum_{j=1}^{3} \frac{\partial T_{ij}^M}{\partial x_j}, \tag{20}$$

where

$$\mathbf{f}^A = \frac{1}{c^2}(n^2 - 1)\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{H}) \tag{21}$$

is the so-called *Abraham force* density.

Equation (17) suggests that the force density acting on the material medium is

$$(\mathbf{f}_M)_i = \sum_{j=1}^{3} \frac{\partial T_{ij}^M}{\partial x_j} - \left(\frac{\partial \mathbf{g}_M}{\partial t}\right)_i, \tag{22}$$

whereas according to Eq. (20) the force density acting on the medium is

$$(\mathbf{f}_A)_i = \sum_{j=1}^{3} \frac{\partial T_{ij}^M}{\partial x_j} - \left(\frac{\partial \mathbf{g}_A}{\partial t}\right)_i, \tag{23}$$

i.e.,

$$\mathbf{f}_A = \mathbf{f}_M + \frac{\partial \mathbf{g}_M}{\partial t} - \frac{\partial \mathbf{g}_A}{\partial t} = \mathbf{f}_M + \mathbf{f}^A. \tag{24}$$

Thus in the Minkowski formulation the force acting on the particles of the dielectric medium is obtained by subtracting $\partial\mathbf{g}_M/\partial t$ from $\Sigma_j \partial T_{ij}^M/\partial x_j$, suggesting that the field momentum density is $\mathbf{g}_M$. In the Abraham interpretation as just described, however, the force on the medium is obtained by subtracting $\partial\mathbf{g}_A/\partial t$ from $\Sigma_j \partial T_{ij}^M/\partial x_j$ under the assumption that the momentum density of the field is $\mathbf{g}_A$. In this interpretation there appears the force density $\mathbf{f}^A$ that, together with the

Lorentz force density $\rho\mathbf{E}+\mathbf{J}\times\mathbf{H}$, gives the total force on the material medium, as is clear from Eq. (20).

Of course both the Minkowski and Abraham momentum densities, $\mathbf{g}_M$ and $\mathbf{g}_A$, are defined in terms of measurable quantities and are themselves therefore measurable in principle. Either momentum density will comport with conservation of linear momentum, the only difference being in how we choose to apportion the total momentum between the field and the material medium. This is of course obvious from the equation

$$\left(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}+\frac{\partial\mathbf{g}_M}{\partial t}\right)=\left(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}+\mathbf{f}^A+\frac{\partial\mathbf{g}_A}{\partial t}\right). \tag{25}$$

On the left-hand side we could interpret $\mathbf{g}_M$ as field momentum density; on the right-hand side we could interpret $\mathbf{g}_A$ as field momentum density, but then, compared with the left side, we have an additional (Abraham) force (and momentum) density associated with the medium. The generally accepted view, which we advocate here, is that $\mathbf{g}_A$ is the momentum density of the field. Once we adopt this viewpoint, "It is…impossible to question [the Abraham force density] notwithstanding that it has as yet not been reliably measured directly. In that way the problem would be solved 'in favor' of the Abraham tensor" [26]. Measurements of the Abraham force density $\mathbf{f}^A$ are discussed in Section 4.

The stress tensor T_{ij}^M is the spatial part of a four-dimensional energy-momentum tensor employed by Minkowski [27,28] in the context of the electrodynamics of moving bodies. For an isotropic medium that is effectively dispersionless at frequencies of interest, and which has no free charges or currents ($\rho=\mathbf{J}=0$), the Minkowski force density (22) reduces to

$$\mathbf{f}_M=-\frac{1}{2}\mathbf{E}^2\nabla\epsilon. \tag{26}$$

Minkowski's energy-momentum tensor (15) is not symmetric. Even the 3×3 matrix T_{ij}^M defined here is not symmetric in the general case of an anisotropic medium. This lack of symmetry led Abraham [29,30] to introduce a different, symmetric energy-momentum tensor, the 3×3 (spatial) part of which is

$$T_{ij}^A=\frac{1}{2}(E_iD_j+E_jD_i)+\frac{1}{2}(H_iB_j+H_jB_i)-\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{H})\delta_{ij}. \tag{27}$$

In terms of this tensor,

$$(\mathbf{f}_A)_i=\sum_{j=1}^{3}\frac{\partial}{\partial x_j}T_{ij}^A-\left(\frac{\partial\mathbf{g}_A}{\partial t}\right)_i=(\mathbf{f}_M)_i+(\mathbf{f}^A)_i, \tag{28}$$

or

$$\mathbf{f}_A=-\frac{1}{2}\mathbf{E}^2\nabla\epsilon+\mathbf{f}^A \tag{29}$$

for an isotropic and dispersionless medium. We remind the reader of the distinction between $\mathbf{f}_A$ and $\mathbf{f}^A$: $\mathbf{f}_A$ is the force density acting on the medium in the Abraham formulation of the stress tensor, whereas $\mathbf{f}^A$ is the "Abraham force density" that appears in addition to the Lorentz force density in this formulation.

We regard the argument leading to Eq. (19) as a strong one for interpreting $\mathbf{g}_A$ as the momentum density *of the field*; more direct arguments based on specific models are presented below. This interpretation appears to be generally accepted. Jackson [31], for example, states that "all workers agree on the definition" of $\mathbf{g}_A$ as the electromagnetic momentum density, and Landau and Lifshitz [32] clearly adopt this definition when they subtract $\partial \mathbf{g}_A / \partial t$ from the total force density to obtain the force density acting on the material medium, as in Eq. (23). Ginzburg [33] writes that "All we have said allows us to take the Abraham tensor to be the 'correct' one …."

A more general formulation of the stress tensor describing electromagnetic forces in a dielectric fluid leads to the force density first derived by Abraham [29,30,34], which has been regarded as "one of the most important results of the electrodynamics of continuous media" [35]:

$$\mathbf{f} = -\nabla P + \nabla\left[\rho\frac{\partial\epsilon}{\partial\rho}\frac{1}{2}\mathbf{E}^2\right] - \frac{1}{2}\mathbf{E}^2\nabla\epsilon + \frac{1}{c^2}(n^2-1)\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}), \tag{30}$$

where ρ is the mass density and P is the pressure in the fluid, which depends on the field only to the extent that the field can affect the density and temperature. The second term on the right-hand side is associated with electrostriction. Under conditions of frequent interest, such as when there is mechanical equilibrium [20,35,36], it is cancelled by the first term, and the net force density reduces to the Abraham force density (29):

$$\mathbf{f} = -\frac{1}{2}\mathbf{E}^2\nabla\epsilon + \frac{1}{c^2}(n^2-1)\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}) = \mathbf{f}_A. \tag{31}$$

Unless otherwise noted, we will assume that this cancellation holds and ignore electrostriction.

2.2. Microscopic Perspective on Force Density

The Lorentz force on a charge distribution of density ρ is $\mathbf{F}=\int[\rho\mathbf{E}+\rho\dot{\mathbf{r}}\times\mathbf{B}]d^3r$. For two oppositely charged $(\pm q)$ point charges at $\mathbf{R}$ and $\mathbf{R}+\mathbf{x}$ this reduces to $\mathbf{F}=q[\mathbf{E}(\mathbf{R}+\mathbf{x})-\mathbf{E}(\mathbf{R})]+q(\dot{\mathbf{R}}+\dot{\mathbf{x}})\times\mathbf{B}(\mathbf{R}+\mathbf{x})-q\dot{\mathbf{R}}\times\mathbf{B}(\mathbf{R})=q\mathbf{x}\cdot\nabla\mathbf{E}(\mathbf{R})+q\dot{\mathbf{x}}\times\mathbf{B}(\mathbf{R})$ as $\mathbf{x}\to 0$. In other words, the force on an electric dipole moment $\mathbf{d}=q\mathbf{x}$ in an electromagnetic field is

$$\mathbf{F} = (\mathbf{d}\cdot\nabla)\mathbf{E} + \dot{\mathbf{d}}\times\mathbf{B}. \tag{32}$$

Introducing the polarizability α, we can express this force as

$$\begin{aligned}\mathbf{F} &= \alpha\left[(\mathbf{E}\cdot\nabla)\mathbf{E} + \frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}\right]\\ &= \alpha\left[\nabla\left(\frac{1}{2}\mathbf{E}^2\right) - \mathbf{E}\times(\nabla\times\mathbf{E}) + \mu_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}) - \mu_0\mathbf{E}\times\frac{\partial\mathbf{H}}{\partial t}\right]\\ &= \alpha\left[\nabla\left(\frac{1}{2}\mathbf{E}^2\right) + \mu_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H})\right].\end{aligned} \tag{33}$$

Using $N\alpha = \epsilon - \epsilon_0$, we obtain the force density for a medium of N dipoles per unit volume:

$$\begin{aligned}\mathbf{f}_{\text{dipoles}} &= (\epsilon - \epsilon_0)\left[\nabla\left(\frac{1}{2}\mathbf{E}^2\right) + \mu_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H})\right] \\ &= \epsilon_0(n^2-1)\,\nabla\left(\frac{1}{2}\mathbf{E}^2\right) + \frac{1}{c^2}(n^2-1)\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}) \\ &= \epsilon_0(n^2-1)\,\nabla\left(\frac{1}{2}\mathbf{E}^2\right) + \mathbf{f}^A, \end{aligned} \tag{34}$$

which obviously differs in general from both $\mathbf{f}_M$ and $\mathbf{f}_A$ as defined by Eqs. (26) and (29), respectively.

