Solutions to Homework 3 - Probability Review

1 Conditional expectation of discrete random variables. Suppose $p(x, y, z)$, the joint pmf of the random variables (RVs) X, Y, and Z, is given by

$$p(1,1,1) = \frac{1}{8},\ p(1,1,2) = \frac{1}{8},\ p(1,2,1) = \frac{1}{16},\ p(1,2,2) = 0,$$
$$p(2,1,1) = \frac{1}{4},\ p(2,1,2) = \frac{3}{16},\ p(2,2,1) = 0,\ p(2,2,2) = \frac{1}{4}.$$

The conditional expectation of X given that $Y = 2$ is defined as

$$\mathbb{E}\left[X|Y=2\right] = \sum_{x=1}^{2} xP[X=x|Y=2]$$

and so we first need to compute the conditional pmf $P[X = x|Y = 2]$. From the definition of conditional probability we find

$$P[X=1|Y=2] = \frac{P[X=1,Y=2]}{P[Y=2]} = \frac{\sum_z P[X=1,Y=2,Z=z]}{\sum_{x,z} P[X=x,Y=2,Z=z]} = \frac{1/16}{1/16+1/4} = \frac{1}{5},$$
$$P[X=2|Y=2] = \frac{P[X=2,Y=2]}{P[Y=2]} = \frac{\sum_z P[X=2,Y=2,Z=z]}{\sum_{x,z} P[X=x,Y=2,Z=z]} = \frac{1/4}{1/16+1/4} = \frac{4}{5}.$$

So the final result is

$$\mathbb{E}\left[X|Y=2\right] = 1 \times \frac{1}{5} + 2 \times \frac{4}{5} = \frac{9}{5}.$$

A similar approach can be followed to obtain the conditional expectation of X given that $Y = 2$ and $Z = 1$. The relevant conditional pmf $P[X = x|Y = 2, Z = 1]$ is

$$P[X=1|Y=2,Z=1] = \frac{P[X=1,Y=2,Z=1]}{P[Y=2,Z=1]} = \frac{1/16}{\sum_x P[X=x,Y=2,Z=1]} = \frac{1/16}{1/16} = 1,$$
$$P[X=2|Y=2,Z=1] = \frac{P[X=2,Y=2,Z=1]}{P[Y=2,Z=1]} = 0.$$

Therefore

$$\mathbb{E}\left[X|Y=2,Z=1\right] = \sum_{x=1}^{2} xP[X=x|Y=2,Z=1] = 1 \times 1 + 2 \times 0 = 1.$$

2 Conditional expectation of continuous random variables. The joint pdf of X and Y is given by

$$f(x,y) = \frac{e^{-x/y}e^{-y}}{y}, \quad x > 0,\ y > 0.$$

The conditional expectation of X given that $Y = y$ is defined as

$$\mathbb{E}\left[X|Y=y\right] = \int_0^\infty xf(x|y)dx$$

and so we first need to compute the conditional pdf $f(x|y)$. We have from the definition that

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

where the marginal pdf $f(y)$ can be obtain from the joint as

$$f(y) = \int_0^\infty f(x,y)dx = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y}dx = \frac{e^{-y}}{y}\int_0^\infty e^{-x/y}dx = -e^{-y}e^{-x/y}\Big|_{x=0}^{x=\infty} = e^{-y}, \quad y > 0.$$

The resulting conditional pdf is

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{e^{-x/y}}{y}$$

so the conditional expectation is given by

$$\mathbb{E}\left[X|Y=y\right] = \int_0^\infty x f(x|y)dx = \int_0^\infty \frac{xe^{-x/y}}{y}dx.$$

We can resort to the method of integration by parts with $u = x$, $du = dx$, $dv = (e^{-x/y}/y)dx$ and $v = -e^{-x/y}$ to finally obtain

$$\mathbb{E}\left[X|Y=y\right] = \int_0^\infty \frac{xe^{-x/y}}{y}dx = \left. -xe^{-x/y}\right|_{x=0}^{x=\infty} + \int_0^\infty e^{-x/y}dx = \left. -ye^{-x/y}\right|_{x=0}^{x=\infty} = y$$

which is the desired result.

3 Expected time to match the initial outcome. Let $X_\mathbb{N} = X_0, X_1, \ldots, X_n, \ldots$ be an i.i.d. sequence of RVs with pmf

$$p(j) = P[X_n = j], \ j = 1, \ldots, m, \quad \sum_{j=1}^m p(j) = 1.$$

Define

$$N = \min\{n > 0 : X_n = X_0\}.$$

We want to calculate $\mathbb{E}\left[N\right]$, which corresponds to the expected time till a RV X_n, $n \geq 1$ equals the initial draw X_0 for the first time. The idea is to condition on the outcome of the initial draw $X_0 = j$, and use iterated expectations to obtain

$$\mathbb{E}\left[N\right] = \sum_{j=1}^m \mathbb{E}\left[N \,|\, X_0 = j\right] P[X_0 = j] = \sum_{j=1}^m \mathbb{E}\left[N \,|\, X_0 = j\right] p(j)$$

where the last equality follows from the definition of pmf of X_0.

