Solutions to Homework 10 - Gaussian processes

1) Arbitrages. Consider bets in different outcomes of a random event. An arbitrage is an opportunity to realize a gain independently of the outcome of the event. Consider for example the upcoming Soccer World Cup (to be held in Russia in 2018) in which the two favorite teams according to British bookies are Germany and Argentina. With the event still so far ahead in time, bookers at this point bias their odds towards the outcome of the most recent World Cup held in Brazil. Suppose you can place bets on Germany (G), Argentina (A) or "neither Germany, nor Argentina" (O). Booker 1 is offering the following odds for these three outcomes:

Germany	**Argentina**	**Other**
6.6:1	8.1:1	1.2:1

A) Arbitrage with single booker. Our goal is to look for arbitrage opportunities by placing mixed bets on all three possible outcomes. We bet x on Germany, y on Argentina and z on other. The total money invested is a given constant $x + y + z = c$. Accordingly, an arbitrage opportunity exists if

$$\begin{aligned}
\text{Germany wins: } & 6.6x - (x + y + z) = 6.6x - c > 0 \ \Rightarrow\ x > c/6.6 \\
\text{Argentina wins: } & 8.1y - (x + y + z) = 8.1y - c > 0 \ \Rightarrow\ y > c/8.1 \\
\text{Other wins: } & 1.2z - (x + y + z) = 1.2z - c > 0 \ \Rightarrow\ z > c/1.2.
\end{aligned}$$

Summing-up all the inequalities in the right-hand-side yields the arbitrage condition

$$x + y + z > c\left(\frac{1}{6.6} + \frac{1}{8.1} + \frac{1}{1.2}\right) \ \Rightarrow\ c > 1.1083c$$

which cannot be satisfied for $c > 0$. Hence, arbitrage is not possible for the odds given above.

B) Arbitrage with many bookers. There are many bookers accepting bets, so it may then be possible to find arbitrages by placing bets on different outcomes with different bookers. In particular, consider three different bookers offering the following odds:

Booker	**Germany**	**Argentina**	**Other**
1	6.6:1	8.1:1	1.2:1
2	6.4:1	7.8:1	1.3:1
3	5.9:1	8.4:1	1.1:1

From the odds given it is apparent that the best strategy is to bet x on Germany from Booker 1, y on Argentina from the Booker 3, and z on other from Booker 2. An arbitrage opportunity exists if

$$\begin{aligned}
\text{Germany wins: } & 6.6x - (x + y + z) = 6.6x - c > 0 \ \Rightarrow\ x > c/6.6 \\
\text{Argentina wins: } & 8.4y - (x + y + z) = 8.4y - c > 0 \ \Rightarrow\ y > c/8.4 \\
\text{Other wins: } & 1.3z - (x + y + z) = 1.3z - c > 0 \ \Rightarrow\ z > c/1.3.
\end{aligned}$$

Summing-up all the inequalities in the right-hand-side yields the arbitrage condition

$$x + y + z > c\left(\frac{1}{6.6} + \frac{1}{8.4} + \frac{1}{1.3}\right) \ \Rightarrow\ c > 1.0398c$$

which cannot be satisfied for $c > 0$. Even with multiple bookers arbitrage is not possible for the odds given above.

2) Option pricing.

A) Derivation of (3) *and* (4). To determine the option's price start with a geometric Brownian motion model for the evolution of the stock price $X(t)$. Specifically, we assume that changes in prices are according to the expression

$$X(t + s) = X(t)e^{Y_t(s)} \tag{1}$$

where $Y_t(s)$ is Gaussian distributed with mean μs and variance $\sigma^2 s$ independently of t. We further assume that relative price changes $Y_t(s)$ in disjoint time intervals are independent. An important observation to make here is to consider a discretization in time steps of fixed duration h, say $h = 1$ day or $h = 1/365$ years, and to define the discrete-time random process $\mathbf{Y}_{\mathbb{N}} = Y_1, Y_2, \ldots, Y_n, \ldots$ as

$$Y_n := \log\big[X(nh)\big] - \log\big[X\big((n-1)h\big)\big] = Y_{(n-1)h}(h). \tag{2}$$

It follows from the model in (1) that RVs Y_n are i.i.d. normals with mean μh and variance $\sigma^2 h$, since different samples $Y_n = Y_{(n-1)h}(h)$ belong to disjoint intervals of length h. This is an important observation because it allows us to infer the parameters μ and σ^2 from empirical data. In fact, the mean μh and variance $\sigma^2 h$ of the normal distribution from where samples Y_n are drawn can be estimated using the sample mean and variance estimators. Accordingly, since h is a fixed known constant the drift parameter μ can be estimated by the scaled sample mean

$$\hat{\mu} = \frac{1}{Nh} \sum_{n=1}^{N} Y_n \tag{3}$$

and the volatility parameter σ can be estimated by the scaled sample variance

$$\hat{\sigma}^2 = \frac{1}{(N-1)h} \sum_{n=1}^{N} \left(Y_n - \hat{\mu}h\right)^2. \tag{4}$$