Gordon [37] attributes this difference to the fact that the local field acting on an electric dipole and the macroscopic field are not the same. To pursue this point further, we follow Gordon and write the force density (30) as

$$\mathbf{f} = -\nabla P + \mathbf{f}_{\text{dipoles}} + \nabla\left\{\left[\rho\frac{d\epsilon}{d\rho} - (\epsilon - \epsilon_0)\right]\frac{1}{2}\mathbf{E}^2\right\}. \tag{35}$$

The last term on the right vanishes if $\epsilon - \epsilon_0$ is simply proportional to the density ρ, i.e., if we assume that the local field and the macroscopic field are the same; this assumption was used in the derivation of Eq. (34). Aside from a force density associated with a pressure gradient, therefore, the difference between $\mathbf{f}$ and $\mathbf{f}_{\text{dipoles}}$ may be attributed to the difference between macroscopic and local fields. If, for instance, we assume the Lorentz–Lorenz local field leading to the Clausius–Mossotti relation,

$$\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} = A\rho, \tag{36}$$

then the difference between $\mathbf{f}$ and $\mathbf{f}_{\text{dipoles}}$ is second order in $\epsilon - \epsilon_0$,

$$\rho\frac{d\epsilon}{d\rho} - (\epsilon - \epsilon_0) = \frac{1}{3\epsilon_0}(\epsilon - \epsilon_0)^2, \tag{37}$$

and is negligible for a sufficiently dilute medium.

The difference between $\mathbf{f}$ and $\mathbf{f}_{\text{dipoles}}$ is closely related to the difference between the "polarization" approach of Kelvin and the "energy" approach of Helmholtz in the theory of the force on a dielectric fluid in an electric field [20,38]. The Kelvin approach, like the derivation here of $\mathbf{f}_{\text{dipoles}}$, ignores the interactions between the dipoles, whereas they are included in the more general Helmholtz theory. For our purposes here we can assume a dielectric medium of sufficiently low density that dipole–dipole interactions are negligible and that local field corrections can be ignored.

2.3. Einstein and Laub

While the Abraham and Minkowski stress tensors are by far the most widely cited and compared, they are not the only ones that have been proposed. The stress tensor originally advocated by Einstein and Laub [39], for instance, is

$$T_{ij}^{\mathrm{EL}} = E_i D_j + H_i B_j - \frac{1}{2}(\epsilon_0 \mathbf{E}^2 + \mu_0 \mathbf{H}^2)\,\delta_{ij} \tag{38}$$

in the case of the purely dielectric, nonmagnetic media of interest here. The field momentum density in the Einstein–Laub formulation is $\mathbf{g}_A$, the Abraham form, and the force density is therefore

$$\mathbf{f}^{\mathrm{EL}} = \sum_{j=1}^{3} \frac{\partial T_{ij}^{\mathrm{EL}}}{\partial x_j} - \left(\frac{\partial \mathbf{g}_A}{\partial t}\right)_i, \tag{39}$$

which is found, by using Maxwell's equations for $\rho = \mathbf{J} = 0$, to be

$$\begin{aligned}\mathbf{f}_{\mathrm{EL}} &= (\mathbf{P}\cdot\nabla)\mathbf{P} + \dot{\mathbf{P}} \times \mathbf{B} = \frac{1}{2}(\mathbf{P}\cdot\mathbf{E}) - \frac{1}{2}\mathbf{E}^2\,\nabla\,\epsilon + \frac{1}{c^2}(n^2-1)\frac{\partial}{\partial T}(\mathbf{E}\times\mathbf{H}) \\ &= \mathbf{f}_A + \frac{1}{2}(\mathbf{P}\cdot\mathbf{E})\end{aligned} \tag{40}$$

at frequencies for which dispersion is negligible. The first equality is physically intuitive: it gives the force density as just the number density N times the force on an individual electric dipole, expressed in terms of the polarization density $\mathbf{P} = N\mathbf{p}$, ignoring any local field corrections. The last equality shows that there is an extra term $\frac{1}{2}(\mathbf{P}\cdot\mathbf{E})$ compared with Abraham's $\mathbf{f}_A$.

The Einstein–Laub tensor has never been generally regarded as a viable alternative to the Minkowski or Abraham tensors. Historically, this is partly due to its implications for the effect of a magnetic field on a conductor with current density $\mathbf{J}$: Einstein and Laub found a force density involving $\mathbf{J}\times\mathbf{H}$ rather than the universally accepted $\mathbf{J}\times\mathbf{B}$. Another reason was Einstein's own rejection of it, a few years after the publication of the Einstein–Laub paper [40].

2.4. Peierls

Peierls [41,42] argued that none of the forms of the stress tensor written thus far is correct and proposed a different form based on a microscopic approach related to the local field corrections discussed earlier. According to him the field momentum has the Abraham form, while the correct total momentum density for a uniform medium is

$$\mathbf{g}_P = \left[\frac{n^2+1}{2} - \frac{\sigma}{2}(n^2-1)^2\right]\mathbf{g}_A = \frac{1}{2}\left[\frac{1}{n} + n - \frac{\sigma}{n}(n^2-1)^2\right]\mathbf{g}_{\mathrm{vac}}, \tag{41}$$

where σ is a numerical coefficient and $\mathbf{g}_{\mathrm{vac}}$ is the momentum density in a vacuum. The stress tensor appropriate to Peierls's theory is [20]

$$T_{ij}^{P} = \left[\epsilon + \frac{\sigma}{\epsilon_0}(\epsilon-\epsilon_0)^2\right]E_iE_j + H_iB_j - \frac{1}{2}\delta_{ij}\left(\left[\epsilon_0 - \frac{\sigma}{\epsilon_0}(\epsilon-\epsilon_0)^2\right]\mathbf{E}^2 + \mu_0\mathbf{H}^2\right). \tag{42}$$

For $n \cong 1$ Eq. (41) reduces to the average of the Minkowski and Abraham momenta on a per-photon basis:

$$p_P = \frac{1}{2}\left(n + \frac{1}{n}\right)\frac{\hbar\omega}{c} = \frac{1}{2}(p_M + p_A). \tag{43}$$

In Peierls's calculation of σ he includes the effects of neighboring dipoles on the local electric field gradient appearing in Eq. (32). He obtains $\sigma = 1/5$, from which we conclude that the total momentum predicted in the case of a uniform medium is

$$p_P = \frac{1}{10n}(4n + 7n^2 - n^4)\frac{\hbar\omega}{c} \tag{44}$$

on a per-photon basis. For n larger than about 2.8, $p_P < 0$, implying a total momentum in the direction opposite to the direction of field propagation [41,42]. While Peierls's arguments seem plausible, the available experimental results cast doubt on the theory, as discussed briefly in Section 4.

2.5. Modifications Due to Dispersion

We now consider the effects of dispersion on photon momentum. Because the Einstein–Laub and Peierls theories are not regarded as viable alternatives to those of Abraham and Minkowski, we will consider specifically only the modifications of p_A and p_M due to dispersion.

Recall that the cycle-averaged electromagnetic energy density at frequencies at which absorption is negligible is given by [43]

$$u = \frac{1}{4}\left[\frac{d}{d\omega}(\epsilon\omega)|\mathbf{E}_\omega|^2 + \mu_0|\mathbf{H}_\omega|^2\right]. \tag{45}$$

Here $\mathbf{E} = \mathbf{E}_\omega(\mathbf{r})e^{-i\omega t}$, $\mathbf{H} = \mathbf{H}_\omega(\mathbf{r})e^{-i\omega t}$, and ϵ is real under the assumption that there is no absorption at frequency ω. u is the total electromagnetic energy density for the (passive) medium and the field. We continue to assume a purely dielectric medium ($\mu = \mu_0$). For a monochromatic plane wave, $|\nabla \times \mathbf{H}_\omega| = k|\mathbf{H}_\omega| = \omega\sqrt{\epsilon\mu_0}|\mathbf{H}_\omega|$, and from the Maxwell equation $\nabla \times \mathbf{H}_\omega = -i\omega\epsilon\mathbf{E}_\omega$ it follows that $|\mathbf{H}_\omega|^2 = (\epsilon/\mu_0)|\mathbf{E}_\omega|^2$. Then Eq. (45) takes the form

$$u = \frac{1}{4}\epsilon_0 n n_g |\mathbf{E}_\omega|^2, \tag{46}$$

with

$$n_g = \frac{d}{d\omega}(n\omega) = n + \omega\frac{dn}{d\omega} \tag{47}$$

the group index.

We now proceed as in the Introduction, Section 1, to express the Abraham and Minkowski momenta in terms of photons. When the field is quantized in a volume V, u is in effect replaced by $q\hbar\omega/V$, where q is the expectation value of the photon number in the volume V, so that $\mathbf{E}_\omega^2$ is effectively replaced by $2\hbar\omega/(\epsilon_0 n n_g V)$ per photon. Thus the Abraham momentum associated with a single photon's worth of energy is

$$p_A = \frac{n}{c}\frac{1}{2}\epsilon_0 \frac{2\hbar\omega}{\epsilon_0 n n_g V} V = \frac{1}{n_g}\frac{\hbar\omega}{c}, \tag{48}$$

and similarly

$$p'_M = \frac{n^2}{n_g}\frac{\hbar\omega}{c} \tag{49}$$

is the Minkowski momentum attributed to a single photon. For reasons that will become clear below, we use a prime to distinguish Eq. (49) from the Minkowski momentum (10). One can arrive at these same expressions more formally by quantizing the fields **E**, **D**, **H**, and **B** in a dispersive medium [44].

