Homework 10 - Gaussian processes

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1) Arbitrages. Consider bets in different outcomes of a random event. An arbitrage is an opportunity to realize a gain independently of the outcome of the event. Consider for example the forthcoming 2026 FIFA World Cup (to be held in US, Canada, and Mexico) in which the two favorite teams according to British bookmakers are Brazil and France. Suppose you can place bets on Brazil (B), France (F) or "neither Brazil, nor France" (O). Booker 1 is offering the following odds for these three outcomes:

Brazil	**France**	**Other**
6.6:1	8.1:1	1.2:1

The odds on Brazil, for example, mean that for each dollar bet on Brazil you are paid 6.6 dollars if Brazil wins the World Cup. The odds on France mean that you get 8.1 dollars for each dollar bet on France, if France wins. The odds on "other" mean that you are paid 1.2 dollars for each dollar bet on the outcome "neither Brazil, nor France" wins, if this indeed comes to happen.

Our goal is to look for arbitrage opportunities by placing mixed bets on all three possible outcomes. We bet x on Brazil, y on France and z on other. The total money invested is a given constant $x+y+z=c$. The earnings associated with this investment strategy are:

$$
\begin{aligned}
\text{Brazil wins:} \quad & 6.6x - (x+y+z) = 6.6x - c, \\
\text{France wins:} \quad & 8.1y - (x+y+z) = 8.1y - c, \\
\text{Other wins:} \quad & 1.2z - (x+y+z) = 1.2y - c.
\end{aligned}
$$

A) Arbitrage with single booker. Assuming Booker 1 is not on the business of giving money away it is not likely that there is an arbitrage opportunity. Show that arbitrage is not possible for the odds given above.

B) Arbitrage with many bookers. There are many bookers accepting bets, so it may then be possible to find arbitrages by placing bets on different outcomes with different bookers. In particular, consider three different bookers offering the following odds:

Booker	**Brazil**	**France**	**Other**
1	6.6:1	8.1:1	1.2:1
2	6.4:1	7.8:1	1.3:1
3	5.9:1	8.4:1	1.1:1

Find if there exits an arbitrage opportunity by combining bets placed with different bookers. If an arbitrage opportunity exists, find a combination of bets that yields an arbitrage.

2) Option pricing. The goal of this problem is to use one year of historic data of a company's stock to determine the price of a European style option. Let us then consider the daily closing price of Cisco Systems (CSCO) between November 25, 2008 and November 24, 2009. The closing price of CSCO for this date range can be obtained from the class's website and is depicted in Fig. 1. This is actual data, as you can corroborate from information available in Google finance (click on 1yr).

To determine the option's price start with a geometric Brownian motion model for the evolution of the stock price $X(t)$. This model presumes that relative variations on the price $X(t)$ can be described

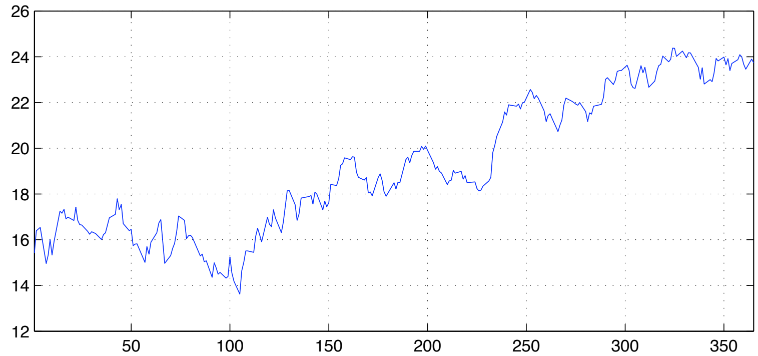

Fig. 1. Daily closing price of Cisco Systems (CSCO) between November 25, 2008 and November 24, 2009.

as a Brownian motion with drift. Specifically, we assume that changes in prices are according to the expression

$$X(t+s) = X(t)e^{Y_t(s)} \tag{1}$$

where $Y_t(s)$ is Gaussian-distributed with mean μs and variance $\sigma^2 s$ independently of t. We further assume that relative price changes $Y_t(s)$ in disjoint time intervals are independent. An important observation to make here is to consider a discretization in time steps of fixed duration h, say $h = 1$ day, and to define the discrete-time random process Y_n as

$$Y_n := \log\big[X(nh)\big] - \log\big[X\big((n-1)h\big)\big] = Y_{(n-1)h}(h). \tag{2}$$

It follows from the model in (1) that RVs Y_n are i.i.d. normals with mean μh and variance $\sigma^2 h$. This is an important observation because it allows us to infer the parameters μ and σ^2 from empirical data. In fact, the drift parameter μ can be estimated by the sample mean

$$\hat{\mu} = \frac{1}{Nh}\sum_{n=1}^{N} Y_n \tag{3}$$

and the volatility parameter σ can be estimated by the sample variance

$$\hat{\sigma}^2 = \frac{1}{(N-1)h}\sum_{n=1}^{N} (Y_n - \hat{\mu}h)^2 \,. \tag{4}$$

In (3) and (4), variables Y_n are computed as in (2) using collected data for the values of $X(nh)$.