B) Determination of drift and volatility. Using the data provided for CSCO, here we estimate the drift and volatility parameters for $X(t)$ using the sample mean and variance expressions in (3) and (4). The Matlab script to carry out these calculations follows:

```
% Load CSCO data
cisco_stock_price

Z=log(close_price);
Y=Z(2:end)-Z(1:end-1);
N=length(Y);
h=1/365;
mu_hat=sum(Y)/(N*h) % Sample mean
sigma_sqr_hat=sum((Y-mu_hat*h).^2)/((N-1)*h) % Sample variance
```

The results obtained are $\hat{\mu} = 0.6275$ and $\hat{\sigma}^2 = 0.2174$. (Notice that in the above code and henceforth we assume the unit for time to be one year, not one day. Daily samples thus correspond to $h = 1/365$ years.)

C) Is geometric Brownian motion a good model? If a geometric Brownian motion with drift $\hat{\mu}$ and volatility $\hat{\sigma^2}$ is a good model for the evolution of CSCO stock price, then the variables Y_n have a $\mathcal{N}\left(\hat{\mu}h, \hat{\sigma}^2 h; y\right)$ pdf. To corroborate this fact, the following Matlab script estimates the pdf of Y_n using a histogram and compares it with the pdf $\mathcal{N}\left(\hat{\mu}h, \hat{\sigma}^2 h; y\right)$. We also carry out the corresponding comparisons for the cdfs. In all cases we select values of Y_n between -0.1 and 0.1, and use a bin size of 0.01 for the histogram.

```
% Load CSCO data
cisco_stock_price

Z=log(close_price);
Y=Z(2:end)-Z(1:end-1);
N=length(Y);
h=1/365;

x=-0.1:0.01:0.1;
n_elements = histc(Y,x);

% Comparison of pdfs
```

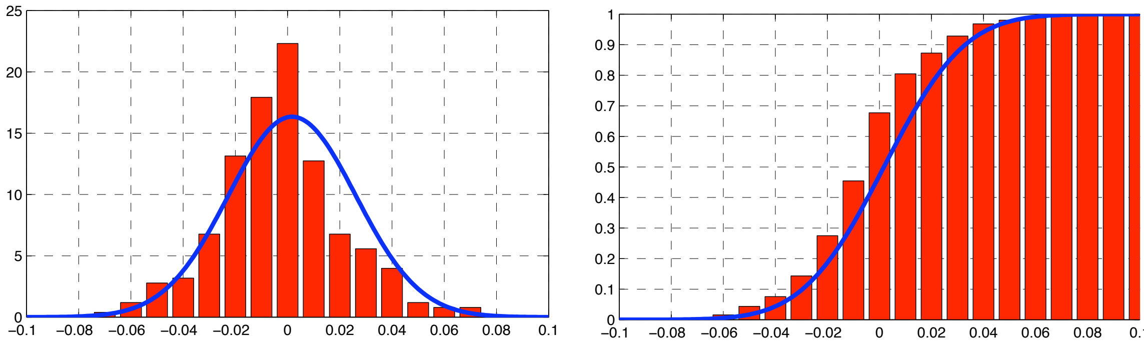

Fig. 1. The comparison of the pdf (left) and cdf (right) shows that the geometric Brownian motion model is acceptable. (Part *C*.)

```
figure(1)
bar(x,n_elements/N/0.01,'r')
hold on
x_padded=-0.1:0.001:0.1;
plot(x_padded,normpdf(x_padded,mu_hat*h,sqrt(sigma_sqr_hat*h)),'Linewidth',3)
grid on
axis([-0.1 0.1 0 25])

% Comparison of cdfs
figure(2)
c_elements = cumsum(n_elements)/N;
bar(x,c_elements,'r')
hold on
x_padded=-0.1:0.001:0.1;
plot(x_padded,normcdf(x_padded,mu_hat*h,sqrt(sigma_sqr_hat*h)),'Linewidth',3)
grid on
axis([-0.1 0.1 0 1])
```

The result of the comparison is depicted in Fig. 1, and the conclusion is that – while not great – the geometric Brownian motion model for $X(t)$ is acceptable.

D) Expected return. The formula for the expected return of an investment on CSCO as a function of time t and the parameters $\hat{\mu}$ and $\hat{\sigma}^2$ is

$$\mathbb{E}\left[\frac{e^{-\alpha t}X(t)}{X(0)} \,\Big|\, X(0)\right] = e^{(\hat{\mu}+\hat{\sigma}^2/2-\alpha)t}.$$

Notice that we have corrected the returns by the return of a risk-free money market investment. (Denoting as α the money market rate of return, the present value of a gain r at time t is $re^{-\alpha t}$.) Thus the expected return for $\alpha = 3.75\%$ and time $t = 1$ year is

$$\mathbb{E}\left[\frac{e^{-\alpha t}X(t)}{X(0)} \,\Big|\, X(0)\right] = e^{(0.6275+0.2174/2-0.0375)\times 1} = 2.01.$$

The rate of return is given by the logarithm of the discounted return, so the probability of having a rate of return of at least 5% in the next year is

$$\mathrm{P}\left[\log\left(\frac{e^{-\alpha}X(1)}{X(0)}\right) \geq 0.05 \,\Big|\, X(0)\right] = \mathrm{P}\left[Y(1) \geq 0.05 + 0.0375\right].$$

Since $Y(1) \sim \mathcal{N}(\hat{\mu}, \hat{\sigma}^2)$ the probability is simply

$$\mathrm{P}\left[Y(1) \geq 0.0875\right] = \Phi\left(\frac{0.0875 - \hat{\mu}}{\hat{\sigma}}\right) = 0.8766$$

where $\Phi(\cdot)$ is the ccdf of a standard Gaussian RV.