Two points are worth noting in connection with these formulas. First, as is well known, the group index n_g can be negative. However, under our assumption that absorption is negligible at frequencies ω of interest, n_g is in fact positive for any passive medium. This follows from the identity

$$n_g = \frac{1}{2}\sqrt{\frac{\epsilon}{\epsilon_0}} + \frac{1}{2}\sqrt{\frac{\epsilon_0}{\epsilon}}\frac{d}{d\omega}\left(\frac{\omega\epsilon}{\epsilon_0}\right). \tag{50}$$

The Kramers–Kronig dispersion formula relates the real (ϵ_R) and imaginary (ϵ_I) parts of the permittivity as follows:

$$\epsilon_R(\omega) - 1 = \frac{2}{\pi}\int_0^\infty \frac{\omega'\epsilon_I(\omega')}{\omega'^2 - \omega^2} d\omega'. \tag{51}$$

In the usual form of this relation the integral on the right-hand side is the Cauchy principal part, but since $\epsilon_I(\omega)=0$ by our assumption of no absorption at frequency ω, what appears in Eq. (51) is an ordinary integral. Therefore

$$\frac{d\epsilon_R}{d\omega} = \frac{4\omega}{\pi}\int_0^\infty \frac{\omega'\epsilon_I(\omega')}{(\omega'^2 - \omega^2)^2} d\omega', \tag{52}$$

which is positive for any passive (nonamplifying) medium, i.e., for any medium for which $\epsilon_I(\omega)>0$ at all frequencies. It then follows from Eq. (50) with $\epsilon=\epsilon_R$ that $n_g>0$. A similar proof can be given for negative-index media [45].

The second point concerns the factor n/n_g in formula (49) for p'_M. To address this point, let us first recall that the equation $\omega(k)=kc/n(\omega)$ for the wave number allows different solutions that we associate with different polariton branches of the coupled matter–field system. Consider, for example, the simplest model of a dielectric medium, for which there is only a single resonance frequency ω_0 such that the real part of the refractive index satisfies

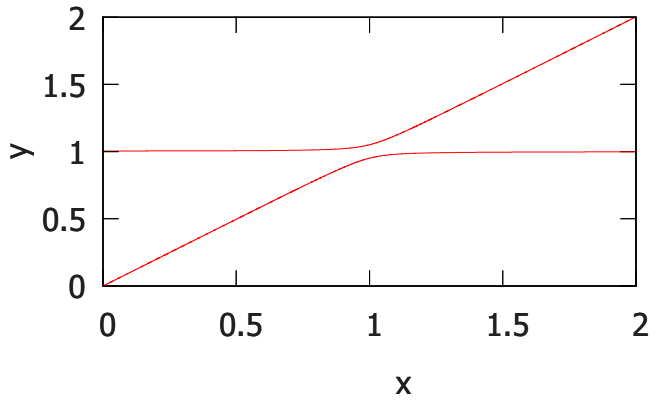
$$n^2(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}. \tag{53}$$

The dispersion equation $k=n(\omega)\omega/c$ takes the form

$$\omega^4 - (\omega_p^2 + \omega_0^2 + k^2c^2)\omega^2 + k^2c^2\omega_0^2 = 0, \tag{54}$$

with two solutions:

Figure 1



$y=\omega/\omega_0$ versus $x=kc/\omega_0$ for the two polariton branches given by Eq. (55). In this example $\omega_p/\omega_0=0.1$.

$$\omega_\pm^2=\frac{1}{2}(\omega_0^2+\omega_p^2+k^2c^2)\pm\frac{1}{2}\sqrt{(\omega_0^2+\omega_p^2+k^2c^2)^2-4k^2c^2\omega_0^2}. \tag{55}$$

Thus in this model there are two different allowed $\omega-k$ curves (polariton branches) separated by a stopband (Fig. 1). More generally the frequency ω appearing in our expressions for p_A, p_M, and p'_M must, of course, be understood to correspond to but one of the (possibly many) polariton branches of the dielectric.

One finds after some algebra that the sum of the ratios n/n_g for the two branches of the single-resonance model is equal to 1. Interestingly, it may be shown more generally that

$$\sum_i \frac{n(\omega_i)}{n_g(\omega_i)}=1, \tag{56}$$

the summation being over *all* the polariton branches of the dielectric [46–48]. As discussed in Section 6, this sum rule has important ramifications relating to whether p'_M as defined by Eq. (49) should be identified as the Minkowski momentum of a photon in a dispersive medium.

3. Examples of Momentum Transfer between Light and Matter

We now turn our attention to a few basic examples of the transfer of momentum between light and matter. These examples will be used to support the interpretation of the Abraham momentum as the momentum of the field, whereas the Minkowski momentum is the momentum that the field imparts to atoms in a dielectric medium. For simplicity we assume in each of these examples that dispersion is negligible at the frequency ω considered.

3.1. The Balazs Block

For our first example we consider a block of mass M, refractive index n, and thickness a. The block is initially at rest on a frictionless surface (Fig. 2). A

Figure 2

A block with refractive index n on a frictionless surface and initially at rest is displaced by Δx when it transmits a photon.

single-photon pulse of frequency ω is incident on the block, which is assumed to be nonabsorbing at frequency ω and to have antireflection coatings on its front and back surfaces. If the photon momentum is p_{in} inside the block and p_{out} in the vacuum outside it, the block will pick up a momentum $MV=p_{\text{out}}-p_{\text{in}}$ when the pulse enters. Outside the block the photon momentum is $p_{\text{out}}=mc$, where $m=E/c^2=\hbar\omega/c^2$ is the mass associated with the photon. Similarly $p_{\text{in}}=mv$, where v is the velocity of light in the block. If there is no dispersion, $v=v_p=c/n$, and the momentum of the photon in the block is evidently given by $p_{\text{in}}=mc/n=\hbar\omega/nc=p_A$, the Abraham photon momentum. The crucial assumption in this argument, originally made in essentially this way by Balazs [49], is that the velocity of light in the dispersionless medium is the phase velocity v_p. Together with momentum conservation, this assumption leads to the conclusion that the momentum of the field has the Abraham form.

This can in principle be tested experimentally. Conservation of momentum requires that $MV=m(c-v)$. When the pulse exits the block, the block recoils and comes to rest and is left with a net displacement

$$\Delta x = V\Delta t = \frac{m}{M}(c-v)\frac{a}{v} = \frac{\hbar\omega}{Mc^2}(n-1)a \tag{57}$$

as a result of the light having passed through it. If the photon momentum inside the block were assumed to have the Minkowski form $n\hbar\omega/c$, however, the displacement of the block would in similar fashion be predicted to be

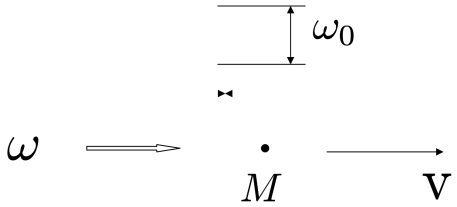
$$\Delta x = \frac{\hbar\omega}{Mc^2}an(1-n). \tag{58}$$

Obviously these two different assumptions about the photon momentum in the medium can lead to different predictions for both the magnitude of the block displacement and its direction.

3.2. The Doppler Effect

In the absence of an experimental test the example just considered does not prove that the momentum of a photon in a dielectric medium is $p_A=\hbar\omega/nc$, but only makes it plausible. We next consider an example where the answer—the Doppler shift in a medium of refractive index n—is known, and see what it says about p_A versus p_M. This example is based on an argument of Fermi's that the Doppler effect is a consequence of recoil [50,51]. Consider an atom of mass M inside a dielectric medium with refractive index $n(\omega)$. The atom is assumed to have a sharply defined transition frequency ω_0 and to be moving initially with a velocity v away from a source of light of frequency ω (Fig. 3). Because the light

Figure 3



An atom having a transition frequency ω_0 and moving with velocity v away from a source of light of frequency ω. The atom is inside a dielectric medium with refractive index $n(\omega)$. Because of the Doppler effect, absorption by the atom requires that $\omega \cong \omega_0(1+nv/c)$.

