PDF III

ABOUT WAVES

NICHOLAS GEORGE
OPTICS 462

TOPICS COMING

SHORT TERM PLAN

MAXWELL'S EQUATIONS: $e^{i\omega t}$, HTD (HARMONIC TIME DEPENDENT)

ENERGY THEOREM: (TIME DEPENDENT CASE TREATED EARLIER)

HTD FORM OF POYNTING'S VECTOR

INTERPRETATION DICKE'S /EMDE JUNCTION

BACK TO CHAPTER 3 WAVE EQUATION SOLUTIONS

ALL ABOUT PLANE WAVES (CHAPTERS 4 & 5)

MAXWELL'S EQUATIONS: FOURIER TRANSFORM FORM Feb 2005

MAIN POINT OF THE LECTURE IS NOW, NAMELY

STARTING FROM

1) $$\nabla \times \underline{\mathcal{E}}(\underline{r},t) = -\mu \frac{\partial \underline{\mathcal{H}}(\underline{r},t)}{\partial t}$$

2) $$\nabla \times \underline{\mathcal{H}}(\underline{r},t) = \underline{\mathcal{J}}(\underline{r},t) + \frac{\partial \underline{\mathcal{D}}(\underline{r},t)}{\partial t}$$

And F.T. both sides wrt to time, as given by

- Fourier Xfm

$$\underline{E}(\underline{r}) = \int_{-\infty}^{\infty} \underline{\mathcal{E}}(\underline{r},t)\, e^{-i2\pi\nu t}\, dt$$

Sometimes the $\underline{E}(\underline{r}) = \underline{E}(\underline{r};\nu)$ is written thusly.

- Inversion

$$\underline{\mathcal{E}}(\underline{r},t) = \int_{-\infty}^{\infty} \underline{E}(\underline{r})\, e^{+i2\pi\nu t}\, d\nu$$

We drop the boundary values as we transform both sides (1), (2) noting that the boundary conditions can be fit later

1)' $$\nabla \times \underline{E}(\underline{r};\nu) = -i2\pi\nu\mu\, \underline{H}(\underline{r};\nu)$$

2)' $$\nabla \times \underline{H}(\underline{r};\nu) = \underline{J}(\underline{r}) + i2\pi\nu\, \underline{D}(\underline{r};\nu)$$

Similarly, we shall see the wave equation and the various potential equations can be expressed in terms of the transform fields.

REFER TO THE SUMMARY CHART FOLLOWING

MAXWELL'S EQUATIONS: STEADY STATE $e^{i\omega t}$

Monochromatic or steady state case is found by simply substituting $C_\omega(\underline{r})e^{i\omega t}$ in which

$$C(\underline{r},t) = \mathrm{Re}\, C_\omega(\underline{r})e^{i\omega t}$$

Note here that we are using ω & $e^{+i\omega t}$

For simple media, we substitute into Maxwell's Equations: (time dependent)

& we find, as follows

See CHP: 1.1 (20-23) uses $\underline{E}_\omega$ $\underline{H}_\omega$ notation.

$$\nabla \times \underline{E}(\underline{r}) = -i\omega\mu\underline{H}$$

$$\nabla \times \underline{H}(\underline{r}) = \underline{J}(\underline{r}) + i\omega\epsilon\underline{E}$$

$$\nabla \cdot \underline{B}(\underline{r}) = 0$$

$$\nabla \cdot \underline{D}(\underline{r}) = \rho(\underline{r})$$

Compare these to $\mathcal{F}$: setting $2\pi\nu = \omega$

The vast majority of texts on EM theory use the notation above

For us, with a basic interest in Fourier Transform methods of analysis,

we will often make use of ~~this~~ the following notational observation:

These equations are formally written in a notation that makes them directly applicable to Fourier Transform form.

Sometimes we use ν to emphasize that we are thinking F.T.

Prob Set 3   Ch 3:   2, 3, 4, 5, 7, 8, 12, 14

Prob Set 4   Handout   Computer Oriented

Prob Set 5   { Ch 4   4, 7, 21, 22   due March 1
{ Ch 5 :   2, 5, 6, 7

MIDTERM   March 6 Monday

Spring Break   March 11 – 19

TWO DISTINCT THEOREMS

ENERGY THEOREMS POYNTING'S VECTOR
REVIEW

I. Real-Valued Time Dependent $\underline{\mathcal{E}}(\underline{r},t)$, $\underline{\mathcal{D}}(\underline{r},t)$, $\underline{\mathcal{H}}$, $\underline{\mathcal{B}}$

$$\nabla\cdot(\underline{\mathcal{E}}\times\underline{\mathcal{H}}) + \underline{\mathcal{E}}\cdot\underline{\mathcal{J}} + \underline{\mathcal{H}}\cdot\frac{\partial \underline{\mathcal{B}}}{\partial t} + \underline{\mathcal{E}}\cdot\frac{\partial \underline{\mathcal{D}}}{\partial t} = 0$$

$$\int_{\text{surf}} \underline{\mathcal{E}}\times\underline{\mathcal{H}}\cdot\underline{n}\,da + \int \underline{\mathcal{E}}\cdot\underline{\mathcal{J}}\,dv + \int\left(\underline{\mathcal{E}}\cdot\frac{\partial \mathcal{D}}{\partial t} + \underline{\mathcal{H}}\cdot\frac{\partial \mathcal{B}}{\partial t}\right)dv = 0$$

$$\frac{\text{energy flow out}}{\text{second}} + \frac{\text{work by fields}}{\text{sec}} + \frac{\text{increase stored fld energy}}{\text{sec}} = 0$$

Poynting Vector $\quad \mathcal{S}(t) = \underline{\mathcal{E}}(\underline{r},t)\times\underline{\mathcal{H}}(\underline{r},t) = \dfrac{\text{net outflow energy}}{\text{time area}}$

One can directly show

$$\text{time average}\ \frac{\text{power}}{\text{area}} = \frac{1}{2}\,\mathrm{Re}\ \underline{E}(\underline{r})\times\underline{H}^*(\underline{r})$$

$\left\{\underline{E}, \underline{H}\right.$ are HTD FORMS

see 1.66 to 1.68 Balanis

Review Table 1.6 Bal

II. $e^{i\omega t}$ HTD FIELDS $\underline{E}(\underline{r}), \underline{H}(\underline{r}), \ldots$

$$\nabla\cdot(\underline{E}\times\underline{H}^*) + \underline{E}\cdot\underline{J}^* + i2\omega\left(\mu H\cdot H^* - \epsilon E\cdot E^*\right)$$

$$\int \underline{E}\times H^*\cdot\underline{n}\,da + \int \underline{\mathcal{E}}\cdot\underline{J}^*dv - i\omega\int\left(\epsilon\underline{E}\cdot\underline{E}^* - \mu\underline{H}\cdot\underline{H}^*\right)dv = 0$$

$\frac{1}{2}$ $\qquad\qquad\qquad$ $\frac{1}{2}$ $\qquad\qquad\qquad$ $\frac{1}{2}$

$\overline{w}_e(\underline{r}) = \frac{1}{4}\epsilon\,\underline{E}(\underline{r})\cdot\underline{E}^*(\underline{r}) \qquad \dfrac{\text{energy average}}{\text{vol}}$ electfld

$\overline{w}_m(\underline{r}) = \frac{1}{4}\mu\,\underline{H}(\underline{r})\cdot\underline{H}^*(\underline{r}) \qquad \dfrac{\text{energy aver}}{\text{vol.}}$ magnfld

$$\int \frac{1}{2}\,\mathrm{Re}\ \underline{E}\times\underline{H}^*\cdot\underline{n}\,da = -\int\frac{1}{2}\mathrm{Re}\ \underline{E}\cdot J^*dv = \frac{\text{average outflow energy}}{\text{second}}$$

Converted to $e^{+i\omega t}$

POYNTING VECTOR THEOREM: HTD CASE

B&P H-der Phy
Sect 2 vol $\underline{XVI}$

Electric energy density : $\frac{\text{energy}}{\text{vol}}$ | Magnetic energy density : $\frac{\text{energy}}{\text{vol}}$

$$w_e(\underline{r},t) = \frac{1}{2}\epsilon\, \underline{\mathcal{E}}(\underline{r},t)\cdot\underline{\mathcal{E}}(\underline{r},t) \qquad w_m(\underline{r},t) = \frac{1}{2}\mu\, \underline{\mathcal{H}}(\underline{r},t)\cdot\underline{\mathcal{H}}(\underline{r},t)$$

Using sinusoidal or htd $e^{i\omega t}$ & time average $\bar{w}_e$, $\bar{w}_m$:

$$\bar{w}_e = \frac{1}{4}\epsilon\, \underline{E}(\underline{r})\cdot\underline{E}^*(\underline{r}) \qquad \bar{w}_m(\underline{r}) = \frac{1}{4}\mu\, \underline{H}(\underline{r})\cdot\underline{H}^*(\underline{r})$$

$$\nabla\times\underline{E} = -i\omega\mu\,\underline{H}$$
$$\nabla\times\underline{H} = \underline{J} + i\omega\epsilon\,\underline{E}$$

$$\nabla\cdot(E\times H^*) = H^*\cdot\nabla\times E - E\cdot\nabla\times H^*$$
$$= H^*\cdot(-i\omega\mu \underline{H}) - \underline{E}\cdot(J^* - i\omega\epsilon E^*)$$
$$= -i\omega\mu\,\underline{H}\cdot\underline{H}^* - \underline{E}\cdot\underline{J}^* + i\omega\epsilon\,\underline{E}\cdot\underline{E}^*$$

$$\nabla\cdot(E\times H^*) = -\underline{J}^*\cdot\underline{E} - i\omega(\mu\,\underline{H}\cdot\underline{H}^* - \epsilon\,\underline{E}\cdot\underline{E}^*)$$

$$\nabla\cdot(\tfrac{1}{2}\underline{E}\times\underline{H}^*) = -\frac{1}{2}\underline{J}^*\cdot\underline{E} - 2i\omega(\bar{w}_m - \bar{w}_e)$$

$$\int_{\pi}\frac{1}{2}E\times H^*\cdot m\,da = -\frac{1}{2}\int_V \underline{J}^*\cdot\underline{E}\,dv - 2i\omega\int_V(\bar{w}_m(r) - \bar{w}_e(r))\,dV$$

F. Emde (1909)

"A connecting link between field theory and circuit theory can be deduced from the above result" reworded slightly from $\underline{\text{R. H. Dicke.}}$

$$e^{i\omega t}$$

$$\left.\begin{aligned} \nabla\times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} \\ \nabla\times \underline{H} &= \underline{J} + \frac{\partial D}{\partial t} \\ F &= \rho\,[\,\underline{E} + \underline{v}\times\underline{B}\,] \end{aligned}\right\}$$ All Script letters

Purcell
Poynting 14-16
132-134
R. H Dicke
Ch 5, vol 8 Prin of
RLS Microwave Circuits

$$\int \mathcal{E}\times\mathcal{H}\cdot\underline{n}\,da + \int \mathcal{E}\cdot\mathcal{J}\,dv + \frac{\partial}{\partial t}\int\left(\tfrac{1}{2}\epsilon\,\mathcal{E}\cdot\mathcal{E} + \tfrac{1}{2}\mu\,\mathcal{H}\cdot\underline{\mathcal{H}}\right)dv = 0$$

power outflow + work by field + $\frac{\text{energy increase}}{\text{sec}}$ into fld = 0

Conservation of EM energy in the volume V ⇒ Poynting's Vector

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Now form

$$\nabla\cdot(E\times H^*) = \underline{H}^*\cdot\nabla\times\underline{E} - \underline{E}\cdot\nabla\times\underline{H}^*$$

$$= -i\omega\mu\,\underline{H}\cdot\underline{H}^* + i\omega\epsilon\,\underline{E}\cdot\underline{E}^* - \underline{E}\cdot\underline{J}^*$$

$$\left\{\begin{aligned} \nabla\times E &= -i\omega\mu H \\ \nabla\times H &= J + i\omega\epsilon E \end{aligned}\right.$$

$$\quad * \qquad\quad * \; - \qquad *$$

$$\frac{1}{2}\oint \underline{E}\times\underline{H}^*\cdot\underline{n}\,da = \frac{-1}{2}\int \underline{E}\cdot\underline{J}^*\,dv + \frac{i\omega}{2}\int\left(\epsilon\,\underline{E}\cdot\underline{E}^* - \mu\,\underline{H}\cdot\underline{H}^*\right)dv$$

At this point to make an energy/power interpretation:

$Re\ \frac{-1}{2}\int E\cdot\underline{J}^*\,dv$ = average $\frac{\text{work}}{\text{second}}$ done by current

$Re\ \frac{1}{2}\int E\times H^*\cdot n\,da$ = average $\frac{\text{energy}}{\text{sec}}$ outflow from the volume

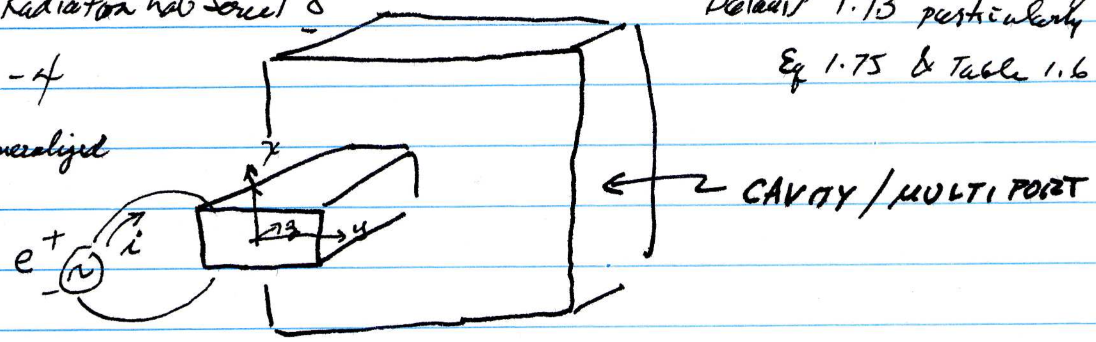
This equation is not derivable from Poynting's Energy ∫ at top.
Entirely distinct energy theorems.
See "General Microwave Circuit Theorems" Ch 5, R. H. Dicke, MIT RLS