A European style option on the stock $X(t)$ is a contract to have the option to buy the stock at a predetermined price on a predetermined future time. The option is described by the strike price K, the strike time t and its price c. Paying c to buy an option at time 0 gives us the opportunity to buy the stock for the strike price K at the strike time t. At time t the worth of the option depends on the value of the stock $X(t)$. If the stock has fallen below the strike price K, i.e., if $X(t) \leq K$ the option becomes worthless. If, on the contrary, the price has risen beyond K, i.e., if $X(t) > K$ we can realize a gain $X(t) - K$ by exercising the option to buy the stock at price K. We can thus write the worth of the option w as

$$w = \big[X(t) - K\big]^+, \tag{5}$$

where $[x]^+ = \max(0, x)$ denotes projection on the positive numbers.

As a final note, throughout this problem we are going to determine returns on different investment strategies. When doing this we will correct our returns by the return of a risk-free money market investment. Denoting as α the money market rate of return, the present value of a gain r at time t is $re^{-\alpha t}$.

A) Derivation of (3) *and* (4). Explain why it is true that the RVs Y_n are i.i.d. normals with mean μh and variance $\sigma^2 h$. Using this fact, argue why μ and σ^2 can be estimated as in (3) and (4).

B) Determination of drift and volatility. Use the data provided for CSCO to determine the drift and volatility parameters for $X(t)$. In the data you are provided, successive stock values differ by one day for weekday transitions, e.g., from Monday to Tuesday, by three days for weekend transitions, i.e., from Friday to Monday, or by other amounts when there are holidays. It is not difficult to account for this nuisance, but you can neglect that in your analysis and assume that all values in the data are separated by one day.

C) Is geometric Brownian motion a good model? If a geometric Brownian motion with drift $\hat{\mu}$ and volatility $\widehat{\sigma^2}$ is a good model for the evolution of CSCO stock price, then the variables Y_n have a $\mathcal{N}\left(\hat{\mu}h, \hat{\sigma}^2 h; y\right)$ pdf. Estimate the pdf of Y_n using a histogram and compare it with the pdf $\mathcal{N}\left(\hat{\mu}h, \hat{\sigma}^2 h; y\right)$. Do this comparison for values of Y_n between -0.1 and 0.1. Use a bin size of 0.01 for your histogram. Are you in awe that the model coincides with reality? Or are you disappointed to see that the model is a useless abstraction? If you are interested, you can research other historical stock series and compound your amazement (or not).

D) Expected return. Compute a formula for the expected return of an investment on CSCO as a function of time t and the parameters $\hat{\mu}$ and $\hat{\sigma}^2$. This return has to be discounted by the money market rate α. Determine this expected return for $\alpha = 3.75\%$ and time $t = 1$ year. What is the probability of having a rate of return of at least 5% in the next year, by investing on CSCO.

E) Risk neutral measure. Determine the risk neutral measure for CSCO's stock.

F) Expected return for risk neutral measure. Assume you are leaving in an alternative reality where the stock's price evolves according to the risk neutral measure. What is the expected discounted rate of return for an investment in CSCO in this alternative reality? What is the non-discounted rate of return?

G) Derive the Black-Scholes formula. The Black-Scholes formula to price an option is obtained by determining the price c that yields zero expected return with respect to the risk neutral measure, i.e., c is chosen as the solution of

$$\mathbb{E}_{\mathbf{q}}\left[e^{-\alpha t}\big[X(t) - K\big]^{+} - c\right] = 0 \tag{6}$$

where the expected value is with respect to the risk neutral measure q, not the actual geometric Brownian motion followed by the stock price $X(t)$. Explain why the expression in (6) yields zero expected return with respect to the risk neutral measure. Obtain a closed form expression for the price c in terms of the risk-free rate of return α, volatility of stock σ^2, the strike price K, option's strike time t, and current price $X(0)$. Notice that the price c is independent of the drift parameter μ.

H) Determine option price. Determine the option price c as a function of t when the strike price coincides with the expected value of the stock, i.e., $K = \mathbb{E}\left[X(t)\right]$. Repeat the calculation when $K = 1.2\mathbb{E}\left[X(t)\right]$ and $K = 0.8\mathbb{E}\left[X(t)\right]$. What is the use of buying options with strike prices $K = 1.2\mathbb{E}\left[X(t)\right]$ and $K = 0.8\mathbb{E}\left[X(t)\right]$?