in the atom's reference frame has a Doppler-shifted frequency $\omega(1-nv/c)$ determined by the phase velocity (c/n) of light in the medium, the atom can absorb a only photon if $\omega(1-nv/c)=\omega_0$, or

$$\omega \cong \omega_0(1+nv/c). \tag{59}$$

We associate with a photon in the medium a momentum p, and consider the implications of (nonrelativistic) energy and momentum conservation. The initial energy is $E_i=\hbar\omega+\frac{1}{2}Mv^2$, and the final energy, after the atom has absorbed a photon, is $\frac{1}{2}Mv'^2+\hbar\omega_0$, where v' is the velocity of the atom after absorption. The initial momentum is $p+Mv$, and the final momentum is just Mv'. Therefore

$$\frac{1}{2}M(v'^2-v^2) \cong Mv(v'-v) = Mv(p/M) = \hbar(\omega-\omega_0), \tag{60}$$

or

$$\omega \cong \omega_0 + \frac{pv}{\hbar}. \tag{61}$$

From Eq. (59) and $\omega \cong \omega_0$ we conclude that

$$p = n\frac{\hbar\omega}{c} = p_M. \tag{62}$$

The example of Fig. 2 suggests that the momentum of the photon is p_A, while the second example seems at first thought to suggest that it is p_M. There is no doubt about the (first-order) Doppler shift in a dielectric medium being $nv\omega/c$, as we have assumed, but does this imply that the momentum of a photon in a dielectric is $n\hbar\omega/c$? We will show in Section 5 that the total force exerted by a single-photon plane monochromatic wave on the particles of a dielectric, including the Abraham force, suggests a momentum density of magnitude

$$p_{\text{med}} = \left(n - \frac{1}{n}\right)\frac{\hbar\omega}{c}\frac{1}{V} \tag{63}$$

if dispersion is negligible. Now from our conclusion from energy and momentum conservation that the known Doppler shift implies that an absorber (or emit-

ter) inside a dielectric recoils with momentum $n\hbar\omega/c$, all we can logically deduce is that a momentum $n\hbar\omega/c$ is taken from (or given to) the *combined system* of the field and the dielectric medium. Given that the medium acquires a momentum density, Eq. (63), from the force exerted on it by the propagating field, and that the atom recoils with momentum $n\hbar\omega/c$, we can attribute to the field, by conservation of momentum, a momentum density

$$n\frac{\hbar\omega}{c}\frac{1}{V}-p_{\text{med}}=\frac{1}{n}\frac{\hbar\omega}{c}\frac{1}{V}=p_A. \tag{64}$$

In other words, the kinetic momentum of the field is the Abraham momentum, consistent with our discussion of the example of Fig. 2. The momentum $n\hbar\omega/c$ evidently gives the momentum not of the field as such but of the combined system of field plus dielectric; it is the momentum density equal to the *total* energy density $u=\hbar\omega/V$ for a monochromatic field divided by the phase velocity c/n of the propagating wave. As discussed in Section 4, experiments on the recoil of objects immersed in dielectric media indicate that the recoil momentum is $n\hbar\omega/c$ per unit of energy $\hbar\omega$ of the field, just as in the Doppler effect.

A consistent interpretation of these examples, therefore, is that the field carries a momentum p_A per photon, but that there is also a momentum p_{med} imparted by the field to the medium. An atom that absorbs or emits a photon of frequency ω in the medium therefore recoils with the momentum $p_A+p_{\text{med}}=p_M$, *as if* the photon momentum were the Minkowski photon momentum p_M. This interpretation is consistent with that of Ginzburg [33], who regards the Minkowski stress tensor as "an auxiliary concept which can be used fully."

4. Overview of Experiments

There are not very many reports of experiments testing the different expressions for electromagnetic momenta in dielectric media. Brevik [20] discusses in considerable detail the implications of some of the older experiments.

The Abraham force $\mathbf{f}^A=(1/c^2)(n^2-1)(\partial/\partial t)(\mathbf{E}\times\mathbf{H})$ obviously plays an important role in comparisons of the Abraham and Minkowski formulations of the stress tensor; recall the remarks following Eq. (25). But $\mathbf{f}^A$ will average to zero over times that are long compared with an optical period, and its effect will not be directly observable. In their discussion of electromagnetic momentum Panofsky and Phillips [52] give another reason for why it is usually unobservable: "Its net impulse due to a finite wave train always vanishes; we shall not discuss its rather complicated interpretation." However, the Abraham force can be measured if the electric and magnetic fields are applied continuously *and* if they vary slowly enough. Such measurements were reported by James [53] and Walker *et al.* [54]. In one of the experiments [54] an annular disk of high permittivity ($\epsilon\approx 3620$), serving as a torsion pendulum, was subjected to a static, vertical magnetic field and a slowly varying (0.4 Hz) radial electric field between its inner and outer surfaces. Then the Abraham force is azimuthal, and the oscillations of the electric field should cause the disk to oscillate about the direction of the magnetic field. Such oscillations were observed, and the measurements confirmed the existence of the Abraham force to an accuracy of about 5%.

Another very important set of experiments was carried out by Jones and Richards [55] and Jones and Leslie [56]. In these experiments the radiation pressure

due to a light beam reflected off a mirror mounted on a torsion balance was measured for different liquids in which the mirror was immersed. The experiments of Jones and Richards confirmed quite accurately ($\approx 1\%$) that the radiation pressure was proportional to the refractive index of the liquid, as would be expected if the recoil of the mirror were $n\hbar\omega/c$ per photon, the Minkowski momentum (10). Since it was (and is) generally believed that the momentum of the field is $p_A=\hbar\omega/nc$ per photon, the results of the Jones–Richards experiments prompted Peierls to remark that "we are at the moment in such a deep state of confusion that we are bound to learn something from it" [57]. Such confusion is due in our view to the fact that the momenta attributable to the field and the recoil of the mirror are not the only momenta that must be accounted for; as discussed following Eq. (62) in connection with the Doppler effect, we must also take account of the momentum of the medium.

Much greater accuracy ($\approx 0.05\%$) in such experiments was reported nearly a quarter of a century later by Jones and Leslie [56]. Garrison and Chiao [44] have calculated, based on the data of Jones and Leslie, that in the case of benzene the observations differ from that predicted by Eq. (49), $[p'_M=(n^2/n_g)(\hbar\omega/c)]$, by 22 standard deviations and from that predicted by Eq. (9), $[p_A=\hbar\omega/nc]$, by 405 standard deviations. Jones and Leslie also performed experiments for the case of oblique reflection, and, contrary to Peierls's prediction [41,42], did not observe any change in the magnitude of the radiation pressure on the mirror when the polarization was changed from perpendicular to the plane of incidence to parallel.

Poynting [58] considered the effect of light incident from a vacuum onto a transparent, nondispersive medium (refractive index $n>1$) and predicted that there should be an outward force at the surface of the medium, opposite to the direction of propagation of the incident field, consistent with momentum conservation and the field momentum inside the medium having a momentum proportional to the refractive index (the Minkowski momentum). The assumption that the momentum inside the medium has the Abraham form ($\propto n$), however, leads to the prediction that there should be an *inward* force on the medium [59]. These starkly different predictions led Ashkin and Dziedzic [60] to study experimentally the effect on an air–water interface of tightly focused incident laser pulses; they found that there is a net outward force regardless of whether the laser radiation is incident at the air–liquid interface from air or from the interior of the liquid. Gordon [37] and Loudon [21] have shown that this outward force is not caused by momentum transfer parallel (or antiparallel) to the direction of propagation of the laser radiation, but rather to radial forces at the liquid surface caused by the finite transverse cross section of the field—a "toothpaste-tube effect" resulting from the pull on the atoms towards the center of a Gaussian beam [cf. Eq. (33)]; in the case of a plane wave the force at the surface would be inward [61]. The Ashkin–Dziedzic experiment has also been analyzed in detail by Brevik [20], who also considers the time development of the effect and reviews earlier work allowing for the compressibility of the liquid.

More recent experiments of Campbell *et al.* [62], like those of Jones and Richards and Jones and Leslie, measure the longitudinal forces and momenta that are the most directly relevant to questions surrounding the Abraham and Minkowski momenta. The dielectric medium in these experiments was a Bose–Einstein condensate of rubidium atoms irradiated by two identical standing-wave pulses separated in time by τ. The pulses were short enough (5 μs) that the motion of each atom is negligible while it is exposed to each pulse. In a standing-wave electric field $\mathbf{E}_o \sin kz \cos \omega t$ an atom at z finds itself in a (cycle-averaged) potential

$U(z)=-(1/2)\alpha(\omega)\mathbf{E}_0^2\sin^2 kz$. If the field is near resonance with a transition of frequency ω_0 and electric dipole matrix element μ_d of the atom, the polarizability may be approximated by

$$\alpha(\omega)=\frac{\mu^2}{\hbar\Delta},\quad \Delta=\omega_0-\omega, \tag{65}$$

so that

$$U(z)=-\frac{\hbar\Omega^2}{\Delta}\sin^2 kz=-\frac{\hbar\Omega^2}{2\Delta}(1-\cos 2kz), \tag{66}$$

where $\Omega=\mu_d E_0/\hbar$ is the Rabi frequency. The evolution operator describing the effect on the atom at z of a pulse of duration t_p may be taken for present purposes to be

$$e^{-i\theta\cos(2kz)}=\sum_{N=-\infty}^{\infty}(-i)^N J_N(\theta)e^{2iNkz}\cong J_0(\theta)-iJ_1(\theta)[e^{2ikz}-e^{-2ikz}] \tag{67}$$

if $\theta=\Omega^2 t_p/2\Delta$ is sufficiently small; $J_N(\theta)$ is the Nth-order Bessel function of the first kind. We assume for simplicity a square pulse. If the atom at z is initially described by the state vector $|0\rangle$ of zero momentum, then after the pulse it is described approximately as a superposition of states with momenta 0 and $\pm 2\hbar k$:

$$|\psi(t_p)\rangle=J_0(\theta)|0\rangle-iJ_1(\theta)[|2\hbar k\rangle-|-2\hbar k\rangle]. \tag{68}$$

The states $|0\rangle$ and $|\pm 2\hbar k\rangle$ may be taken to have energies 0 and $(2\hbar k)^2/2m+\Delta E_m\equiv 4\hbar\omega_r+\Delta E_m\equiv\hbar\omega_{rm}$, respectively, where ΔE_m is a mean-field energy shift, which is independent of the refractive index [62]. After a time interval τ before the second pulse is applied, therefore,

$$|\psi(t_p+\tau)\rangle=J_0(\theta)|0\rangle-iJ_1(\theta)[|2\hbar k\rangle-|-2\hbar k\rangle]e^{-4i\omega_{rm}\tau}. \tag{69}$$