EQUIVALENT CIRCUIT - HTD

Consider a microwave circuit with one port or an optical junction with one port - the extension to m-port is straightforward

Dicke: Radiation Lab Series 8
p132-4
generalized

Belanis 1.73 particularly
Eq 1.75 & Table 1.6



From

$$\frac{1}{2}\int \underline{E}(\underline{r}) \times \underline{H}^*(\underline{r}) \cdot \underline{n}\, da = -\frac{1}{2}\int \underline{E}\cdot\underline{J}^* dv + \frac{i\omega}{2}\int(\epsilon \underline{E}\cdot\underline{E}^* - \mu \underline{H}\cdot\underline{H}^*) dv$$

Power in

$$\frac{1}{2} e i^* = -\frac{1}{2}\int E \times H^* \cdot \underline{n}\, da = \frac{1}{2}\int E\cdot \underline{J}^* dv + \frac{i\omega}{2}\int(\mu \underline{H}\cdot\underline{H}^* - \epsilon \underline{E}\cdot\underline{E}^*) dv$$

Power flow in has a real part dissipative & an imaginary part Reactive

$\frac{1}{2} i i^* (R + jX)$ = joule losses & stored energy L, C
magnetic fld, electric field

$$\frac{1}{2} e i^* = P_{dissipated} + \frac{i\omega}{2}\int(\mu \underline{H}\circ\underline{H}^* - \epsilon \underline{E}\circ\underline{E}^*) dv$$

$\bar{w}_m(\underline{r}) = \frac{1}{4}\mu \underline{H}\circ\underline{H}^*$ average energy stored mag fld / vol

$\bar{w}_e(\underline{r}) = \frac{1}{4}\epsilon \underline{E}\cdot\underline{E}^*$ aver energy stored elect fld / vol

This above equation/interpretation is a fundamental result letting us
① characterize an optical or microwave ~~junction~~ junction region as "an equivalent circuit" being "inductive" when the $\bar{w}_m(\underline{r}) > \bar{w}_e(\underline{r})$ summed throughout the volume. ... capacitive ... $\int \bar{w}_e > \int \bar{w}_m$

② We can deduce this by looking at the Imag. part of $E \times H^*$.

$$-\frac{1}{2}\int E \times H^* \circ n\, da = \frac{1}{2} e i^* = P_D + 2j\omega(W_H - W_E) = II^*(R + jX)$$

WAVE EQUATION

PLAN- SHORT TERM

1. CAREFUL DETAILED STUDY OF PLANE WAVES
    - 1 MEDIUM
    - INTERFACE
    - LOSSY (OHMIC)
    - REFLECTION, TRANSMISSION
2. WAVE EQUATION, BALANIS CHAPTER 3
    - VECTOR WAVE EQUATION
    - CARTESIAN
    - CYLINDRICAL
    - SPHERICAL

CARTESIAN $\left.\begin{array}{l}\text{3-26 a, b,c}\\ \text{3-27}\end{array}\right\}$ & Table 3.1

CYLINDRICAL E_z IS DETAILED 3.54C

P119 GOOD SUMMARY 3.65-69 & Table 3.2

SPHERICAL: EXPLAIN TE^r TM^r WAVE MODES CH 10

(ONLY) SCALAR WAVE EQUATION IN SPHERICAL COORDINATES

P124 GOOD SUMMARY 3.84-92 & Table 3.3

VECTOR WAVE EQUATION

$$\nabla^2 \boldsymbol{E}(x,y,z) + k^2 \boldsymbol{E}(x,y,z) = 0$$

WHERE $k^2 = (2\pi\nu)^2 \mu\varepsilon$

3 SCALAR EQUIVALENTS

SINCE

$$\boldsymbol{E}(x,y,z) = \hat{e}_x E_x + \hat{e}_y E_y + \hat{e}_z E_z$$

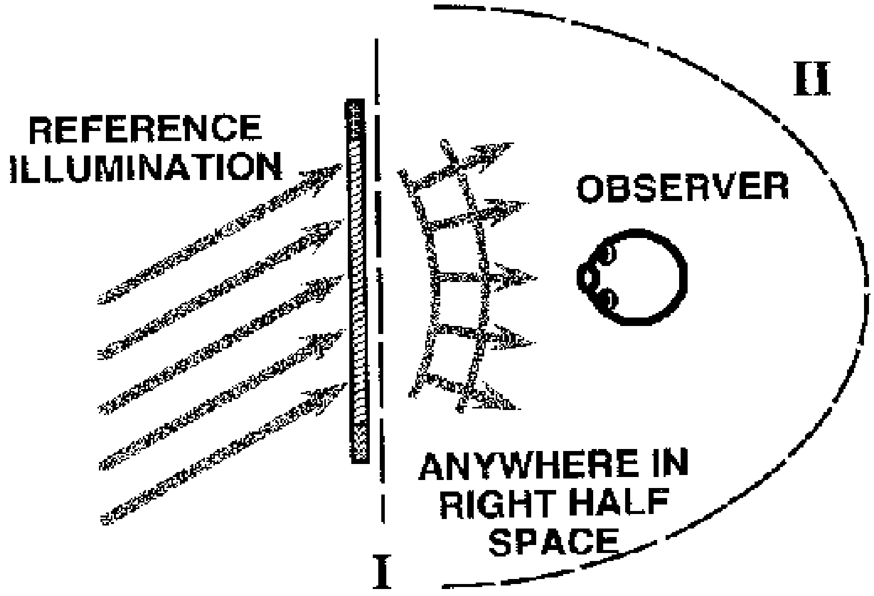
$$\left.\begin{aligned} \nabla^2 E_x + k^2 E_x &= 0 \\ \nabla^2 E_y + k^2 E_y &= 0 \\ \nabla^2 E_z + k^2 E_z &= 0 \end{aligned}\right\}$$

ASSERTION: IF WE FIND AN EXACT SOLUTION OF MAXWELL'S EQUATIONS, THEN IT CAN BE SET UP IN REAL LIFE

ANALOGY: APPLY GENERATOR TO AN ELECTRONIC CIRCUIT AND CALCULATE CURRENTS AND VOLTAGES

EXISTENCE QUESTION

GOING BACK TO A TIME BEFORE GABOR'S REALIZATION WOULD THE EXISTENCE OF '3-D' RESTORATION USING A PLANAR DIFFRACTING MEDIUM HAVE BEEN THOUGHT REASONABLE?



YES - SINCE THE UNIQUENESS THEOREM FOR $e^{i\omega t}$ CASE STATES THAT IF E – TANGENTIAL ON I – I SPECIFIED AND RADIATION CONDITION ON II, THEN INTERIOR E_{OBS}, H_{OBS} ARE SPECIFIED UNIQUELY

EXISTENCE QUESTION

GOING BACK TO A TIME BEFORE GABOR'S REALIZATION

UNIQUENESS THEOREM FOR MAXWELL'S EQUATIONS

ISOTROPIC, LINEAR, PASSIVE

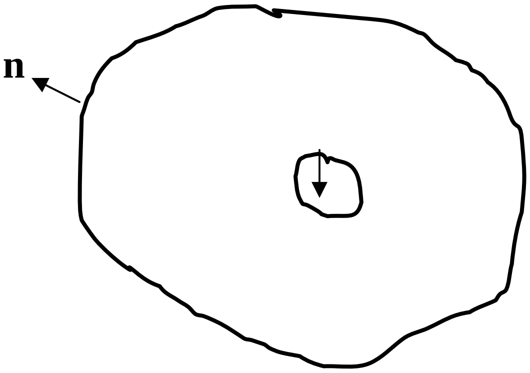
ε, μ, σ $\sigma \geq 0$

$\boldsymbol{E}(\boldsymbol{r}, t), \boldsymbol{H}(\boldsymbol{r}, t)$

TH: IF E, H & E', H' ARE SOLUTION PAIRS OF MAXWELL'S EQUATIONS AND IF (E-E') x **n** OR (**H-H'**) x **n**=0 ON THE BOUDARY AND IF

AT t=0, $\Delta\boldsymbol{E}$ =0, $\Delta\boldsymbol{H} = 0$ in volume

Then $\Delta\boldsymbol{E}$ =0, $\Delta\boldsymbol{H} = 0$ throughout V, all times



Ref. W.R. SMYTHE
3rd EDIT 12.10

COR: $\widetilde{\boldsymbol{E}}(\boldsymbol{r}; \nu), \widetilde{\boldsymbol{H}}(\boldsymbol{r}; \nu)$

ELECTROMAGNETIC THEORY

- WHAT IS THE BEST ILLUSTRATION OF PROPAGATION TO START:

STUDY OF PLANE WAVE PROPAGATION

1. MAXWELL'S EQUATIONS

SIG. REP. : REAL-VALUED VARIABLES, $\mathcal{E}(x,y,z,t)$

$$\nabla\times\mathcal{E}=-\mu\frac{\partial\mathcal{H}}{\partial t} \qquad \nabla\bullet\mathcal{D}=\rho$$

$$\nabla\bullet\mathcal{B}=0$$

$$\nabla\times\mathcal{H}=\mathcal{J}+\varepsilon\frac{\partial\mathcal{E}}{\partial t} \qquad \nabla\bullet\mathcal{J}=-\frac{\partial\rho}{\partial t}$$

HOMOGENEOUS ISOTROPIC DIELECTRIC

- **HARMONIC TIME DEPENDENCE**

$$\mathcal{E}(x,y,z,t)\rightarrow \boldsymbol{E}(x,y,z)e^{i2\pi\nu t}$$

$$\left.\begin{aligned}\nabla\times\boldsymbol{E}&=-i2\pi\nu\mu\boldsymbol{H}\\ \nabla\times\boldsymbol{H}&=i2\pi\nu\varepsilon\boldsymbol{E}\end{aligned}\right\}$$

$$\left.\begin{aligned}\nabla^2\boldsymbol{E}+(2\pi\nu)^2\mu\varepsilon\boldsymbol{E}&=0\\ \boldsymbol{H}&=\frac{-1}{i2\pi\nu\mu}\nabla\times\boldsymbol{E}\end{aligned}\right\}\quad\bullet$$

BACK TO CHAPTER 3

WAVE EQUATION

READ BALANIS CH 3 EMPHASIZE FOLLOWING

HTD $e^{i\omega t}$ 3.4.1 RECTANGULAR COORDINATES

TABLE 3.1

3.4.2 CYLINDRICAL COORDINATES

TABLE 3.2

3.4.3 SPHERICAL COORDINATES

TABLE 3.3

STUDY FUNCTIONS: Series forms, Curves

ABRAMOWITZ & STEGUN

SMYTHE

GRADSHTEYN & RYZHIK

MORSE & FESHBACH

PROB SET III : 3.2, 3.3, 3.4, 3.5, 3.7, 3.8, 3.12, 3.14

WAVE EQUATION

SIMPLE MEDIUM μ, ϵ, σ scalar constants
source free
ohmic $\mathcal{J} = \sigma \mathcal{E}$

Previously, we wrote

$$\nabla^2 \underline{\mathcal{E}}(\underline{r},t) - \mu\sigma \frac{\partial \mathcal{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = 0$$ 3-12 Balanis

HTD: $\underline{\mathcal{E}}(r,t) = \mathrm{Re}\ \underline{E}(\underline{r}) e^{i\omega t}$

$$\nabla^2 \underline{E}(\underline{r}) - i\omega\mu\sigma \underline{E}(\underline{r}) + \omega^2\mu\epsilon \underline{E}(\underline{r}) = 0$$

Define: $\gamma^2 = i\omega\mu\sigma - \omega^2\mu\epsilon$ 3.17c
$\gamma = \alpha + i\beta$ 3.17d

$$\nabla^2 \underline{E}(\underline{r}) - \gamma^2 \underline{E} = 0$$

lossless case: $\gamma^2 = -\omega^2\mu\epsilon = -\beta^2$
$\beta^2 = \omega^2\mu\epsilon = k^2$ $\left\{ \text{"many authors use } k\text{"} \right.$
$\beta = \pm\ \omega\sqrt{\mu\epsilon}$

WAVE EQUATION   HTD   SIMPLE MEDIUM

$$\nabla^2 \underline{E}(x,y,z) + k^2 \underline{E}(x,y,z) = 0 \qquad k^2 = \omega^2 \mu\epsilon$$

Source Free

3.4.1 Balanis

SEPARATION OF VARIABLES

CARTESIAN COODINATE SYSTEM

Since $\nabla^2(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z) = \hat{x}\nabla^2 E_x + \hat{y}\nabla^2 E_y + \hat{z}\nabla^2 E_z$

one has 3 separate equations – "uncoupled"

$$\nabla^2 E_{x,y,z} + k^2 E_{x,y,z} = 0$$

ASSUME SOLUTIONS: $E_x(x,y,z) = XYZ$ $\quad\begin{cases} X = X(x) \\ Y = Y(y) \\ Z = Z(z) \end{cases}$

$$\ddot{X}YZ + X\ddot{Y}Z + XY\ddot{Z} + k^2 XYZ = 0$$

$$\underset{\substack{\parallel \\ -k_x^2}}{\frac{\ddot{X}}{X}} + \underset{\substack{\parallel \\ -k_y^2}}{\frac{\ddot{Y}}{Y}} + \underset{\substack{\parallel \\ -k_z^2}}{\frac{\ddot{Z}}{Z}} + k^2 = 0$$

$$-k_x^2 - k_y^2 - k_z^2 + k^2 = 0 \quad \Big\} \text{ CONSTRAINT EQUATION}$$

$$e^{\pm ik_x x} \qquad e^{\pm ik_y y} \qquad e^{\pm ik_z z}$$

$$E_x(x,y,z) = \begin{Bmatrix} \sin k_x x & \cos k_x x \\ e^{+ik_x x} & e^{-ik_x x} \end{Bmatrix} \begin{Bmatrix} \sin k_y y & \cos k_y y \\ e^{+ik_y y} & e^{-ik_y y} \end{Bmatrix} \begin{Bmatrix} \sin k_z z & \cos k_z z \\ e^{+ik_z z} & e^{-ik_z z} \end{Bmatrix}$$