E) Risk neutral measure. The risk neutral measure for CSCO's stock is a Brownian motion with drift $\mu = \alpha - \hat{\sigma}^2/2 = -0.0712$, and variance parameter $\sigma^2 = \hat{\sigma}^2 = 0.2174$.

F) Expected return for risk neutral measure. If we assume we are living in an alternative reality where the stock's price evolves according to the risk neutral measure, then the expected discounted rate of return for an investment in CSCO is 0. The non-discounted rate of return is α.

G) Derive the Black-Scholes formula. A European style option on the stock $X(t)$ is a contract to have the option to buy the stock at a predetermined price on a predetermined future time. The option is described by the strike price K, the strike time t and its price c. Paying c to buy an option at time 0 gives us the opportunity to buy the stock for the strike price K at the strike time t. At time t the worth of the option depends on the value of the stock $X(t)$. If the stock has fallen below the strike price K, i.e., if $X(t) \leq K$ the option becomes worthless. If, on the contrary, the price has risen beyond K, i.e., if $X(t) > K$ we can realize a gain $X(t) - K$ by exercising the option to buy the stock at price K. We can thus write the worth of the option w as

$$w = \big[X(t) - K\big]^+, \tag{5}$$

where $[x]^+ = \max(0, x)$ denotes projection on the positive numbers.

In this context, the Black-Scholes formula to price an option is obtained by determining the price c that yields zero expected return (hence no arbitrage opportunities) with respect to the risk neutral measure $\mathbf{q}$, i.e., c is chosen as the solution of

$$\mathbb{E}_{\mathbf{q}}\left[e^{-\alpha t}\big[X(t) - K\big]^+ - c\right] = 0.$$

Note that the expected value is with respect to the risk neutral measure $\mathbf{q}$ specified in part *E*, not the actual geometric Brownian motion followed by the stock price $X(t)$. As derived in class, the closed form expression for the option's price c is

$$c = X(0)\Phi\left(\frac{\log(K/X(0)) - \mu t}{\sqrt{\sigma^2 t}} - \sqrt{\sigma^2 t}\right) - e^{-\alpha t} K \Phi\left(\frac{\log(K/X(0)) - \mu t}{\sqrt{\sigma^2 t}}\right)$$

which depends on the risk-free rate of return α, volatility of stock σ^2, the strike price K, option's strike time t, and current price $X(0)$. Notice that the price c is independent of the drift parameter μ.

H) Determine option price. Using the Black-Scholes formula, the following Matlab script calculates the option price c for $t = 1$ year, when the strike price coincides with the expected value of the stock by the strike time, i.e., $K = \mathbb{E}\left[X(t)\right]$. The code also repeats the calculation when $K = 1.2\mathbb{E}\left[X(t)\right]$ and $K = 0.8\mathbb{E}\left[X(t)\right]$.

```
% Load CSCO data
cisco_stock_price

Z=log(close_price);
Y=Z(2:end)-Z(1:end-1);
N=length(Y);
h=1/365;
mu_hat=sum(Y)/(N*h) % Sample mean
sigma_sqr_hat=sum((Y-mu_hat*h).^2)/((N-1)*h) % Sample variance

alpha=0.0375;

X_0=close_price(1,1);
EX=X_0*exp(mu_hat+sigma_sqr_hat/2);
K=[0.8 1 1.2]*EX;
a=(log(K/X_0)-(alpha-sigma_sqr_hat/2))/(sqrt(sigma_sqr_hat));
b=a-sqrt(sigma_sqr_hat);
Q_a=1-normcdf(a,0,1);
Q_b=1-normcdf(b,0,1);
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```
c=X_0*Q_b-exp(-alpha)*K.*Q_a;
```

The price for $K = \mathbb{E}[X(t)]$ is $c = 0.2941$. Likewise we obtaine $c = 0.7190$ for $K = 0.8\mathbb{E}[X(t)]$, while $c = 0.1243$ for $K = 1.2\mathbb{E}[X(t)]$. Options with strike prices $K = 1.2\mathbb{E}[X(t)]$ and $K = 0.8\mathbb{E}[X(t)]$ are products that could be more or less appealing to investors depending on their risk preferences. For instance, for $K = 0.8\mathbb{E}[X(t)]$ it is more likely that by the strike time the option will be exercised (meaning that K is below $X(t)$), thus realizing a gain. Of course, because this option bears the least risk it is the most expensive and its corresponding c is the largest.