After the second pulse is applied the atom is described by the state vector

$$|\psi\rangle=[J_0^2(\theta)+2J_1^2(\theta)e^{-4i\omega_r\tau}]|0\rangle+\ \ldots, \tag{70}$$

where we write explicitly only the zero-momentum component of $|\psi\rangle$. The probability that an atom is in the zero-momentum state of the Bose condensate after the two pulses are applied is therefore

$$p_0=J_0^4(\theta)+4[J_1^4(\theta)+J_0^2(\theta)J_1^2(\theta)]\cos^2(4\omega_{rm}\tau). \tag{71}$$

The number of atoms in the zero-momentum state is therefore an oscillatory function of the pulse separation τ. The frequency

$$\omega_{rm}=\frac{\hbar k^2}{2m}+\Delta E_m/\hbar=n^2(\omega)\frac{\hbar\omega^2}{2mc^2}+\Delta E_m/\hbar, \tag{72}$$

which implies that the recoil momenta $\pm 2\hbar k=\pm n(\omega)(2\hbar\omega/c)$. In other words, if the fraction of atoms in the zero-momentum state were determined experimentally to oscillate at the frequency ω_{rm}, it could be inferred that the recoil momentum has the Minkowski form, $\propto n(\omega)$. By resonant absorption imaging of the momentum distribution of the condensate with different time delays τ and detunings Δ, it was confirmed that the recoil momentum is proportional to $n(\omega)$,

and furthermore that it does not depend on $dn/d\omega$ [62]. In other words, the recoil momentum was found to be given by Eq. (10) rather than by Eq. (49) [or (9)]. This is discussed further in the following section.

Finally, we mention experiments by She *et al.* [63] in which it was observed that laser radiation propagating through a nanometer silica fiber produces an inward force on the end face. These authors interpret this as confirmation of the Abraham expression for the momentum. Whether this conclusion is justified in light of other possible interpretations of the observed force is still under discussion [64–66], and in our view further experiments and analyses are necessary before firm conclusions can be drawn.

5. Analyses of Momentum Transfer between Light and Matter

As noted in the Introduction (Section 1), the Minkowski momentum $p_M = n\hbar\omega/c$ per photon correctly characterizes the recoil observed in a dielectric medium, although the number of experiments is small [55,56,62]. Even aside from the Abraham–Minkowski controversy, some interesting conceptual points arise in interpreting the experimental results [62,64–66]. Prior to the experiments of Campbell *et al.*, for example, there were two arguments that could lead one to expect that the recoil of the atoms should be $\hbar\omega/c$, independent of the refractive index [62]. One argument was that each atom in the very dilute medium is effectively localized in a vacuum and therefore should experience the recoil $\hbar\omega/c$ of an atom absorbing a photon in a vacuum. The second argument [67,62] was that the force on an atom is $\nabla(\mathbf{d}\cdot\mathbf{E})$, and that the gradient acting on $\mathbf{E}$ results in a factor $k=n\omega/c$ while $\mathbf{E}$ itself is inversely proportional to n (Section 1); therefore the force acting on an atom in a dielectric medium should be independent of n. Both arguments are invalidated by the experiments.

The field acting on any atom in a dilute medium is a superposition of the field incident on the medium plus the fields scattered from other atoms of the medium. The field in the medium therefore experiences a phase delay (or advance) along the direction of propagation. A plane wave can propagate as $\exp(ikz) = \exp(in\omega z/c)$, as assumed in the brief analysis reviewed at the end of the preceding section, even if the distance between atoms is large compared with a wavelength [68]. In such a situation, contrary to the first argument above, an atom at a position z along the propagation direction will experience a field and a force that depends on n, even if it is localized in a vacuum within a dilute medium.

The second argument is invalid, as is clear from the brief analysis leading to Eq. (72). Consider again the force, Eq. (32), on an atom in which there is an *induced* electric dipole moment

$$\mathbf{d}(\mathbf{r},t) = \mathbf{d}_0(\mathbf{r})e^{-i\omega t} \tag{73}$$

in the monochromatic field

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r})e^{-i\omega t}, \quad \mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r})e^{-i\omega t}. \tag{74}$$

Here $\mathbf{d}_0(\mathbf{r}) = \alpha(\omega)\mathbf{E}_0(\mathbf{r})$, and we allow for possible absorption by the dipole by taking the polarizability $\alpha(\omega)$ to be complex $[\alpha(\omega) = \alpha_R(\omega) + i\alpha_I(\omega)]$. It follows

from Eq. (73) and $\mathbf{B}_0=-(i/\omega)\nabla\times\mathbf{E}_0$ that the cycle-averaged z component, for instance, of the Lorentz force on the dipole is [69]

$$F_z(\mathbf{r})=\frac{1}{4}\alpha_{\mathrm{R}}(\omega)\frac{\partial}{\partial z}|\mathbf{E}_0(\mathbf{r})|^2-\frac{1}{2}\alpha_{\mathrm{I}}(\omega)\mathrm{Im}\left[E_{0x}(\mathbf{r})\frac{\partial E_{0x}^*}{\partial z}+E_{0y}(\mathbf{r})\frac{\partial E_{0y}^*}{\partial z}+E_{0z}(\mathbf{r})\frac{\partial E_{0z}^*}{\partial z}\right]. \tag{75}$$

When the field frequency is far from any absorption resonance of the guest atom, the force on the atom is given by the first term on the right-hand side, together with the scattering force due to the effect of radiative reaction on the polarizability and the optical theorem, as recalled briefly below. This case has already been discussed for a standing-wave field as in the experiments of Campbell *et al.*

To see more clearly why the argument that the recoil momentum is independent of n [67] is incorrect, consider a plane wave with a frequency close to an absorption resonance of a guest atom in a dielectric host with index of refraction $n(\omega)$, so that the term proportional to $\alpha_I(\omega)$ makes the dominant contribution to the force (75). Assuming $\mathbf{E}_0(\mathbf{r})=\hat{x}E_0\exp(ikz)$, $k=n(\omega)\omega/c$, we obtain

$$F_z=\frac{1}{2}n(\omega)\frac{\omega}{c}\alpha_I(\omega)|E_0|^2=n(\omega)\frac{\hbar\omega}{c}R_{\mathrm{abs}}, \tag{76}$$

where R_{abs} is the absorption rate: the force on the atom is equal to the absorption rate times the momentum $n(\omega)\hbar\omega/c$, implying that the atom recoils with the (Minkowski) momentum $n(\omega)\hbar\omega/c$ when it absorbs a photon of energy $\hbar\omega$ [70–72]. While the force is inversely proportional to n, since $|\mathbf{E}_0|^2$ is inversely proportional to n^2 (Section 1), the recoil *momentum* is proportional to n. The argument in [67] does not take account of the fact that the dipole moment is induced by the field, and therefore it presumes that $|\mathbf{E}_0|$ rather than $|\mathbf{E}_0|^2$ determines the force, which, because it involves the gradient of the field, is then found to be independent of n; from this it is concluded that the recoil momentum is also independent of n.

The recoil momentum in stimulated emission is, of course, also $n(\omega)\hbar\omega/c$ in magnitude. The calculation of the recoil momentum when an atom undergoes spontaneous emission of a photon of frequency ω is a bit more complicated than for absorption or stimulated emission, and leads again to the conclusion that the recoil momentum is $n(\omega)\hbar\omega/c$ [71].

Regarding Eq. (75), there is a contribution to the term proportional to $\alpha_I(\omega)$ even in the absence of absorption. This contribution is due simply to the scattering of radiation and is required for consistency with energy conservation and the optical theorem [69,73,74]. It is responsible among other things for the scattering force in the theory of laser tweezers [72,75].

5.1. Momentum Density in the Medium

We now consider the momentum density imparted to a dilute, nonabsorbing dielectric medium as a result of the propagation of a quasi-monochromatic field

$$\mathbf{E} = \mathcal{E}_0(\mathbf{r},t)e^{-i\omega t} = e^{-i\omega t}\int_{-\infty}^{\infty} d\Delta\tilde{\mathcal{E}}_0(\mathbf{r},\Delta)e^{-i\Delta t}, \tag{77}$$

$|\partial\mathcal{E}_0/\partial t| \ll \omega|\mathcal{E}_0|$. The induced electric dipole moment of an atom in this field is $\mathbf{d} = \alpha(\omega)\mathbf{E}_\omega \exp(-i\omega t)$, the polarizability $\alpha(\omega)$ being real according to our assumption that the medium is nonabsorbing. The Lorentz force on each atom is

$$\mathbf{F} = (\mathbf{d}\cdot\nabla)\mathbf{E} + \dot{\mathbf{d}}\times\mathbf{B} = (\mathbf{d}\cdot\nabla)\mathbf{E} + \mathbf{d}\times(\nabla\times\mathbf{E}) + \frac{\partial}{\partial t}(\mathbf{d}\times\mathbf{B}) \equiv \mathbf{F}_E + \mathbf{F}_B, \tag{78}$$

where we define

$$\mathbf{F}_E = (\mathbf{d}\cdot\nabla)\mathbf{E} + \mathbf{d}\times(\nabla\times\mathbf{E}), \tag{79}$$

$$\mathbf{F}_B = \frac{\partial}{\partial t}(\mathbf{d}\times\mathbf{B}). \tag{80}$$