From physical problem, one needs to learn how to pick appropriate functions, linearly independent

6 constants less 1 = 5

Study Tables in Balanis   x, y, z   TABLE 3-1

VECTOR

CYLINDRICAL COORDS - WAVE EQUATION HTD

ASSERT LOSSLESS

In reading Balanis 3.4.2 we note that he starts out to form

$$\nabla^2 \underline{E}(\rho, \phi, z) + k^2 \underline{E}(\rho, \phi, z) = 0$$

Writing the Laplacian in cylindrical coordinates, one finds the problem inherently more complicated than that for the Cartesian case - since

$$\nabla^2 [\hat{\rho} E_\rho + \hat{\phi} E_\phi + \hat{z} E_z]$$

the unit vectors $\hat{\rho}$, $\hat{\phi}$ do not "factor" out, i.e., do not commute with ∇^2

In a complicated calculation that he omits, he finds 3.54a
3.54b
3.54c

BUT HE NEVER USES 3.54a, 3.54b again IN THIS CHAPTER

Later we will approach this problem using vector potentials. For now, we want to study the solution of

a scalar wave equation in cylindrical coordinates

How to find 3.54c EASILY : { FOR E_z or H_z we proceed in same way

$$\hat{e}_z \cdot \left(\nabla^2 \underline{E}(\rho, \phi, z) + k^2 \underline{E}(\rho, \phi, z) \right) = 0$$

$\hat{z}$ commutes

$$\nabla^2 E_z(\rho, \phi, z) + k^2 E_z(\rho, \phi, z) = 0$$

Following his notation, lets use the scalar $\psi(\rho, \phi, z) = E_z$ only

$$\nabla^2 \psi(\rho, \phi, z) + k^2 \psi(\rho, \phi, z) = 0$$

SCALAR WAVE EQUATION

CYLINDRICAL COORDS - CONT. — SEPARATION OF VARIABLES

$$\psi(\rho,\phi,z) = \mathcal{P}\,\Phi\,Z \qquad \mathcal{P}(\rho),\ \Phi(\phi)\ Z(z)$$

$$\nabla^2 \mathcal{P}\Phi Z + k^2 \mathcal{P}\Phi Z = 0$$

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right)\mathcal{P}\Phi Z = -k^2 \mathcal{P}\Phi Z$$

$$\frac{\cancel{\Phi Z}\,\ddot{\mathcal{P}}}{\mathcal{P}} + \frac{\frac{\cancel{\Phi Z}}{\rho}\dot{\mathcal{P}}}{\mathcal{P}} + \frac{\frac{\cancel{\mathcal{P}Z}}{\rho^2}\ddot{\Phi}}{\Phi} + \underbrace{\frac{\cancel{\mathcal{P}\Phi}\,\ddot{Z}}{Z}}_{\substack{\shortparallel \\ -k_z^2}} = -k^2$$

$$\rho^2\frac{\ddot{\mathcal{P}}}{\mathcal{P}} + \rho\frac{\dot{\mathcal{P}}}{\mathcal{P}} + \underbrace{\frac{\ddot{\Phi}}{\Phi}}_{\substack{\shortparallel \\ -m^2}} + \underbrace{(k^2-k_z^2)}_{\shortparallel}\rho^2 = 0$$

$$\rho^2\ddot{\mathcal{P}} + \rho\dot{\mathcal{P}} + (\underbrace{(k^2-k_z^2)}_{\substack{\shortparallel \\ k_\rho^2}}\rho^2 - m^2)\mathcal{P} = 0 \qquad \text{Bessel's D.Eq.}$$

$$\boxed{k^2 = k_\rho^2 + k_z^2}$$

$$\rho^2\ddot{\mathcal{P}} + \rho\dot{\mathcal{P}} + ((k_\rho\rho)^2 - m^2)\mathcal{P} = 0 \qquad \begin{matrix}(3\text{-}66A)\\(3\text{-}64)\\B\end{matrix}$$

- $\mathcal{P}(\rho) = \begin{cases} J_m(k_\rho\rho)\ \&\ Y_m(k_\rho\rho) & \text{Bessel Functions} \\ \text{OR} \\ H_m^{(1)}(k_\rho\rho)\ \&\ H_m^{(2)}(k_\rho\rho) & \text{Hankel Functions}\end{cases}$

$$\ddot{\Phi} = -m^2\Phi$$

- $\Phi(\phi) = \begin{cases} e^{\pm im\phi} \\ \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(m\phi \end{cases}$

$$\ddot{Z} = -k_z^2 Z$$

- $Z(z) = \begin{cases} e^{\pm ik_z z} \\ \sin k_z z \qquad \cos k_z z \end{cases}$

$m = \text{integer} \Rightarrow J_m, H_m^{(1)}$
Y_m

HANKEL FUNCTIONS

$e^{i\omega t}$

BALANIS

$$H_p^{(1)}(\xi) = J_p(\xi) + i\,Y_p(\xi)$$

J_p: Bessel Function of first kind

Y_p: Bessel Function of second kind

$$Y_m(\xi) = \frac{J_m(\xi)\cos(m\pi) - J_{-m}(\xi)}{\sin n\pi}$$

$$H_p^{(2)}(\xi) = J_p(\xi) - i\,Y_p(\xi)$$

Asymptotic Forms: $H_p^{(1)}(\xi) \rightarrow \left(\frac{2}{\pi\xi}\right)^{1/2} e^{i(\xi - p\frac{\pi}{2} - \frac{\pi}{4})}$ INCOMING $\xi \to \infty$

$H_p^{(2)}(\xi) \rightarrow \left(\frac{2}{\pi\xi}\right)^{1/2} e^{-i(\xi - p\frac{\pi}{2} - \frac{\pi}{4})}$ OUTGOING $\xi \to \infty$

CIRCULAR METALLIC WAVEGUIDE: WAVE EQ FOR E_z & H_z

Recall:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k_\rho^2\right]\begin{Bmatrix} E_z \\ \text{or} \\ H_z \end{Bmatrix} = 0$$

$$k_\rho^2 = \omega^2\mu\epsilon - k_z^2$$

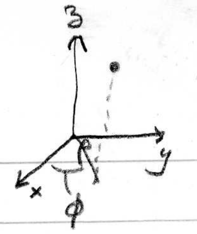
with $\xi = k_\rho \rho$, we have Bessel Equation

$$\left[\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d}{d\xi}\right) + \left(1 - \frac{m^2}{\xi^2}\right)\right] B(\xi) = 0$$

$e^{+i\omega t}$

	$\xi \to 0$	$\xi \to \infty$
$H_m^{(1)}(\xi)$	J_m okay	INWARD
$H_m^{(2)}(\xi)$	$\left.\begin{matrix} Y_m \\ H_m^{(1)} \\ H_m^{(2)} \end{matrix}\right\} \to \infty$	OUTWARD

LAPLACE'S EQUATION IN CYLINDRICAL COORDS —



(ρ, ϕ, z)

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V = \mathbb{R}\,\Phi\,Z$$

$$\frac{1}{\Phi}\ddot{\Phi} = -n^2 \ ; \quad \frac{\ddot{Z}}{Z} = k^2 \ ; \quad \rho\frac{d}{d\rho}\left(\rho\frac{d\mathbb{R}}{d\rho}\right) + (k^2\rho^2 - n^2)\mathbb{R} = 0$$

Special case : $k=0$, $n=0$

$$V_{00} = (M \ln\rho + N)(Cz + D)(A\phi + B)$$

$$\Phi = Ae^{in\phi} + Be^{-in\phi} \ ; \quad A_1 \cos n\phi + B_1 \sin n\phi$$

$$Z = Ce^{kz} + De^{-kz} \ ; \quad C' \cosh kz + D' \sinh kz$$

$$\mathbb{R}(v) = A' J_m(k\rho) + B' Y_m(k\rho)$$

Bessel's Functions
First kind — Second kind
order n

$J_0(0) = 1$, $J_{m\geq 1}(0) = 0$ $\quad Y_m(0) = \infty$

$J_m(\infty) = 0$ $\quad Y_m(\infty) = 0$

PLOTS J_m $\quad v = k\rho = 0$ to 10 or 20 are nice problem
Y_m

Spherical Bessel Functions

$$j_m(v) = \left(\frac{\pi}{2v}\right)^{1/2} J_{m+1}(v) \qquad n_n(v) = \left(\frac{\pi}{2v}\right)^{1/2} Y_{m+\frac{1}{2}}(v)$$

MODIFIED BESSEL EQUATION : let above $k = ik_1$ ...

$$V = R_n^{\circ}(k_1\rho)\, e^{\pm in\phi}\, e^{\pm ik_1 z}$$

$$\hookrightarrow = A\, I_m(k_1\rho) + B\, K_m(k_1\rho)$$

LAPLACE'S EQUATION - CYLINDRICAL COORDS

JACKSON 3.7

RADIAL EQUATION $$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0$$

Separation of Variables $\mathbb{R}\ \Theta\ \mathbb{Z}$

$$\left[A\, J_\nu(k\rho) + B\, N_\nu(k\rho)\right] e^{\pm kz}\ e^{\pm i\nu\phi}$$

Bessel Function First Kind

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n}$$

compare to SMYTHE 5.293.3

$$\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}x\right)^{\nu+2n}}{n!\ \Gamma(\overset{\nu}{m}+n+1)}$$

$$\left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{2n}$$ Same

Bessel Function Second Kind also called NEUMANN FUNCTION

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$ Stratton, Schelkunoff

$$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$ IDENTICAL Watson

Sm 5.293.4

$Y_\nu(0) = \infty$

$Y_\nu(\infty) = 0$

BESSEL FUNCTION Third Kind call HANKEL FUNCTIONS

$$H_\nu^{(1)}(x) = J_\nu(x) + i\,N_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - i\,N_\nu(x)$$

Jackson 3.86

Sm 5.294.10

Same

Useful Cylindrical Traveling Waves

$$\mathbb{R} = A H_\nu^{(1)} + B H_\nu^{(2)}$$

HELMHOLTZ WAVE EQUATION - CYLINDRICAL COORDS - $e^{-i\omega t}$

KONG

$$\nabla^2 \psi + k^2 \psi = 0$$

$$\psi = P \, \Phi \, Z$$

$$\left. \begin{matrix} J_m(k_\rho \rho) \\ Y_m(k_\rho \rho) \end{matrix} \right\} \qquad \left. \begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix} \right\} \qquad \left. e^{\pm i k_z z} \right\}$$

$$\left. \begin{matrix} H_m^{(1)}(k_\rho \rho) \\ H_m^{(2)}(k_\rho \rho) \end{matrix} \right\} \qquad \left. e^{\pm i m \phi} \right\}$$

$k_o = n k$

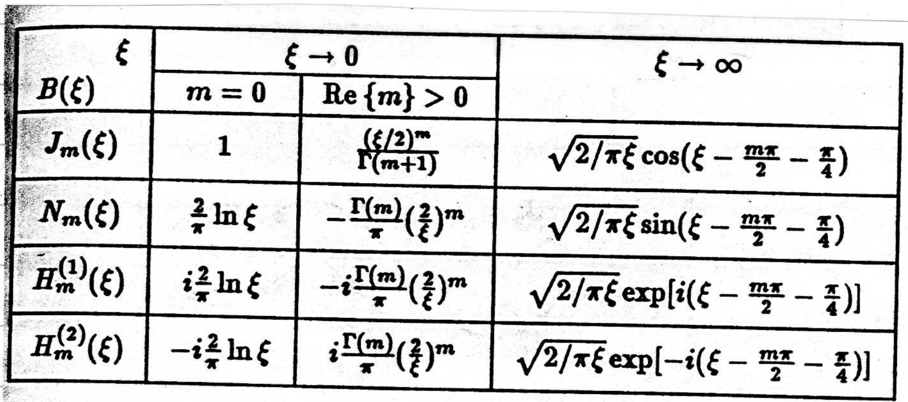
$$k^2 = k_\rho^2 + k_z^2$$

$$Y_m(0) \to \infty$$

$$H_m^{(1)}(\underset{\xi}{\underset{\parallel}{k\rho}}) \longrightarrow \sqrt{\frac{2}{\pi \xi}} \, e^{i(\xi - \frac{m\pi}{2} - \frac{\pi}{4})}$$

outward travelling wave

watch out $e^{-i\omega t}$



ξ $B(\xi)$	$\xi \to 0$		$\xi \to \infty$
	$m = 0$	$\mathrm{Re}\{m\} > 0$	
$J_m(\xi)$	1	$\frac{(\xi/2)^m}{\Gamma(m+1)}$	$\sqrt{2/\pi\xi}\cos(\xi - \frac{m\pi}{2} - \frac{\pi}{4})$
$N_m(\xi)$	$\frac{2}{\pi}\ln\xi$	$-\frac{\Gamma(m)}{\pi}(\frac{2}{\xi})^m$	$\sqrt{2/\pi\xi}\sin(\xi - \frac{m\pi}{2} - \frac{\pi}{4})$
$H_m^{(1)}(\xi)$	$i\frac{2}{\pi}\ln\xi$	$-i\frac{\Gamma(m)}{\pi}(\frac{2}{\xi})^m$	$\sqrt{2/\pi\xi}\exp[i(\xi - \frac{m\pi}{2} - \frac{\pi}{4})]$
$H_m^{(2)}(\xi)$	$-i\frac{2}{\pi}\ln\xi$	$i\frac{\Gamma(m)}{\pi}(\frac{2}{\xi})^m$	$\sqrt{2/\pi\xi}\exp[-i(\xi - \frac{m\pi}{2} - \frac{\pi}{4})]$

Table 3.6.1 Limiting values of J_m, N_m, $H_m^{(1)}$ and $H_m^{(2)}$.

p177 KONG (2nd Edit 1990)

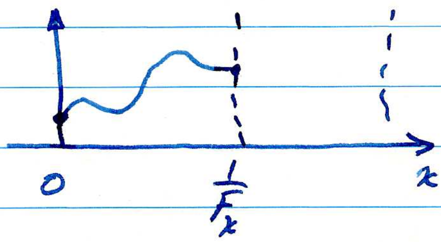
FOURIER SERIES

p107
Papoulis S&T wAO

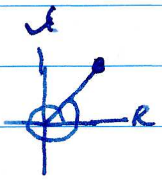
Consider a periodic function $g(x)$, i.e.,

$$g(x+\tfrac{1}{F_x}) = g(x)$$

$\mathcal{F}$ both sides $\int_{-\infty}^{\infty} g(x+\tfrac{1}{F_x})\, e^{-i2\pi f_x(x+\frac{1}{F_x})+i2\pi\frac{f_x}{F_x}}\, d(x)$



• $G(f_x)\, e^{i2\pi\frac{f_x}{F_x}} = G(f_x)$



$G(f_x) = 0$ except when
$f_x = m F_x$, m integer

• $G(f_x) = \sum_{n=-\infty}^{\infty} \beta_m\, \delta(f_x - m F_x)$!!