We approximate the induced electric dipole moment of an atom at $\mathbf{r}$ as

$$\begin{aligned}\mathbf{d}(\mathbf{r},t) &= \int_{-\infty}^{\infty} d\Delta\,\alpha(\omega+\Delta)\tilde{\mathcal{E}}_0(\mathbf{r},\Delta)e^{-i(\omega+\Delta)t} \\ &\cong \int_{-\infty}^{\infty} d\Delta[\alpha(\omega)+\Delta\alpha'(\omega)]\tilde{\mathcal{E}}_0(\mathbf{r},\Delta)e^{-i(\omega+\Delta)t} \\ &= \left[\alpha(\omega)\mathcal{E}_0(\mathbf{r},t) + i\alpha'(\omega)\frac{\partial\mathcal{E}_0}{\partial t}\right]e^{-i\omega t}.\end{aligned} \tag{81}$$

Here $\alpha' = d\alpha/d\omega$, and we presume that material dispersion and pulse durations are such that terms $(d^m\alpha/d\omega^m)\partial^m\mathcal{E}_0/\partial t^m$ can be neglected for $m \geq 2$. Using Eq. (81) in Eq. (79), we obtain, after some straightforward manipulations and cycle-averaging, the force

$$\mathbf{F}_E = \nabla\left[\frac{1}{4}\alpha(\omega)|\mathcal{E}|^2\right] + \frac{1}{4}\alpha'(\omega)\mathbf{k}\frac{\partial}{\partial t}|\mathcal{E}|^2, \tag{82}$$

where $\mathcal{E}$ and $\mathbf{k}$ are defined by writing $\mathcal{E}_0(\mathbf{r},t) = \mathcal{E}(\mathbf{r},t)e^{i\mathbf{k}\cdot\mathbf{r}}$. Since the refractive index n of a dilute medium is given in terms of α by $n^2 - 1 = N\alpha/\epsilon_0$, N being the density of dipoles in the dielectric, we have $\alpha' = (2n\epsilon_0/N)(dn/d\omega)$ and

$$\mathbf{F}_E = \nabla\left[\frac{1}{4}\alpha(\omega)|\mathcal{E}|^2\right] + \frac{\epsilon_0}{2N}\mathbf{k}n\frac{dn}{d\omega}\frac{\partial}{\partial t}|\mathcal{E}|^2. \tag{83}$$

The first term is the dipole force associated with the energy $W = -\frac{1}{2}\alpha(\omega)\mathbf{E}^2$ involved in inducing an electric dipole moment:

$$W = -\int_0^{\mathbf{E}} \mathbf{d}\cdot d\mathbf{E} = -\alpha(\omega)\int_0^{\mathbf{E}} \mathbf{E}\cdot d\mathbf{E} = -\frac{1}{2}\alpha(\omega)\mathbf{E}^2. \tag{84}$$

The second force in Eq. (83) owes its existence to dispersion ($dn/d\omega \neq 0$). It is in the direction of propagation of the field and implies for a uniform density N of atoms per unit volume a momentum density of magnitude

$$p_D = \frac{1}{2}\epsilon_0 n^2 \frac{dn}{d\omega}\frac{\omega}{c}|\mathcal{E}|^2 = \frac{1}{2}\frac{\epsilon_0}{c}n^2(n_g - n)|\mathcal{E}|^2, \tag{85}$$

since $k = n(\omega)\omega/c$. This momentum density comes specifically from the dispersion of the medium. It coincides with a result contained in Eq. (52) of a paper by Nelson [76], and a similar result was obtained earlier by Washimi and Karpman [77,78]. (Nelson's paper is noteworthy in that it is based on a Lagrangian formulation in which electrostrictive effects are included.)

The force $\mathbf{F}_B$ defined by Eq. (80), similarly, implies a momentum density

$$\mathbf{P}^A = N\mathbf{d} \times \mathbf{B}. \tag{86}$$

As the notation suggests, this momentum density is associated with the Abraham force density, Eq. (21). A straightforward evaluation of $\mathbf{P}^A$, using Eq. (81) and $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$, yields

$$\mathbf{P}^A = \frac{1}{2}\epsilon_0(n^2 - 1)\frac{\mathbf{k}}{\omega}|\mathcal{E}|^2, \quad P^A = \frac{1}{2}\frac{\epsilon_0}{c}n(n^2 - 1)|\mathcal{E}|^2, \tag{87}$$

when we assume that the field can be described approximately as a plane wave ($\mathbf{k}\cdot\mathbf{E} = 0$) and use the approximation $|\dot{\mathcal{E}}_0| \ll \omega|\mathcal{E}_0|$. If the dipole force on the right-hand side of (83) can be ignored (see below), then $\mathbf{F}_E$ reduces to the derivative with respect to time of a quantity we can identify as a momentum density, and the momentum density obtained by adding it to P^A is

$$P_{\text{med}} = P_D + P^A = \frac{\epsilon_0}{2c}[n^2(n_g - n) + n(n^2 - 1)]|\mathcal{E}|^2 = \frac{\epsilon_0}{2c}n(nn_g - 1)|\mathcal{E}|^2. \tag{88}$$

The total momentum density for the field and the medium is obtained by adding to Eq. (88) the Abraham momentum density $P_A (= g_A)$ of the field. According to Eq. (19), $P_A = (\epsilon_0/2c)n|\mathcal{E}|^2$, and so the total momentum density is

$$P_A + P_D + P^A = \frac{\epsilon_0}{2c}[n + n(nn_g - 1)]|\mathcal{E}|^2 = \frac{\epsilon_0}{2c}n^2 n_g|\mathcal{E}|^2 \tag{89}$$

if the dipole force is negligible. To express these results in terms of single photons, we once again replace $|\mathcal{E}_0|^2$ with $2\hbar\omega/(\epsilon_0 n n_g V)$; then Eq. (89) becomes

$$p_A + p_D + p^A = n\frac{\hbar\omega}{c}\frac{1}{V}; \tag{90}$$

i.e., the total momentum per photon is $n(\omega)\hbar\omega/c$. This is consistent with the experimental results of Jones and Leslie [56] and Campbell *et al.* [62] showing that the recoil momentum of an atom has the Minkowski form $n(\omega)\hbar\omega/c$ and does not depend on $dn/d\omega$. The momentum density of the medium per photon follows from Eq. (88):

$$p_{\text{med}} = p_D + p^A = \frac{\epsilon_0}{2c}n(nn_g - 1)\frac{2\hbar\omega}{nn_g\epsilon_0 V} = \left(n - \frac{1}{n_g}\right)\frac{\hbar\omega}{c}\frac{1}{V}, \tag{91}$$

and this plus the (Abraham) momentum density of the field per photon ($\hbar\omega/n_g cV$) is the total momentum density. When dispersion is neglected, Eq. (91) reduces to (63).

Formula (88) implies that the "mechanical" momentum [37] attributable to the medium is carried along with the propagating field, and that this momentum is just the difference between the Minkowski and Abraham momentum densities. This result for the "forward bodily impulse" p_{med} was also assumed by Jones [57], for instance. A "pseudomomentum" obtained by Loudon *et al.* [79] also has exactly this form in the absence of electrostriction. In our derivation of it we have neglected the dipole force, $\mathbf{F}_{\text{dipole}}=\nabla[(1/4)\alpha(\omega)|\mathcal{E}|^2]$, which for the dilute medium assumed can be related to electrostriction as discussed below. F_{dipole} can be thought of in terms of the process of absorption of a photon with wave vector $\mathbf{k}_1$ accompanied by stimulated emission of a photon with wave vector $\mathbf{k}_2$, resulting in a momentum $\hbar(\mathbf{k}_1-\mathbf{k}_2)$ imparted to the atom. As such it is attributable to a redistribution of photons among all the $\mathbf{k}$ states composing the field. However it is interpreted, it does not lend itself to as straightforward an interpretation as the radiation pressure force associated with one-photon absorption and emission processes. As remarked by Gordon and Ashkin [80], "The photon concept does *not* seem particularly helpful in understanding this part of the force on the atom."

The dipole force complicates the identification of the momentum density carried by the medium. Ignoring it, we derived Eq. (91), which neatly comports with momentum conservation: in the emission of a photon by an guest atom in a dielectric medium, for example, the (Minkowski) recoil momentum of the atom $(n\hbar\omega/c)$ equals the (Abraham) momentum of the emitted photon $(\hbar\omega/n_g c)$ plus the momentum imparted to the host medium (p_{med}). From a macroscopic perspective, the existence of dipole forces on the atoms of the medium is reflected in the second term on the right-hand side of Eq. (30), which is associated with electrostriction. Thus, in the simplest case of a dilute fluid in which $\epsilon-\epsilon_0$ is proportional to the density ρ, this term is just the total dipole force density

$$\nabla\left[\rho\frac{\partial\epsilon}{\partial\rho}\frac{1}{2}\mathbf{E}^2\right]=\nabla\left[(\epsilon-\epsilon_0)\frac{1}{2}\mathbf{E}^2\right]=\nabla\left[N\alpha\frac{1}{2}\mathbf{E}^2\right] \tag{92}$$

for the medium of N atoms per unit volume. As noted in connection with Eq. (30), in mechanical equilibrium this force density is cancelled by a pressure gradient, in which case we can in effect ignore the dipole force, as was implicit when we wrote formulas such as Eqs. (22) and (23) for the force density in terms of the stress tensor T_{ij}^M.

Suppose, on the other hand, that the field is in the form of a pulse that is too short for mechanical equilibrium to be realized. In particular, consider for simplicity a plane-wave pulse propagating with phase velocity c/n in the z direction in a dispersionless dilute fluid, so that the dipole force is

$$\begin{aligned}\nabla\left[N\alpha\frac{1}{2}\mathbf{E}^2\right]&=\frac{\partial}{\partial z}\left[N\alpha\frac{1}{2}\mathbf{E}^2\right]\hat{z}=-\frac{n}{c}\frac{\partial}{\partial t}\left[N\alpha\frac{1}{2}\mathbf{E}^2\right]\hat{z}=-\frac{n}{c}\frac{\partial}{\partial t}\frac{1}{2}(\epsilon-\epsilon_0)|\mathbf{E}|^2\\&=-\frac{\epsilon_0}{2c}n(n^2-1)\frac{\partial}{\partial t}|\mathbf{E}|^2.\end{aligned} \tag{93}$$

In this case we must retain the first term on the right-hand side of Eq. (83), which implies a cycle-averaged momentum density $-(n/c)\left[N\alpha\frac{1}{4}|\mathcal{E}|^2\right]$ that must be added to Eq. (88) to give the actual momentum density of the (dispersionless) medium:

$$P'_{\text{med}} = \frac{\epsilon_0}{2c} n(n^2-1)|\mathcal{E}|^2 - \frac{\epsilon_0}{4c} n(n^2-1)|\mathcal{E}|^2 = \frac{\epsilon_0}{4c} n(n^2-1)|\mathcal{E}|^2. \tag{94}$$

Thus the dipole force has the effect of reducing the momentum density assigned to the material medium. We can rewrite Eq. (94) as

$$P'_{\text{med}} = \frac{1}{2c^2}(n^2-1)|\mathbf{E}\times\mathbf{H}|, \tag{95}$$

or, on a single-photon basis,

$$p'_{\text{med}} = \frac{\hbar\omega}{2c}\left(n - \frac{1}{n}\right), \tag{96}$$

which is half the momentum, Eq. (91), when $n=n_g$ in the latter formula. Equation (95) is exactly the form obtained by Shockley [81] and Haus [82] when $\mu=\mu_0$ and the medium is assumed to be dispersionless. Haus refers to Eq. (96) as the momentum density "assigned to the material in the presence of a plane wave packet." Robinson [19] strongly emphasizes the reduction in the momentum density assigned to the medium from Eq. (91) to Eq. (94) when the electrostrictive term we associate with the dipole force is included; instead of being equal to the difference between the Minkowski and Abraham momenta, the momentum assigned to the medium is reduced to half this difference. Robinson also emphasizes that Eq. (95) lacks general validity, as it is based on the model of a "simple fluid" in which the susceptibility is proportional to the density.

The question as to the correct form of the mechanical momentum assigned to the medium thus appears to have no general answer. However, the observable forces and recoils exerted on objects in dielectric media can be calculated without explicit consideration of what momentum should be assigned to the medium—the force on an atom in a dielectric, for example, is unambiguously the sum of the radiation pressure and dipole forces. In particular, Loudon *et al.* have calculated forces and momenta for a number of cases of interest based the Lorentz force, assuming the well-known field momentum in a vacuum *but requiring no assumptions about the form of the field momentum in a dielectric* (see, for example, [61,21,25,24]).

5.2. Dielectric Surfaces

In the spirit of Shockley's "try simplest cases" [81], we have restricted ourselves thus far in this section to an idealized homogeneous dielectric medium. An obvious example of inhomogeneity is an interface between two dielectric media. Suppose a single-photon field is incident normally from vacuum onto a nondispersive dielectric medium with (real) refractive index n at the field frequency ω. Before the field enters the medium the total momentum of the system is $p_i = \hbar\omega/c$ in the direction of field propagation; afterwards it is

$$p_f = -|R|^2\frac{\hbar\omega}{c} + n|T|^2\frac{\hbar\omega}{nc} + p_s, \tag{97}$$

where R and T are, respectively, the reflection and the transmission coefficients and p_s is the momentum imparted to the medium by the field at its surface; in the second term we have assumed that the momentum of the photon inside the di-

electric medium is the Abraham momentum $\hbar\omega/nc$. Conservation of momentum and $|R|^2=(n-1)^2/(n+1)^2$, $|T|^2=4/(n+1)^2$ imply that the momentum imparted to the surface is

$$p_s=\frac{2\hbar\omega}{c}\frac{n-1}{n+1}, \tag{98}$$

while the fraction of energy transmitted into the medium is

$$f=n|T|^2=\frac{4n}{(n+1)^2}. \tag{99}$$

The momentum imparted to the surface per transmitted photon is therefore [61]

$$p_s/f=\frac{\hbar\omega}{c}\frac{n^2-1}{2n}=\frac{\hbar\omega}{2c}\left(n-\frac{1}{n}\right). \tag{100}$$

Loudon [61] has calculated the radiation pressure exerted by a normally incident pulse on a dielectric medium from the Lorentz force. The medium is assumed to be nondispersive but can be absorbing, and the field (and the Lorentz force) are treated fully quantum mechanically. In the case of a transparent, semi-infinite, rigid-body medium he obtains, for the total momentum per photon transferred from the field to the dielectric medium,

$$p=\left[\frac{1}{2}\left(n-\frac{1}{n}\right)+\frac{1}{n}\right]\frac{\hbar\omega}{c}=\frac{1}{2}\left(n+\frac{1}{n}\right)\frac{\hbar\omega}{c} \tag{101}$$

when this is expressed per transmitted photon as in Eq. (100). The first term in brackets on the left-hand side of this equation is interpreted as a *surface* contribution and is identical to Eq. (100). It implies an *inward* force on the dielectric surface, in contrast to the outward force that had been predicted by Poynting [58], as mentioned earlier. The second term is a *bulk* contribution [61]. Note that the surface contribution turns out to be identical to the Shockley–Haus mechanical momentum, Eq. (96), and that the total momentum, Eq. (101), is identical to the total momentum per photon inferred from the papers of Shockley and Haus. The same end results for normal incidence, with interpretations similar to those of Shockley and Haus, have been obtained by Mansuripur [83].

Loudon has also examined the case in which there is an antireflection coating at the surface of the dielectric [21]. In this case the surface contribution is found to be $(\hbar\omega/c)(n-1/n)$, the same as Eq. (91) with $n=n_g$, and the bulk contribution is $\hbar\omega/nc$, giving a total momentum $n\hbar\omega/c$.

5.3. Momentum Exchange between a Light Pulse and an Induced Dipole

We next review an analysis by Hinds and Barnett [84] that helps to solidify the interpretation of the Abraham momentum as the momentum of the field. We consider the momentum exchange between a plane-wave pulse, propagating in the z direction in a medium with real refractive index $n(\omega)$, and a single particle with real polarizability $\alpha(\omega)$ in the medium. The force (in the z direction) on the particle is the sum of $\mathbf{F}_E$, defined by Eq. (82), and $\mathbf{F}_B$, defined by Eq. (80), with the result that the cycle-averaged force on the particle has the magnitude

$$F=\frac{1}{4}\alpha(\omega)\frac{\partial}{\partial z}\mathcal{E}^2+\frac{1}{4}\alpha'(\omega)n(\omega)\frac{\omega}{c}\frac{\partial}{\partial t}\mathcal{E}^2+\frac{1}{2c}\alpha(\omega)n(\omega)\frac{\partial}{\partial t}\mathcal{E}^2, \tag{102}$$

which reduces to Eq. (75) when $\alpha(\omega)$ is real and $\partial|\mathcal{E}|^2/\partial t=0$. The electric field is taken to be

$$\mathbf{E}(z,t)=\mathcal{E}(t-z/v_g)\cos(\omega t-kz), \tag{103}$$

with carrier phase velocity $c/n(\omega)$ and envelope group velocity $v_g=c/n_g$. The momentum of the particle at z at time T is

$$\begin{aligned}p&=\int_{-\infty}^{T}F dt=\frac{1}{4}\alpha\int_{-\infty}^{T}\frac{\partial}{\partial z}\mathcal{E}^2(t-z/v_g)dt+\frac{1}{4c}\alpha' n\omega\int_{-\infty}^{T}\frac{\partial}{\partial t}\mathcal{E}^2(t-z/v_g)dt\\&\quad+\frac{1}{2c}\alpha n\int_{-\infty}^{T}\frac{\partial}{\partial t}\mathcal{E}^2(t-z/v_g)dt=-\frac{1}{4}\alpha\frac{1}{v_g}\mathcal{E}^2+\frac{1}{4c}\alpha'\omega\mathcal{E}^2+\frac{1}{2}\alpha\frac{n}{c}\mathcal{E}^2\\&=\frac{1}{4c}[(2n-n_g)\alpha+n\omega\alpha']\mathcal{E}^2(T-z/v_g).\end{aligned} \tag{104}$$

Consider the force on a two-level atom that is due to a pulse of light in free space [84]. In this case $n_b=n_{bg}=1$, and Eq. (104) reduces to

$$p=\frac{1}{4c}[\alpha+\omega\alpha']\mathcal{E}^2. \tag{105}$$

To express this in terms of photons occupying volume V in the neighborhood of the atom in free space, we once again replace $|\mathcal{E}|^2$ with $2q\hbar\omega/\epsilon_0 V$, where q is the number of photons. Then

$$p=\frac{1}{2c}[\alpha+\omega\alpha']\frac{\hbar\omega}{\epsilon_0 V}q. \tag{106}$$

Here $\alpha=\epsilon_0(n^2-1)/N$, where n is the refractive index in the case of N particles per unit volume. Then

$$p=\frac{1}{2c}\left[\frac{\epsilon_0(n^2-1)}{N}+\frac{2\epsilon_0 n}{N}\omega\frac{dn}{d\omega}\right]\frac{\hbar\omega}{c}q\cong\left[n-1+\omega\frac{dn}{d\omega}\right]\frac{\hbar\omega}{c}q\equiv K\frac{\hbar\omega}{c}q. \tag{107}$$

This momentum imparted to the particle implies a change in field momentum per photon equal to

$$\frac{\hbar\omega}{c}[1-K]\cong\frac{\hbar\omega}{c}\frac{1}{1+K}=\frac{\hbar\omega}{n_g c} \tag{108}$$

for $|K|\ll 1$, corresponding to the Abraham momentum for the field. In other words, the force on the particle is consistent with the field having the Abraham momentum.