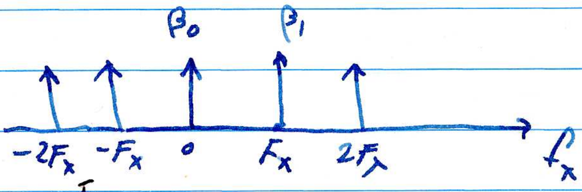
$\int \delta(f_x - nF_x)\, e^{i2\pi f_x x}\, df_x$

$$g(x) = \sum_{n=\infty}^{\infty} \beta_m\, e^{i2\pi m F_x x}$$

$$\int_{-\frac{1}{2F_x}}^{\frac{1}{2F_x}} g(x)\, e^{-i2\pi m F_x x}\, dx = \sum \int_{-\frac{1}{2F_x}}^{\frac{1}{2F_x}} \beta_n\, e^{i2\pi(n-m)F_x x}\, dx$$

$$0 = \frac{\left(e^{i2\pi(n-m)F_x\frac{1}{2F_x}} - e^{-i2\pi(n-m)F_x\frac{1}{2F_x}}\right)}{i2\pi(n-m)F_x} \quad , n \neq m$$

• $F_x \int_{-\frac{1}{2F_x}}^{\frac{1}{2F_x}} g(x)\, e^{-i2\pi m f_x x}\, dx = \beta_m$



$\Big\{$ Lovely Fourier Series
see Papoulis

Also $\int_0^{\frac{1}{F_x}} = e^{i2\pi(n-m)\frac{F_x}{F_x}} - 1$ $\quad n \neq m$

$= 1 - 1 = 0$

• $\int_0^{\frac{1}{F_x}} g\, e^{-i2\pi m F_x x} = \beta_m \frac{1}{F_x}$ $\quad n = m$

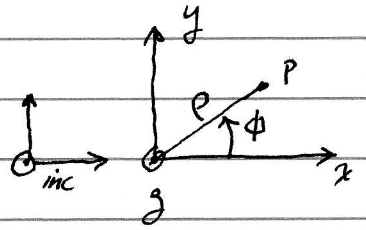
Problem: Plane Wave, Orthogonal Functions, Expansion

Find an expression for a plane wave in cylindrical coordinates

Consider

$P(\rho, \phi, z)$

$e^{-i\beta x} = e^{-i\beta\rho\cos\phi}$



Noting too, the periodicity in $\phi \Rightarrow$ Fourier Series

$$\cancel{e^{-i\beta\rho\cos\phi} = \sum_{m=-\infty}^{\infty} \alpha_m e^{i2\pi m\phi}}$$

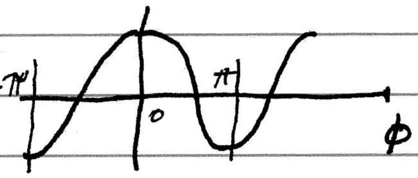
Motivation: What form?

Wave Eq. $P(\rho)\,\Phi\,Z(z)$

$J_m(k_\rho\rho), Y_m$, H_m^1, H_m^2 $\Big\}$ $\sin m\phi, \cos m\phi$, $e^{\pm im\phi}$

$$\cancel{\alpha_m = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\beta\rho\cos\phi - i2\pi m\phi} d\phi}$$

$$e^{-i\beta\rho\cos\phi} = \sum_{m=-\infty}^{\infty} \alpha_m e^{im\phi}$$



$$\alpha_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\beta\rho\cos\phi - im\phi} d\phi$$

$$\alpha_m = \frac{1}{2\pi}\Big(\cos(\beta\rho\cos\phi - m\phi) - i\sin(\beta\rho\cos\phi - m\phi)\Big)$$

$$\underbrace{\cos\ \cos}\ + \underset{\text{odd}}{\sin\ \sin} \quad -i(\underbrace{s\ c}_{\text{even}} - \underset{\text{odd}}{c\ s})$$

$$2\pi\alpha_m = \int_{-\pi}^{\pi} \cos(\beta\rho\cos\phi)\cos m\phi\, d\phi$$ G/R 3.71 18

$$= 2\pi\cos\left(\frac{m\pi}{2}\right) J_m(\beta\rho)$$ G/R 3.71 - 13

$$= -i\int^{\pi} \sin(\beta\rho\cos\phi)\cos m\phi\, d\phi = -i2\pi\sin\frac{n\pi}{2} J_m(\beta\rho)$$

$$\alpha_m = \frac{1^{-\pi}}{e^{i\frac{\pi}{2}m}} = \frac{1}{(j)^m} \qquad \sum_{m=-\infty}^{\infty} J_m(\beta\rho) e^{im(\phi-\frac{\pi}{2})}$$ or

$$e^{-i\beta\rho\cos\phi} = \sum_{m=-\infty}^{\infty}\left(\frac{1}{j}^m\right) J_m(\beta\rho) e^{im\phi}$$ $\Big\{$ Bal 11-81a, Sm 12.17(6) 31

SPHERICAL COORDINATES

3.4.3 Bal.

$$\nabla^2 \underline{E} + k^2 \underline{E} = 0$$

$$\nabla^2(\hat{r} E_r + \hat{\theta} E_\theta + \hat{\phi} E_\phi) + k^2(\hat{r} E_r + \hat{\theta} E_\theta + \hat{\phi} E_\phi = 0$$

After lengthy math manipulations, one finds 3 coupled equations scalar ok but containing E_r, E_θ, E_ϕ See 3.74 a b c

Later with introduction TE^r ; TM^r there is a way

Hence, as math problem consider the scalar equation for $\psi(r,\theta,\phi)$:

$$\nabla^2 \psi(r,\theta,\phi) + k^2 \psi(r,\theta,\phi) = 0$$

Assume separable $\psi(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} + k^2 \psi = 0$$

$$\frac{\Theta\Phi}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{R\Phi}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) + \frac{R\Theta}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} + k^2 R\Theta\Phi = 0$$

$$\overset{\checkmark}{\frac{\sin^2\theta}{R}}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{\Theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) + \underbrace{\boxed{\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}}}_{= -m^2} + \overset{\checkmark}{k^2} r^2 \sin^2\theta = 0$$

m integer

$\div \sin^2\theta$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + (kr)^2 + \underbrace{\boxed{\frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) - \left(\frac{m}{\sin\theta}\right)^2}}_{= -n(n+1)} = 0$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + [(kr)^2 - n(n+1)]\,R(r) = 0 \qquad \text{3.86 (a)}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) + \left[n(n+1) - \left(\frac{m}{\sin\theta}\right)^2\right]\Theta(\theta) = 0 \qquad \text{3.86 (b)}$$

$$\frac{\partial^2 \Phi}{\partial \phi^2} + m^2 \Phi = 0 \qquad \text{3.86 (c)}$$

SPHERICAL BESSEL FUNCTIONS

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R_s}{\partial r}\right) + \left[(kr)^2 - m(m+1)\right] R_s(r) = 0$$

STANDING WAVE FORM

spherical Bessel functions

$$j_m(kr) = \sqrt{\frac{\pi}{2kr}}\; J_{m+\frac{1}{2}}(kr)$$

first kind

$$y_m(kr) = \sqrt{\frac{\pi}{2kr}}\; Y'_{m+\frac{1}{2}}(kr)$$

second kind

RADIAL TRAVELING WAVES

spherical Hankel functions

$$h_m^{(1)}(kr) = \sqrt{\frac{\pi}{2kr}}\; H^{(1)}_{m+\frac{1}{2}}(kr)$$

first kind

$$h_m^{(2)}(kr) = \sqrt{\frac{\pi}{2kr}}\; H^{(2)}_{m+\frac{1}{2}}(kr)$$

second kind

We see the form of the equation for $R_s(r)$ which is called the spherical Bessel equation. It is related to the ~~cylindrical~~ Bessel equation arising in cylindrical coordinates. One can find a solution by transforming the equation for $R_s(r)$ as follows:

Let $R_s(r) = \dfrac{R(r)}{r^{1/2}}$

And find the corresponding d.e. for $R(r)$

As detailed on the next page, we find that the d.e. for $R(kr)$ is Bessel's equation

order $(m+\frac{1}{2})$ not m

Ref. Kong 3.7c $\left\{\begin{array}{l}(17)\\(18a)\end{array}\right.$

SM 12.12 (4), (5), (6)

10.05 (9)

SPHERICAL COORDINATES: DETAILS

Spherical $\mathbb{R}_s(r)$ & Cylindrical $\mathbb{P}(\rho)$

$\mathbb{R}_s(r)$:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \mathbb{R}_s}{\partial r}\right) + \left[(kr)^2 - m(m+1)\right]\mathbb{R}_s(r) = 0$$

Let $\mathbb{R}_s(r) = \dfrac{\mathbb{R}(r)}{\sqrt{r}}$

$$j_m(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{m+\frac{1}{2}}(kr)$$

$$y_m(kr) = \sqrt{\frac{\pi}{2kr}}\, Y_{m+\frac{1}{2}}(kr)$$

etc for $h_m^{(1)}, h_m^{(2)}$

$$\frac{\partial \mathbb{R}_s}{\partial r} = \frac{-\frac{1}{2}\mathbb{R}}{r^{3/2}} + \frac{1}{r^{1/2}}\dot{\mathbb{R}}$$

$$r^2\frac{\partial R_s}{\partial r} = \frac{-r^{\frac{1}{2}}\mathbb{R}}{2} + r^{3/2}\dot{\mathbb{R}}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \mathbb{R}_s}{\partial r}\right) = -\frac{1}{4}\frac{\mathbb{R}}{r^{1/2}} - \frac{r^{\frac{1}{2}}\dot{\mathbb{R}}}{2} + \frac{3}{2}r^{\frac{1}{2}}\dot{\mathbb{R}} + r^{\frac{3}{2}}\ddot{\mathbb{R}}$$

$$\frac{-\mathbb{R}}{4r^{\frac{1}{2}}} \overset{\times}{+} r^{\frac{1}{2}}\dot{\mathbb{R}} \overset{\checkmark}{+} r^{\frac{3}{2}}\ddot{\mathbb{R}} + \left[(kr)^2 - m(m+1)\right]\frac{\mathbb{R}}{r^{\frac{1}{2}}} = 0$$

$$\ddot{\mathbb{R}} + \frac{1}{r}\dot{\mathbb{R}} - \frac{\mathbb{R}}{4r^2} + \left[(kr)^2 - m(m+1)\right]\frac{\mathbb{R}}{r^2} = 0$$

$$\ddot{\mathbb{R}} + \frac{1}{r}\dot{\mathbb{R}} + \left[(kr)^2 - m(m+1) - \frac{1}{4}\right]\frac{\mathbb{R}}{r^2} = 0$$

Bessel's equation in kr is as follows

$$\dot{\mathbb{R}} = \frac{d\,\mathbb{R}(kr)}{dr} = \frac{d\,\mathbb{R}(kr)}{d(kr)}\,k$$

$$\ddot{\mathbb{R}} = \frac{d^2\mathbb{R}(kr)}{dr^2} = \frac{d^2\mathbb{R}(kr)}{d(kr)^2}\,k^2$$

$$k^2\frac{d^2\mathbb{R}(kr)}{d(kr)^2} + \frac{k}{r}\frac{d\,\mathbb{R}(kr)}{dkr} + \left[(kr)^2 - m(m+1) - \frac{1}{4}\right]\frac{\mathbb{R}(kr)}{r^2} = 0$$

$$(kr)^2\frac{d^2\mathbb{R}(kr)}{d(kr)^2} + \frac{(kr)^2}{\cancel{kr}}\frac{d\,\mathbb{R}(kr)}{d(kr)} + \left[(kr)^2 - \overline{\underline{m(m+1) - \frac{1}{4}}}\right]\frac{\mathbb{R}(kr)}{\cancel{(kr)^2}} = 0$$