In the case of a polarizable particle in a host dielectric rather than in free space it follows from Eq. (104) that

$$p = \frac{I}{2\epsilon_0 c^2}\left[\left(2 - \frac{n_g}{n}\right)\alpha + \omega\alpha'\right], \tag{109}$$

where the intensity $I=(1/2)c\epsilon_0 n\mathcal{E}^2$. If dispersion in the medium and in the polarizability of the guest particle are negligible, we can set $n_g=n$ and $\alpha'=0$, and then Eq. (109) may be shown to be equivalent to Eq. (2.7) of Gordon's paper when that equation is applied to the case of a single atom in the medium.

The force on a polarizable particle can be large in a slow-light medium (n_g large) because the gradient of the field, Eq. (103), responsible for the dipole force on the particle is large. If, for example, $\alpha'(\omega)\cong 0$ and $n_g \gg n$, the momentum (104) is approximately

$$p = \frac{n_g}{4c}\frac{\mu^2\mathcal{E}^2}{4\hbar(\omega-\omega_0)}, \tag{110}$$

when we use formula (65) for the polarizability of a two-state atom. The force $F=dp/dt$ obtained from this expression is identical to Eq. (3) of a paper by Harris [85] that explores some consequences of this force.

6. Kinetic and Canonical Momenta

In Section 3 we associated an inertial mass $E/c^2=\hbar\omega/c^2$ with a photon of frequency ω. Multiplication of this mass by the group velocity gives the Abraham momentum $p_A=\hbar\omega/n_g c$ of the photon. In this sense p_A can be regarded as the *kinetic* momentum of a photon [44,48].

Canonical momentum is the momentum p appearing in canonical commutation relations such as $[x,p]=i\hbar$ in quantum mechanics (or in Poisson brackets such as $\{x,p\}=1$ in classical mechanics), and it differs in general from kinetic momentum. For a particle of charge q and mass m in an electromagnetic field, for example, the kinetic momentum is $m\mathbf{v}$, whereas the canonical momentum $\mathbf{p}=m\mathbf{v}+q\mathbf{A}$, where $\mathbf{v}$ is the particle velocity and $\mathbf{A}$ is the vector potential. This canonical momentum is the generator of spatial translations, e.g.,

$$e^{i\hat{\mathbf{p}}\cdot\mathbf{a}/\hbar}F(\hat{\mathbf{x}})e^{-i\hat{\mathbf{p}}\cdot\mathbf{a}/\hbar} = F(\hat{\mathbf{x}}+\mathbf{a}) \tag{111}$$

for any real displacement vector $\mathbf{a}$ and for any function $F(\hat{\mathbf{x}})$ that can be expressed as a power series in $\hat{\mathbf{x}}$. (Here we follow the practice of using a circumflex to designate an operator.) For the electromagnetic field in a homogeneous dielectric medium in which the atoms may be assumed to be held fixed in position, the canonical momentum may be defined as the generator of spatial translations of the field operators [44]. Garrison and Chiao [44] found that the momentum corresponding to $p_M=n\hbar\omega/c$ per photon can be identified with the canonical momentum of the field. Barnett [48] showed more generally that, for the momentum operator $\hat{\mathbf{p}}_{\mathrm{Min}}$ identified with the Minkowski momentum $\mathbf{p}_{\mathrm{Min}}=\int dV(\mathbf{D}\times\mathbf{B})$,

$$e^{-i\hat{\mathbf{p}}_{\mathrm{Min}}\cdot\mathbf{a}/\hbar}\hat{\mathbf{A}}(\mathbf{r})e^{i\hat{\mathbf{p}}_{\mathrm{Min}}\cdot\mathbf{a}/\hbar} = \hat{\mathbf{A}}(\mathbf{r}+\mathbf{a}), \tag{112}$$

where $\hat{\mathbf{A}}(\mathbf{r})$ is the (Coulomb-gauge) vector potential operator of nonrelativistic, macroscopic quantum electrodynamics. In other words, the Minkowski momentum is the canonical momentum of the field [48].

We noted earlier that the single-photon momentum corresponding to the Minkowski momentum $\int dV(\mathbf{D}\times\mathbf{B})$ is $p'_M=(n^2/n_g)(\hbar\omega/c)$ in the case of a dispersive medium. More precisely, this is the expectation value of the Minkowski momentum for a one-photon state. It is *not* the canonical momentum [44,48]. How is this to be reconciled with Eq. (112), from which it is deduced that the Minkowski momentum is the canonical momentum of the field, regardless of whether or not we allow for dispersion? This question has been answered by Barnett [48], based on the general identity (56) and the commutator

$$[\hat{p}^j_{\mathrm{Min}},\hat{A}^k(\mathbf{r})]=i\hbar\frac{\partial}{\partial x^j}\hat{A}^k(\mathbf{r}), \tag{113}$$

which is equivalent to Eq. (112). As reviewed in Section 2, the dispersion equation $k=n(\omega)\omega/c$ allows different polariton branches, and commutators such as Eq. (113) require a summation over all polariton branches [46,47], each contributing a factor $n(\omega)/n_g(\omega)$. When we sum over all the polariton branches and use Eq. (56), the index dependence of the commutator (113) is found to involve only the factor $n(\omega)$ that comes from the derivative on the right-hand side. If we restricted ourselves to one particular polariton branch, however, the factor $n(\omega)/n_g(\omega)$ for this one branch appears, and then the momentum at frequency ω would in effect be $[n^2(\omega)/n_g(\omega)](\hbar\omega/c)$ as in Eq. (49). This explains why Eq. (49)—which we obtained under the implicit restriction to a single polariton branch—is not the canonical momentum, even though it is the single-photon expectation value of the Minkowski momentum $\int dV(\mathbf{D}\times\mathbf{B})$, which *is* the canonical momentum of the field.

In [44,71,72], and in this review, the momentum $(n^2/n_g)(\hbar\omega/c)$ has been regarded as the single-photon Minkowski momentum in a dispersive medium, whereas Barnett calls $n\hbar\omega/c$ the single-photon Minkowski momentum, regardless of dispersion. Thus Bradshaw *et al.* [72] state that in a dispersive medium the Minkowski momentum does not give the recoil momentum of an absorbing atom, for example, whereas according to Barnett's terminology it does. We note also that the calculations of forces and recoil momenta reviewed in Section 5 presume narrow-bandwidth fields for which the restriction to a single polariton branch is appropriate and which are not inconsistent with Barnett's conclusions regarding the kinetic and canonical momenta of the field, with which we are in full agreement.

7. Summary

We have reviewed various expressions for the stress tensor and for the momentum of light in dielectric media, with heavy emphasis on the two most widely used formulations, those of Abraham and Minkowski. The Abraham momentum is generally taken to be the momentum of the field, and we have adhered to this viewpoint, reinforcing it by consideration of the Balazs block model that suggests the interpretation of the Abraham momentum as a kinetic momentum, i.e., $p_A=(E/c^2)v_g=\hbar\omega/n_gc$ per photon. The Abraham formulation of the Maxwell stress tensor introduces an Abraham force that acts in addition to the Lorentz force, and we briefly reviewed experiments confirming the existence of this force.

The Minkowski momentum $p_M = n(\omega)\hbar\omega/c$ of a single photon is equal to the recoil momentum of an atom that absorbs or emits a photon in a host dielectric medium. We showed that it is necessary for momentum conservation that we account not only for this recoil momentum but also for the (Abraham) momentum of the field and the momentum transferred to the medium, the latter propagating with the field. Depending on assumptions made with respect to electrostriction and the dipole force on the atoms of the medium, the momentum of the medium can have different forms, and there appears to be no general expression for it. We summarized work on the momentum transferred to a dielectric medium, showing how this momentum may be separated into surface and bulk contributions.

When dispersion is included, the single-photon Minkowski momentum derived from the Minkowski momentum density $\mathbf{g}_M = \mathbf{D} \times \mathbf{B}$ is $p'_M = (n^2/n_g)\hbar\omega/c$. However, the recoil momentum of an absorbing or emitting atom in the dielectric is not p'_M but p_M. This is shown by the experiments we reviewed as well as the calculations in Section 5.

The Abraham and Minkowski momenta have different physical interpretations, as shown by the examples we have reviewed. Furthermore, as has recently been shown, the Abraham momentum represents a kinetic momentum, whereas the Minkowski momentum is the canonical momentum of the field in a dielectric medium. Neither is more correct than the other.

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