Sols: $J_p(kr)$, $Y_p(kr)$

$H_p^{(1)}(kr)$, $H_p^{(2)}(kr)$

$$p^2 = m(m+1) + \frac{1}{4}$$

$$p^2 = m^2 + m + \frac{1}{4}$$

$$p = \pm\left(m + \tfrac{1}{2}\right)^2 \qquad \underline{\text{Yes}}$$

TABLE 3-1
Wave functions, zeroes, and infinities of plane wave functions in rectangular coordinates

Wave type	Wave functions	Zeroes of wave functions	Infinities of wave functions
Traveling waves	$e^{-j\beta x}$ for $+x$ travel	$\beta x \to -j\infty$	$\beta x \to +j\infty$
	$e^{+j\beta x}$ for $-x$ travel	$\beta x \to +j\infty$	$\beta x \to -j\infty$
Standing waves	$\cos(\beta x)$ for $\pm x$	$\beta x = \pm(n + \frac{1}{2})\pi$	$\beta x \to \pm j\infty$
	$\sin(\beta x)$ for $\pm x$	$\beta x = \pm n\pi$ $n = 0, 1, 2, \ldots$	$\beta x \to \pm j\infty$
Evanescent waves	$e^{-\alpha x}$ for $+x$	$\alpha x \to +\infty$	$\alpha x \to -\infty$
	$e^{+\alpha x}$ for $-x$	$\alpha x \to -\infty$	$\alpha x \to +\infty$
	$\cosh(\alpha x)$ for $\pm x$	$\alpha x = \pm j(n + \frac{1}{2})\pi$	$\alpha x \to \pm\infty$
	$\sinh(\alpha x)$ for $\pm x$	$\alpha x = \pm jn\pi$ $n = 0, 1, 2, \ldots$	$\alpha x \to \pm\infty$
Attenuating traveling waves	$e^{-\gamma x} = e^{-\alpha x}e^{-j\beta x}$ for $+x$ travel	$\gamma x \to +\infty$	$\gamma x \to -\infty$
	$e^{+\gamma x} = e^{+\alpha x}e^{+j\beta x}$ for $-x$ travel	$\gamma x \to -\infty$	$\gamma x \to +\infty$
Attenuating standing waves	$\cos(\gamma x) = \cos(\alpha x)\cosh(\beta x)$ $-j\sin(\alpha x)\sinh(\beta x)$ for $\pm x$	$\gamma x = \pm j(n + \frac{1}{2})\pi$	$\gamma x \to \pm j\infty$
	$\sin(\gamma x) = \sin(\alpha x)\cosh(\beta x)$ $+j\cos(\alpha x)\sinh(\beta x)$ for $\pm x$	$\gamma x = \pm jn\pi$ $n = 0, 1, 2, \ldots$	$\gamma x \to \pm j\infty$

TABLE 3-2
Wave functions, zeroes, and infinities for radial wave functions in cylindrical coordinates

Wave type	Wave functions	Zeroes of wave functions	Infinities of wave functions
Traveling waves	$H_m^{(1)}(\beta\rho) = J_m(\beta\rho) + jY_m(\beta\rho)$ for $-\rho$ travel	$\beta\rho \to +j\infty$	$\beta\rho = 0$ $\beta\rho \to -j\infty$
	$H_m^{(2)}(\beta\rho) = J_m(\beta\rho) - jY_m(\beta\rho)$ for $+\rho$ travel	$\beta\rho \to -j\infty$	$\beta\rho = 0$ $\beta\rho \to +j\infty$
Standing waves	$J_m(\beta\rho)$ for $\pm\rho$	Infinite number (see Table 9-2)	$\beta\rho \to \pm j\infty$
	$Y_m(\beta\rho)$ for $\pm\rho$	Infinite number	$\beta\rho = 0$ $\beta\rho \to \pm j\infty$
Evanescent waves	$K_m(\alpha\rho) = \frac{\pi}{2}(-j)^{m+1}H_m^{(2)}(-j\alpha\rho)$ for $+\rho$	$\alpha\rho \to +\infty$	
	$I_m(\alpha\rho) = j^m J_m(-j\alpha\rho)$ for $-\rho$		$\alpha\rho \to +\infty$ for integer orders
Attenuating traveling waves	$H_m^{(1)}(\gamma\rho) = H_m^{(1)}(\alpha\rho + j\beta\rho)$ for $-\rho$ travel	$\gamma\rho \to +j\infty$	$\gamma\rho \to -j\infty$
	$H_m^{(2)}(\gamma\rho) = H_m^{(2)}(\alpha\rho + j\beta\rho)$ for $+\rho$ travel	$\gamma\rho \to -j\infty$	$\gamma\rho \to +j\infty$
Attenuating standing waves	$J_m(\gamma\rho) = J_m(\alpha\rho + j\beta\rho)$ for $\pm\rho$	Infinite number	$\gamma\rho \to \pm j\infty$
	$Y_m(\gamma\rho) = Y_m(\alpha\rho + j\beta\rho)$ for $\pm\rho$	Infinite number	$\gamma\rho \to \pm j\infty$

TABLE 3-3
Wave functions, zeroes, and infinites for radial waves in spherical coordinates

Wave type	Wave functions	Zeroes of wave functions	Infinities of wave functions
Traveling waves	$h_n^{(1)}(\beta r) = j_n(\beta r) + jy_n(\beta r)$ for $-r$ travel	$\beta r \to +j\infty$	$\beta r = 0$ $\beta r \to -j\infty$
	$h_n^{(2)}(\beta r) = j_n(\beta r) - jy_n(\beta r)$ for $+r$ travel	$\beta r \to -j\infty$	$\beta r = 0$ $\beta r \to +j\infty$
Standing waves	$j_n(\beta r)$ for $\pm r$	Infinite number	$\beta r \to \pm j\infty$
	$y_n(\beta r)$ for $\pm r$	Infinite number	$\beta r = 0$ $\beta r \to \pm j\infty$

In (3-87a) $j_n(\beta r)$ and $y_n(\beta r)$ are referred to, respectively, as the *spherical Bessel functions* of the first and second kind. They are used to represent radial standing waves, and they are related, respectively, to the corresponding regular Bessel functions $J_{n+1/2}(\beta r)$ and $Y_{n+1/2}(\beta r)$ by

$$j_n(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, J_{n+1/2}(\beta r) \tag{3-90a}$$

$$y_n(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, Y_{n+1/2}(\beta r) \tag{3-90b}$$

In (3-87b) $h_n^{(1)}(\beta r)$ and $h_n^{(2)}(\beta r)$ are referred to, respectively, as the *spherical Hankel functions* of the first and second kind. They are used to represent radial traveling waves, and they are related, respectively, to the regular Hankel functions $H_{n+1/2}^{(1)}(\beta r)$ and $H_{n+1/2}^{(2)}(\beta r)$ by

$$h_n^{(1)}(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, H_{n+1/2}^{(1)}(\beta r) \tag{3-91a}$$

$$h_n^{(2)}(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, H_{n+1/2}^{(2)}(\beta r) \tag{3-91b}$$

Wave functions used to represent radial traveling and standing waves in spherical coordinates are listed in Table 3-3. More details on the spherical Bessel and Hankel functions can be found in Appendix IV.

In (3-88a) and (3-88b) $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$ are referred to, respectively, as the *associated Legendre functions* of the first and second kind (more details can be found in Appendix V).

The appropriate solution forms of f, g, and h will depend on the problem in question. For example, a typical solution for $\psi(r, \theta, \phi)$ of (3-85) to represent the fields within a sphere as shown in Figure 3-7 may take the form

$$\psi_1(r, \theta, \phi) = [A_1 j_n(\beta r) + B_1 y_n(\beta r)] \times [C_2 P_n^m(\cos\theta) + D_2 Q_n^m(\cos\theta)][C_3\cos(m\phi) + D_3\sin(m\phi)] \tag{3-92}$$

WAVE EQUATION

3.7
B

Show that Bessel & Hankel functions represent
standing   traveling waves respectively – use asymptotic forms

CYLINDRICAL WAVES Table 3.2 / Class Notes

$$\Psi(\rho,\phi,z) = \mathcal{P}\,\Phi\,\mathbb{Z}\,, \quad \mathcal{P} \quad \begin{cases} J_m(k_\rho \rho)\,,\ Y_m(k_\rho \rho) \\ \text{OR} \\ H_m^{(1)}(k_\rho \rho)\,,\ H_m^{(2)}(k_\rho \rho) \end{cases}$$

For Hankel functions $H_m^{(2)}(k_\rho \rho)$ outward paired with $e^{i\omega t}$

$$H_m^{(2)}(k_\rho \rho) \rightarrow \sqrt{\frac{2}{\pi k_\rho \rho}}\, e^{-i[k_\rho \rho - m(\pi/2) - \pi/4] + i\omega t} \qquad (IV-16)$$

Balanis

$e^{-ik_\rho \rho + i\omega t}$ form is an outgoing traveling wave

For $J_m(k_\rho \rho) \rightarrow \sqrt{\frac{2}{\pi k_\rho \rho}} \cos\left(k_\rho \rho - \frac{\pi}{4} - \frac{m\pi}{2}\right) e^{i\omega t}$ $\underline{IV}$-13 B.

$$Y_m(k_\rho \rho) \rightarrow \sqrt{\frac{2}{\pi k_\rho \rho}} \sin\left(k_\rho \rho - \frac{\pi}{4} - \frac{m\pi}{2}\right) e^{i\omega t}$$

$$\left(e^{ik_\rho \rho + i\phi_m} \pm e^{-ik_\rho \rho - i\phi_m}\right) e^{i\omega t}$$

This will form a standing wave $\rightleftarrows$ that has harmonic time variation composed of an incoming and an outgoing in $J_m(k_\rho \rho)$ by itself.

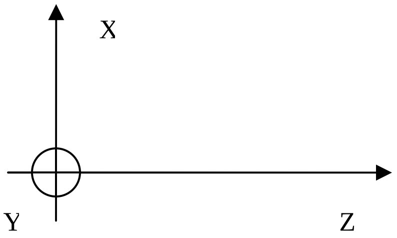
Ditto for $Y_m(k_\rho \rho)$

Also, one can look at $\underline{\underline{Re}}\ J_m(k_\rho \rho) e^{i\omega t} = \sqrt{\frac{2}{\pi k_\rho \rho}} \cos\left(k_\rho \rho - \frac{\pi}{4} - \frac{m\pi}{2}\right) \cos \omega t$

which is ~~an~~ a modulated standing wave pattern.

UNIFORM PLANE WAVES

This is an extremely important, yet simple case in which we are able to solve Maxwell's equations directly without recourse to potentials.



Source free, HTD, $e^{i\omega t}$, OHMIC $\boldsymbol{J} = \sigma \boldsymbol{E}$

(1)

$$\nabla \times \boldsymbol{E} = -i\omega\mu \boldsymbol{H}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + i\omega\varepsilon \boldsymbol{E} \equiv (\sigma + i\omega\varepsilon)\boldsymbol{E} = i\omega\varepsilon' \boldsymbol{E}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

i) Assert: only have z spatial variation

ii) Take $\boldsymbol{E}$ $(E_x, 0, 0)$

$$-i\omega\mu(\hat{x}H_x + \hat{y}H_y + \hat{z}H_z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix}$$

$\therefore H_x = 0$ & $H_z = 0$

$$\therefore H_y = -\hat{y}\frac{\partial E_x}{\partial z}\frac{1}{i\omega\mu}$$

(2) $\nabla^2 \boldsymbol{E} + (k^2 - i\omega\mu\sigma)\boldsymbol{E} = 0$ or $i\omega\varepsilon' = i\omega\varepsilon + \sigma$

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} + k'^2 E_x = 0}$$

$$\varepsilon' = \varepsilon + \frac{\sigma}{i\omega}$$

$$E_x = Ae^{-ik'z} + Be^{ik'z}$$

$$\varepsilon' = \varepsilon - i\frac{\sigma}{\omega}$$

$$k'^2 = \omega^2 \mu\varepsilon'$$

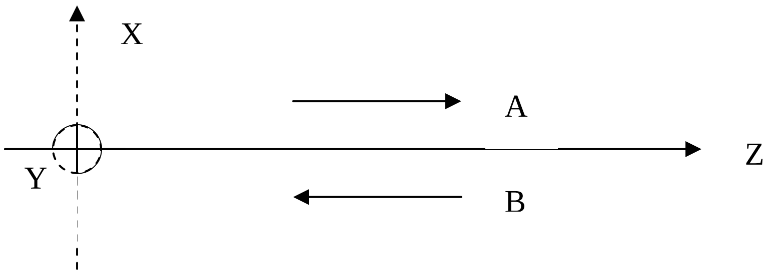
$H_y = \sqrt{\frac{\varepsilon'}{\mu}}(Ae^{-ik'z} - Be^{ik'z})$, from $-\frac{1}{i\omega\mu}\frac{\partial E_x}{\partial z} = \frac{ik}{i\omega\mu}(Ae^{-ik'z} - Be^{ik'z})$

$$\frac{k'}{\omega\mu} = \frac{\omega\sqrt{\mu\varepsilon'}}{\omega\mu} = \sqrt{\frac{\varepsilon'}{\mu}}$$

Admittance of wave $y = \sqrt{\frac{\varepsilon'}{\mu}} = \frac{1}{\text{ohm}}$

UNIFORM PLANE WAVE

ILLUSTRATES: WAVEFRONTS
EQUI-PHASE SURFACES
PROPAGATION



WE HAVE FOUND

- **SPECIFIC SOLUTION THAT HAS** $\boldsymbol{E} = (E_x, 0, 0)$ **AND NO X,Y VARIATIONS, AND** $\boldsymbol{H} = (0, H_y, 0)$
-

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0$$

$$\frac{1}{\eta} = g = \sqrt{\frac{\varepsilon}{\mu}} = \frac{n}{377}\left(\frac{1}{\text{OHM}}\right)$$

$$H_x = 0$$

$$H_z = 0$$

$$H_y = \frac{-1}{i2\pi\nu\mu}\frac{\partial E_x}{\partial z}$$

- $E_x = [Ae^{-ikz} + Be^{ikz}]e^{i2\pi vt}$
- $H_y = \frac{1}{\eta}[Ae^{-ikz} - Be^{ikz}]e^{i2\pi vt}$

PLANE WAVE SOLUTION

ILLUSTRATES: SIGNAL REPRESENTATION

WAVEFRONTS

PROPAGATION

COMPARE COMPLEX REPRESENTATION

LET

$$A = |A| e^{i\phi_A}$$

$$B = |B| e^{i\phi_B}$$

$$E_x = [|A| e^{-ikz+i\phi_A} + |B| e^{ikz+i\phi_B}] e^{i2\pi\nu t}$$

- $\mathcal{E}_x = |A| \cos[2\pi\nu t - kz + \phi_A] + |B| \cos[2\pi\nu t + kz + \phi_B]$
- $\mathcal{H}_y = \frac{|A|}{\eta} \cos[2\pi\nu t - kz + \phi_A] - \frac{|B|}{\eta} \cos[2\pi\nu t + kz + \phi_B]$

- **ONE NEEDS TO STUDY BOTH SOLUTION FORMS TO GAIN INSIGHT INTO PLANE WAVES, PHASE FRONTS, AND PROPAGATION**

PLANE WAVE SOLUTION

- HOW DO WE ILLUSTRATE THE WAVE VELOCITY?

DRAW $\mathcal{E}_X = |A| \cos[2\pi\nu t - kz + \phi_A]$ ———

AND AT A SLIGHTLY LATER TIME $t \to t + \Delta t$

$\mathcal{E}_X = |A| \cos[2\pi\nu(t + \Delta t) - kz + \phi_A]$ – – – – –

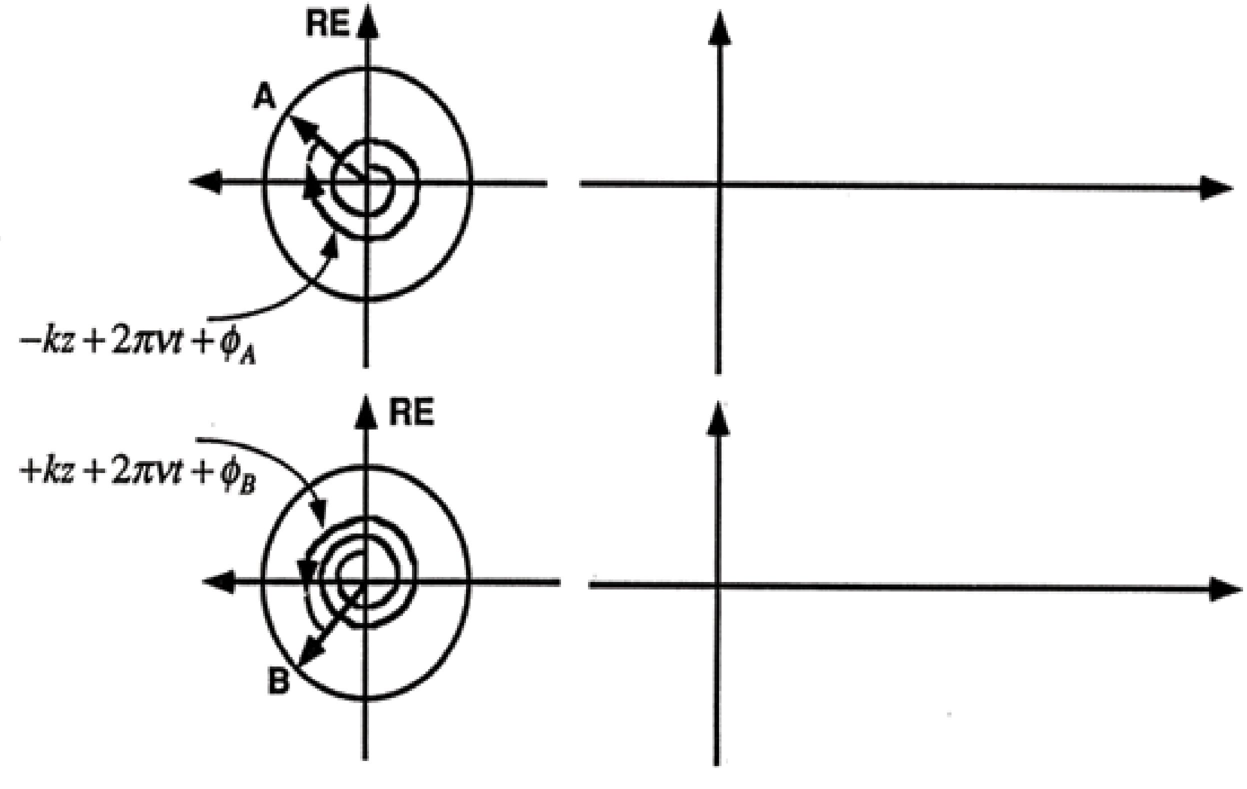
→ WAVE TRAVEL $\frac{dz}{dt} = \frac{2\pi\nu}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$ •

$$v = \frac{1}{n\sqrt{\mu\varepsilon}} = \frac{c}{n}$$ •

PLANE WAVE SOLUTION

NOW WITH THE COMPLEX PHASE NOTATION HOW DO WE ILLUSTRATE THE WAVE VELOCITY

$$E_X = [|A| e^{-ikz + i\phi_A} + |B| e^{ikz + i\phi_B}] e^{i2\pi vt}$$

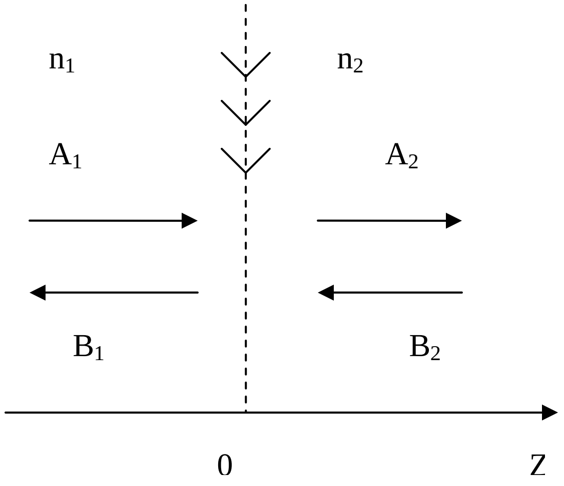


DIELECTRIC INTERFACE TOPIC

PLANE WAVE - NORMALLY INCIDENT

The plane wave solution and its various cases are closely analogous to the famous old problem of solving the telegrapher’s equation. Many of you have studied this earlier as undergraduates and you should review this material, particularly the Smith chart.

AT A DIELECTRIC INTERFACE



$$A_1 e^{-ik_{,1}z} + B_1 e^{ik_{,1}z} : E_x : A_2 e^{-ik_{,2}z}$$

$$n_1[A_1 e^{-ik_{,1}z} - B_1 e^{ik_{,1}z}] : H_y : n_2 A_2 e^{-ik_{,2}z}$$

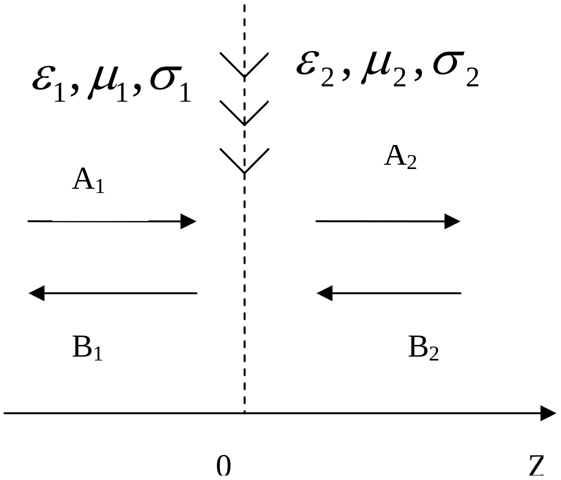
- **TANGENTIAL E, H CONTINOUS**

$$\Gamma = \frac{B_1}{A_1} = \frac{n_1 - n_2}{n_1 + n_2}$$

$$T = \frac{A_2}{A_1} = \frac{2n_1}{n_1 + n_2}$$

CONDUCTING INTERFACES- NORMAL INCIDENCE

CONSIDER



GENERATOR

WITH COMPLEX NUMBERS

$$\varepsilon_1' = \varepsilon_1(1+\frac{\sigma_1}{i\omega\varepsilon_1}), \qquad \varepsilon_2' = \varepsilon_2(1+\frac{\sigma_2}{i\omega\varepsilon_2})$$

$$g_1' = (\frac{\varepsilon_1'}{\mu_1})^{1/2} \qquad g_2' = (\frac{\varepsilon_2'}{\mu_2})^{1/2}$$

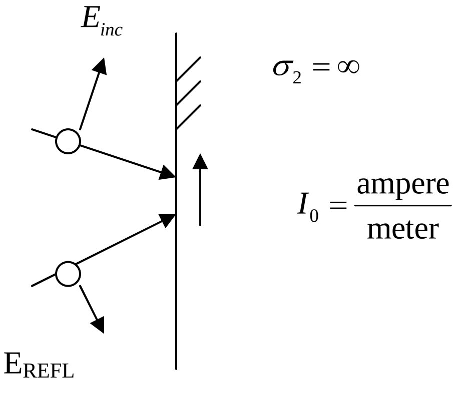
MATCHING TANGENTIAL E & H, AS BEFORE YIELDS

$$\frac{B_1}{A_1} = \Gamma_0' = \frac{g_1' - g_2'}{g_1' + g_2'}$$

$$\frac{A_2}{A_1} = T_0' = \frac{2g_1'}{g_1' + g_2'}$$

WITH AN INTERFACE AT A PERFECT CONDUCTOR, i.e., $\sigma_2 \to \infty$:

$$\Gamma_0' = \frac{1-(\frac{g_2'}{g_1'})}{1+(\frac{g_2'}{g_1'})} = -1$$



THERE IS A π PHASE SHIFT IN THE ELECTRIC FIELD.

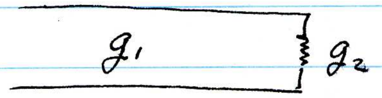
TRANSMISSION LINE FORMS
STANDING WAVE PICTURE
SMITH CHART

In the solution of the second order linear differential equation, one can select two forms $\left\{ e^{\pm ikz} \right\}$ or $\left\{ \begin{matrix} \sin kz \\ \cos kz \end{matrix} \right\}$

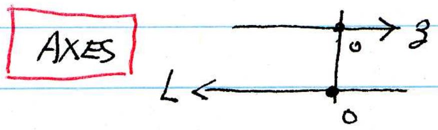
They are of course equivalent although the algebra connecting them can become an effort. For problems involving one or two sections or an input/output, the traveling wave form is probably the more convenient and more commonly used. In the study of multilayers for the layers themselves, the sine/cosine form is commonly used, as we shall see in the next few lectures. Here we use the exponential form to explain standing waves, reflection coefficient, VSWR, and the Smith Chart (all as a review):

$$E_x = A e^{-ikz} + B e^{+ikz} \qquad \text{with } e^{+i\omega t} \text{ implicit}$$
$$H_y = g_1 [A e^{-ikz} - B e^{ikz}]$$

Setting $\Gamma_0 = B/A$ gives



$$E_x = A e^{-ikz} [1 + \Gamma_0 e^{i2kz}]$$



$$\begin{cases} E_x = E_{inc} [1 + \Gamma_0 e^{-i2kL}] \\ H_y = g_1 E_{inc} [1 - \Gamma_0 e^{-i2kL}] \end{cases}$$

$$\Gamma_0 = \frac{g_1 - g_2}{g_1 + g_2}, \qquad T = \frac{2g_1}{g_1 + g_2}$$

$$\text{VSWR} = \frac{1 + |\Gamma_0|}{1 - |\Gamma_0|}$$

EQUIVALENCE OF TRANSMISSION/MULTILAYER SOLUTIONS

COMPARE THE TRAVELING WAVE FORM & STANDING WAVE FORMS:

$$E_x = A e^{-ikz} + B e^{+ikz} = a \cos kz + b \sin kz$$

LET $a = |a| e^{i\alpha}$

$b = |b| e^{i\beta}$

$$E_x = |a| e^{i\alpha} \left(\frac{e^{\overset{\checkmark}{i}kz} + e^{-ikz}}{2} \right) + |b| e^{i\beta} \left(\frac{e^{\overset{\checkmark}{\mp} ikz} - e^{-ikz}}{2i} \right)$$

$$\emptyset \quad E_x = \left(|a| \frac{e^{i\alpha}}{2} + |b| \frac{e^{i\beta}}{2i} \right) e^{ikz} + \boxed{\left(|a| \frac{e^{i\alpha}}{2} - |b| \frac{e^{i\beta}}{2i} \right) e^{-ikz}}$$

first

$$\qquad\qquad \| \qquad\qquad\qquad\qquad\qquad \|$$

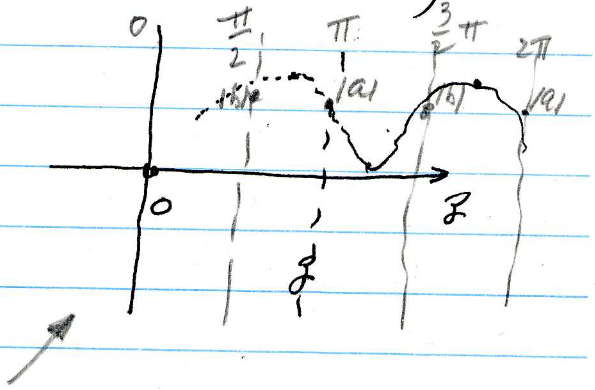
$$\qquad\qquad B \qquad\qquad\qquad\qquad\qquad A$$

THIS ESTABLISHES THE EQUIVALENCE

$$\Gamma_0 = \frac{B}{A}$$

$$1 + |\Gamma_0|, \quad 1 - |\Gamma_0|$$

USING THE $z = 0$ NOTATION AT THE ENTRANCE TO EACH MULTILAYER, ONE CAN WRITE THE FIELDS AS FOLLOWS:



$$E_x(z) = [a \cos kz + b \sin kz] e^{+i\omega t}$$

$$E_x(z,t) = |a| e^{i(\alpha + \omega t)} \cos kz + |b| e^{i(\beta + \omega t)} \sin kz$$

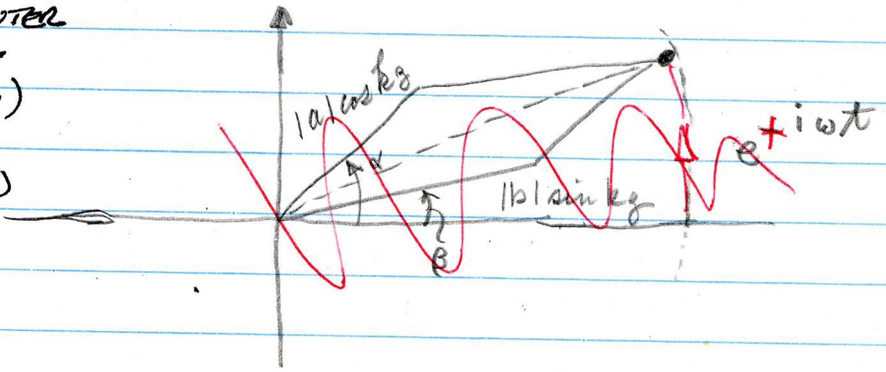
Take the real part:

$$\mathrm{Re}\, E_x(z,t) = |a| \cos(kz) \cos(\omega t + \alpha) + |b| \sin(kz) \cos(\omega t + \beta)$$

OKAY FOR COMPUTER

~~TAKE MAX ABS $E_x(z)$~~

~~TAKE MIN ABS $E_x(z)$~~



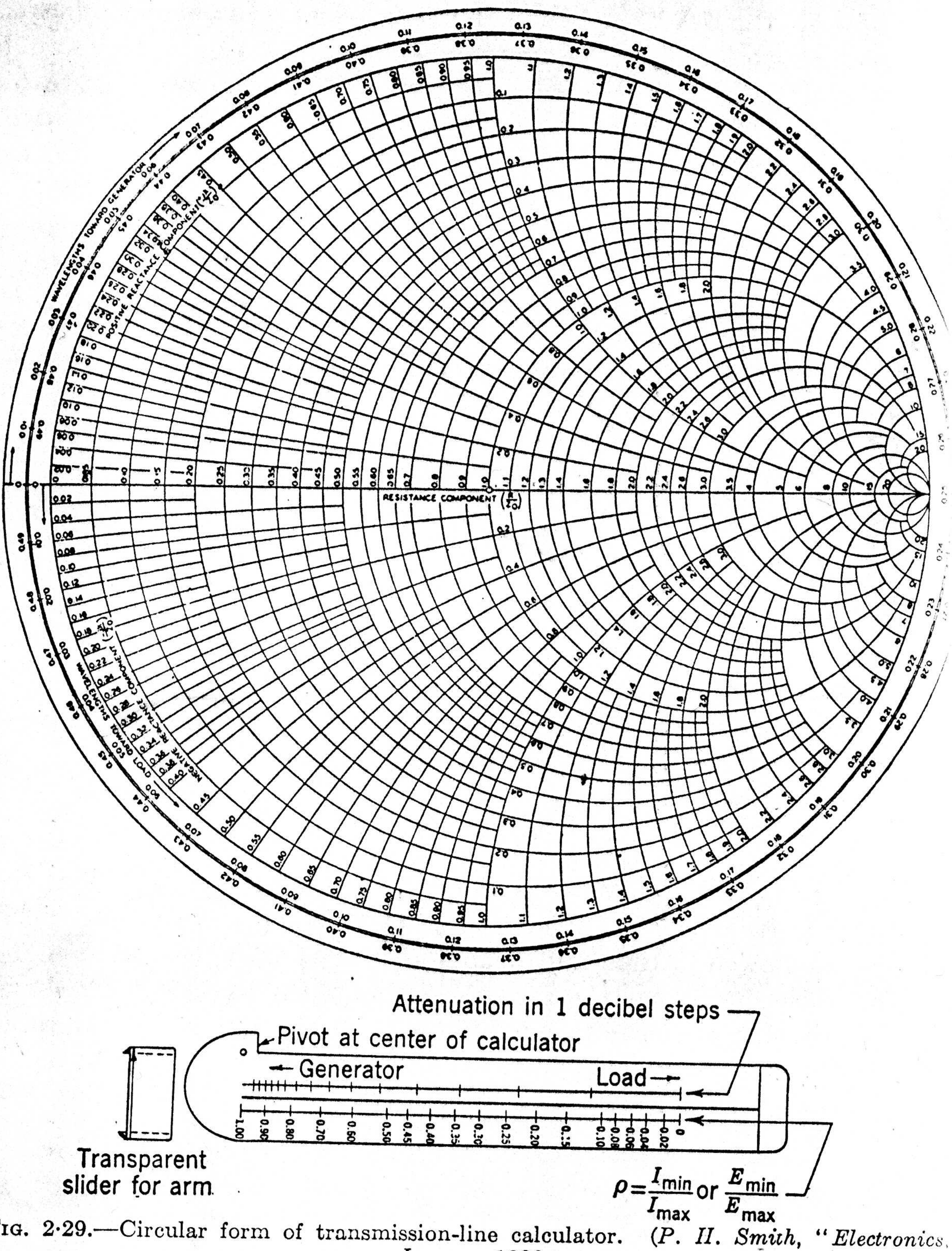
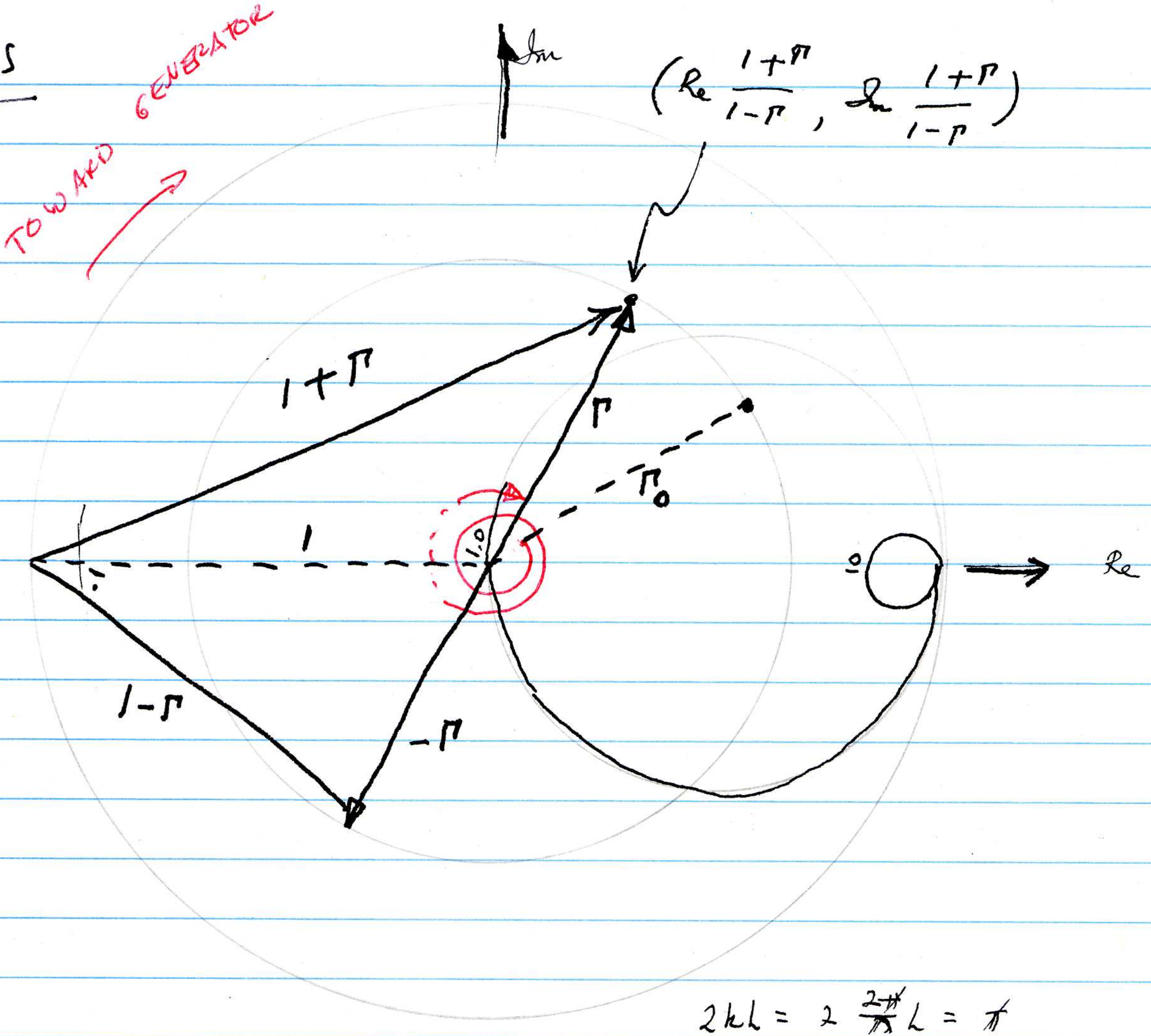


FIG. 2·29.—Circular form of transmission-line calculator. (*P. H. Smith, "Electronics, January* 1939.)

over minimum, giving a number in the range unity to infinity; others use the minimum over maximum and obtained numbers in the range unity t zero. There seems to be emerging a general preference in favor of usin the voltage standing-wave ratio exceeding unity,

DETAILS



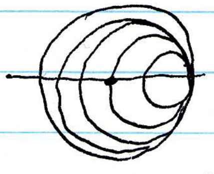
$$2kL = 2\,\frac{2\pi}{\lambda}\,L = \pi$$

$$L = \lambda/4 \qquad z \rightarrow \frac{1}{z}$$

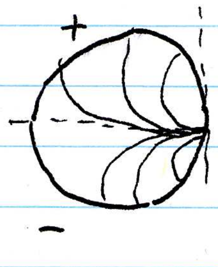
$$\frac{E_x}{E_{inc}} = 1 + \Gamma_0 e^{-ikL2}$$

$$\frac{H_y}{E_{inc}} = 1 - \Gamma_0 e^{-ikL2}$$

Circles of constant resistance



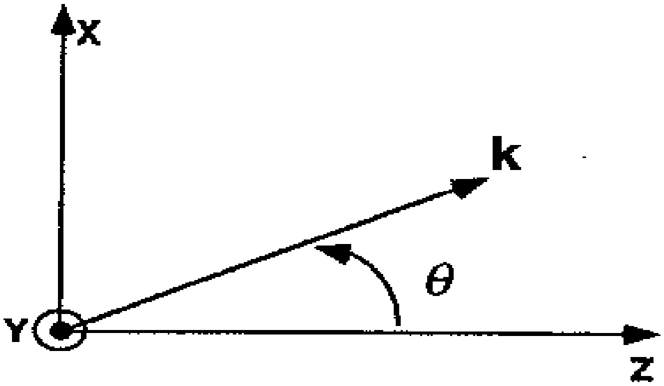
Circles of constant reactance



PLANE WAVE SOLUTION

TWO IMPORTANT RESULTS

PLANE WAVE AT AN ANGLE



CONSIDER E_y POLARIZATION

Let

$$\boldsymbol{K} = \hat{e}_X k \sin\theta + \hat{e}_y k \cos\theta$$

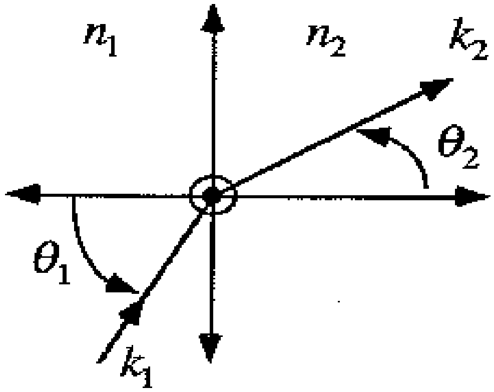
$$E_y = Ae^{-ikx\sin\theta - iky\cos\theta + i2\pi vt}$$

At Z=0

$$E_y = Ae^{-ikx\sin\theta + i2\pi vt}$$

SNELL'S LAW

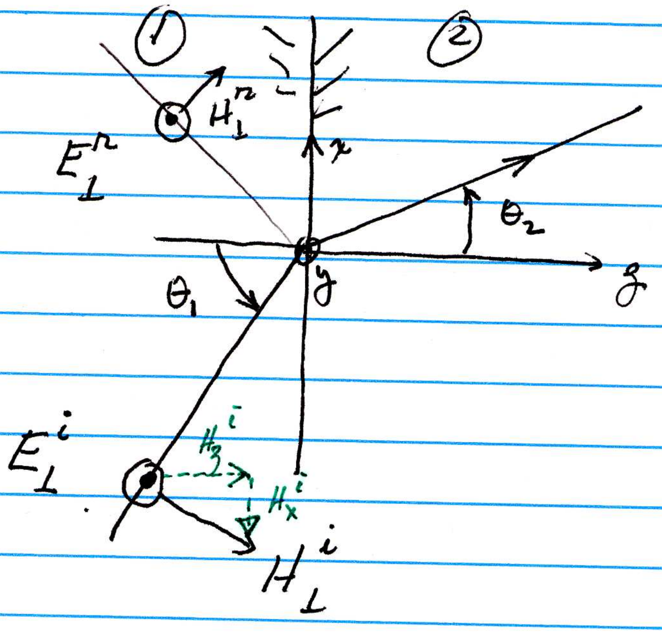
- $n_1 \sin\theta_1 = n_2 \sin\theta_2$



FRESNEL EQUAS : REFLECTION & TRANSMISSION

REFLECTION   E-WAVE   TM CASE   OUTLINE

READ TEXT



no y variation

FROM SEP OF VARIABLES FOR WAVE EQUATION

FIRST WRITE INCIDENT WAVES USING $X \cancel{Y} Z$

$\beta_x^2+\beta_z^2=\beta^2$

$$E_\perp^i = \hat{y}\, E_0\, e^{-i\beta_1(x\sin\theta_1 + z\cos\theta_1)}$$

FROM

$$-i\omega\mu \underline{H} = \nabla\times\underline{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cancel{E_x}^{0} & E_y & \cancel{E_z}^{0} \end{vmatrix}$$

$$\underline{H} = -\frac{1}{i\omega\mu}\left[-\frac{\partial E_y}{\partial z}\hat{x} + \hat{y}\,0 + \hat{z}\frac{\partial E_y}{\partial x}\right]$$

$$H_x = \frac{1}{i\omega\mu}\frac{\partial E_y}{\partial z} = \frac{-i\beta_1\cos\theta_1}{i\omega\mu} E_0\, e^{-i\beta_1(x\sin\theta_1 + z\cos\theta_1)}$$

$$H_x = -\underline{g_1}\cos\theta_1\, E_0\, e^{-i\beta_1(x\sin\theta_1 + z\cos\theta_1)}$$

$$H_z = \frac{1}{-i\omega\mu}\frac{\partial E_y}{\partial x} = \frac{-i\beta_1\sin\theta_1 E_0}{-i\omega\mu}\, e^{-i\beta_1(x\sin\theta_1 + z\cos\theta_1)}$$

$$H_z = \underline{g_1\sin\theta_1}\, E_0\, e^{-i\beta_1(x\sin\theta_1 + z\cos\theta_1)}$$

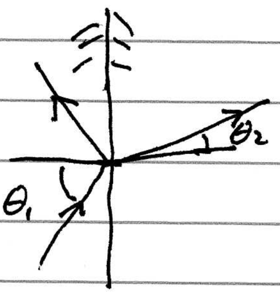
See Bal 5.10

TM CASE

DEFINITIONS

$$\Gamma_\perp = \frac{E^r}{E^i}$$ y oriented

$$T_\perp = \frac{E_{2\perp}}{E_{1\perp}}$$



ESTABLISH:

$$\theta_{1r} = \theta_1$$

$$\beta_1 \sin\theta_1 = \beta_2 \sin\theta_2$$

$$\Gamma_\perp = \frac{\eta_2\cos\theta_1 - \eta_1\cos\theta_2}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2} \longrightarrow \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} ; \quad \theta_1 = 0$$

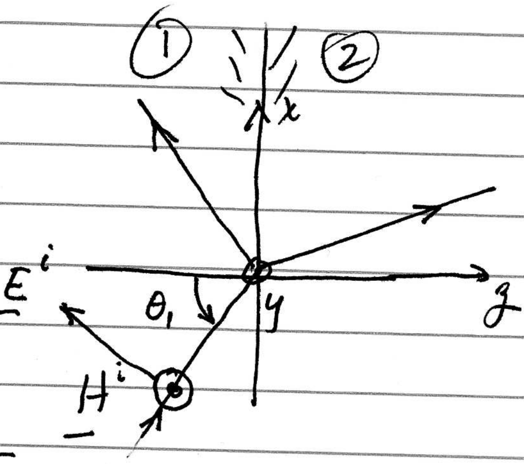
$$T_\perp = \frac{2\eta_2\cos\theta_1}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2} \longrightarrow \frac{2\eta_2}{\eta_2 + \eta_1}$$

WE HAVE ONLY OUTLINED THE SOLUTION IN CLASS
THE DERIVATION IN BALANIS FOR Γ AND FOR T
IS WELL WRITTEN AND IT NEEDS TO BE STUDIED.
YOU MIGHT MAKE UP A FEW COMPUTER ORIENTED
PROBLEMS & SOLUTIONS TO PRACTICE THIS MATERIAL.

IF YOU UNDERSTAND THIS MATERIAL, THE THIN FILM MULTILAYER
WILL BE QUITE SIMPLE FOR YOU.

H-WAVE TE CASE

INCIDENT $\underline{H}$ along y axis



TO START THE DERIVATION, WE WRITE THE $\underline{H}$ INCIDENT AS A SOLN $X \cancel{Y} Z$

no y variation

THEN WE USE

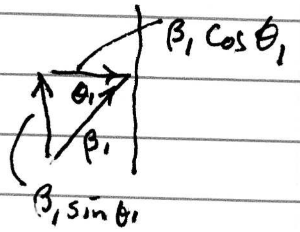
$$\nabla \times \underline{H} = (\sigma + i\omega\epsilon)\underline{E} = i\omega\epsilon'\underline{E}$$ TO FIND $\underline{E}$

E_0 = total amplitude of incident electric field

$H_{\parallel}^{i}$ = " " " magnetic field incident

$$\frac{E_0}{H_{\parallel}^{i}} = \frac{1}{\eta_1}$$

$$\underline{H}_{\parallel}^{i} = \hat{y}\,\frac{E_0}{\eta_1}\,e^{-i(\beta_1)(x\sin\theta_i + z\cos\theta_i)}$$



$$\hat{x}\,E_x^{i} = \hat{x}\,E_0\cos\theta_1\,e^{-i\,\underline{\beta}_1\cdot\underline{r}}$$

$$\hat{z}\,E_z^{i} = -\hat{z}\,E_0\sin\theta_1\,e^{-i\,\underline{\beta}_1\cdot\underline{r}}$$

define

$$\Gamma_{\parallel} = \frac{E_{\parallel}^{r}}{E_0}$$ $\dfrac{\text{total Elec fld ampl. reflected}}{\text{total Elec fld ampl. incident}}$

$$T_{\parallel} = \frac{E_{\parallel}^{t}}{E_0}$$ Solutions 5-24 p 191

FRESNEL EQUATIONS REFLECTION TRANSMISSION
SINGLE INTERFACE

READ CAREFULLY

E-WAVE TM CASE PLOTS ON FIG 5-3

H-WAVE TE CASE PLOTS ON FIG 5-5

SNELL'S LAW

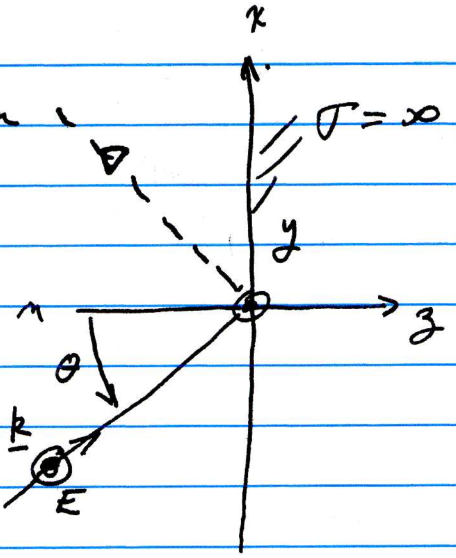
BREWSTER ANGLE TOTAL TRANSMISSION H-WAVE
E IN PLANE OF INCIDENCE

TOTAL REFLECTION - CRITICAL ANGLE

MULTILAYERS - THIN FILMS

REVIEW & CONSOLIDATION

Consider a plane wave incident on a perfect conductor with the E vector polarized along y



Wave equation solutions can be written

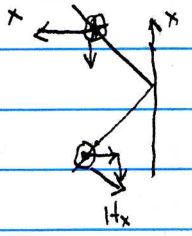
$$E_y = XYZ = XZ$$

$$E_y = \left(a e^{-ikz\cos\theta} + b e^{ikz\overset{\cos}{\cancel{\sin}}\theta} \right) e^{-ikx\sin\theta}$$

Corresponding to this we have $\underline{H}$ given by

$$\underline{H} = \frac{-1}{i\omega\mu} \nabla \times \underline{E} = \frac{-1}{i\omega\mu} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

$$H_x = \frac{+1}{i\omega\mu} \frac{\partial E_y}{\partial z}$$



$$H_x = +\frac{1\,(+ik\cos\theta)}{i\omega\mu} \overset{\checkmark}{\left(-a e^{-ikz\cos\theta} + b e^{ikz\cos\theta} \right)} e^{-ikx\sin\theta}$$

$$\frac{k}{\omega\mu} = \frac{\omega\sqrt{\mu\epsilon}}{\omega\mu} = \sqrt{\frac{\epsilon}{\mu}} = g_o$$

$$H_y = 0 \;, \quad H_z = \frac{\partial E_y}{\partial x} \frac{1}{i\omega\mu}$$

$$H_z = \left(a e^{-ikz\cos\theta} \overset{\times}{+} b e^{ikz\cos\theta} \right) \frac{(+ik\sin\theta)}{i\omega\mu} e^{-ikx\sin\theta}$$

As we wrote the E_y term, there is both $+z$ and $-z$ waves

At the interface, $E_y = 0 \Rightarrow b = -a$

the reflection coefficient $\Gamma = \frac{b}{a} = -1$

The H_x term is $-(a+a) = -2a$ at the interface

it has a wave admittance: $g_o \cos\theta$

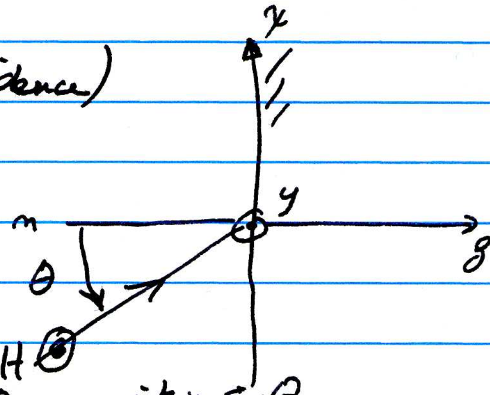
The H_z term is $a - a = 0$ null @ $z = 0$ just like E_y

{Important Physical
{Notion
{Functional Dependence

REFLECTION AT CONDUCTING INTERFACE

H- WAVE (⊥ plane of incidence)

$H = \mathbb{X}\,\mathbb{Z}$



NEED TO PRESERVE MEANING a, b *

START WITH –

$$H_z = \left(ce^{-ikz\cos\theta} + de^{ikz\cos\theta}\right)e^{-ikx\sin\theta}$$

$$E = \frac{1}{i\omega\epsilon}\nabla\times\underline{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} \quad \begin{matrix} -\hat{x}\frac{\partial H_y}{\partial z} \\ \hat{z}\frac{\partial H_y}{\partial x} \end{matrix}$$

$$E_x = \frac{--\cancel{i}k\cos\theta}{\cancel{i}\omega\epsilon}\left(ce^{-ikz\cos\theta} - de^{ikz\cos\theta}\right)e^{-ikx\sin\theta}$$

$$E_x = \left(ae^{-ikz\cos\theta} + be^{ikz\cos\theta}\right)e^{-ikx\sin\theta} \qquad \frac{\cos\theta}{z}c = a \ *$$

$$H_y = \frac{z}{\cos\theta}\left(ae^{-ikz\cos\theta} - be^{ikz\cos\theta}\right)e^{-ikx\sin\theta} \qquad -\frac{\cos\theta}{z}d = b \ *$$

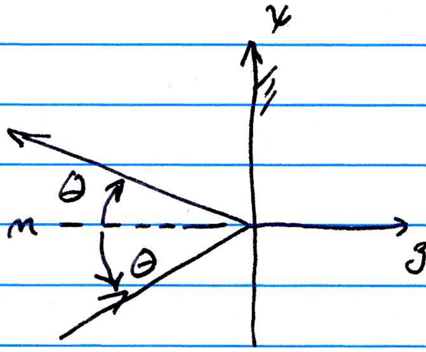
$$E_z = \frac{-\cancel{k}\sin\theta\,\cancel{z}}{i\omega\epsilon\cos\theta}\left(ae^{-ikz\cos\theta} - be^{ikz\cos\theta}\right)e^{-ikx\sin\theta}$$

For the perfect conductor, we see @ $z=0$, $E_x=0$, $b=-a$

The H_y is $a+a = 2a$

The E_z is $a+a = 2a$

ALL AS EXPECTED



The ∡ reflection = ∡ incidence

Need only tangential condition + ME ~~Bo~~ & the normal conditions follow.

"I am the wisest man alive, for I know one thing, and that is that I know nothing."
"By all means, marry. If you get a good wife, you'll become happy; if you get a bad one, you'll become a philosopher."
"Wisdom begins in wonder." ***Socrates***

"No evil can happen to a good man, neither in life nor after death." ***Plato***

"What's in a name? That which we call a rose by any other word would smell as sweet."
"My words fly up, my thoughts remain below: words without thoughts never to heaven go."
Shakespeare

"A picture is worth a thousand words." ***Napoleon Bonaparte***

"A man should look for what is, and not for what he thinks should be."
"A person who never made a mistake never tried anything new."
"I know quite certainly that I myself have no special talent; curiosity, obsession and dogged endurance, combined with self-criticism have brought me to my ideas." ***Albert Einstein***

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