Towards obtaining the conditional expectation $\mathbb{E}\left[N \,|\, X_0 = j\right]$, the key observation is that the conditional distribution of N given that $X_0 = j$ is geometric with parameter $p(j)$. This is because each X_n, $n \geq 1$ can be viewed as an independent trial, and we have a success ($X_n = X_0 = j$) with probability $p(j)$. Since N counts the number of trials till $X_n = j$ for the first time, then N is conditionally geometric. From this one readily obtains $\mathbb{E}\left[N \,|\, X_0 = j\right] = 1/p(j)$, and

$$\mathbb{E}\left[N\right] = \sum_{j=1}^m \mathbb{E}\left[N \,|\, X_0 = j\right] p(j) = \sum_{j=1}^m \frac{1}{p(j)}p(j) = m$$

as we wanted to show.

4 The trapped miner. Let N denote the total number of doors selected before the miner reaches safety. It is clear that $N \sim \text{Geo}(1/3)$, where $p = 1/3$ is the probability of opening the correct door that leads to the exit of the mine (safety). Also, let T_i denote the travel time corresponding to the $i-$th choice, $i \geq 1$. We recall that T_i can take the values 2, 3 and 5 with equal probability. Finally, let X denote the time when the miner reaches safety.

Having defined these RVs it follows that

$$X = \sum_{i=1}^N T_i$$

where both the T_i's and N are random (and not independent).

To calculate $\mathbb{E}\left[N\right]$, recall that for a RV $X \sim \text{Geo}(p)$ then its expected value is $\mathbb{E}\left[X\right] = 1/p$. Since $N \sim \text{Geo}(1/3)$, then $E[N] = 3$. Moving on to $\mathbb{E}\left[T_N\right]$, the key observation is that $i = N$ corresponds to time where the miner picks the first door and reaches safety (for the first time, of course). Then we have that

$$\mathbb{E}\left[T_N\right] = \mathbb{E}\left[T_i|N=i\right] = \mathbb{E}\left[T_i|T_i=2\right] = 2.$$

The idea is to obtain $\mathbb{E}\left[X\right]$ using iterated expectations, after first conditioning on $N = n$. The conditional expectation $\mathbb{E}\left[X \,|\, N = n\right]$ is

$$\mathbb{E}\left[\sum_{i=1}^{N} T_i \Big| N = n\right] = \mathbb{E}\left[\sum_{i=1}^{n} T_i \Big| N = n\right] = \sum_{i=1}^{n} \mathbb{E}\left[T_i | N = n\right]$$

$$= \sum_{i=1}^{n-1} \mathbb{E}\left[T_i | N = n\right] + \mathbb{E}\left[T_n | N = n\right]$$

$$= \sum_{i=1}^{n-1} \mathbb{E}\left[T_1 | T_1 \neq 2\right] + \mathbb{E}\left[T_n | T_n = 2\right] = (n-1)\mathbb{E}\left[T_1 | T_1 \neq 2\right] + 2$$

$$= (n-1)\left[3 \times \frac{1}{2} + 5 \times \frac{1}{2}\right] + 2 = 4n - 2.$$

Note that for the first equality of the first line we used the fact that $N = n$ and then applied the linearity of expectation. In obtaining the third line we used that given $N = n$, for $i < n$ we know that $T_i \neq 2$ and the T_i's are i.i.d. with the following conditional pmf

$$P[T_i = 3 | T_i \neq 2] = 1/2, \quad P[T_i = 5 | T_i \neq 2] = 1/2.$$

Using all the previous results we can easily compute $\mathbb{E}[X]$ by calculating the expectation of the RV $\mathbb{E}\left[X \mid N\right] = 4N - 2$ over the distribution of N. In fact, we have that

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{N} T_i\right] = \sum_{n=1}^{\infty} \mathbb{E}\left[\sum_{i=1}^{N} T_i | N = n\right] P[N = n]$$

$$= \sum_{n=1}^{\infty} (4n - 2)P[N = n] = 4 \sum_{n=1}^{\infty} nP[N = n] - 2 \sum_{n=1}^{\infty} P[N = n]$$

$$= 4\mathbb{E}[N] - 2 = 4 \times 3 - 2 = 10.$$

5 Limit of a sequence of random variables. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of RVs, each with mean μ and variance σ^2. Consider the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (X_{2i+1} + X_{2i})^2.$$

Because $X_{\mathbb{N}}$ is an i.i.d. sequence of RVs, then observe that $Y_{\mathbb{N}} = (X_3 + X_2)^2, (X_5 + X_4)^2, \ldots, (X_{2n+1} + X_{2n})^2, \ldots$ is also i.i.d.. By the strong law of large numbers the limit exists and is equal to

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (X_{2i+1} + X_{2i})^2 = \mathbb{E}\left[(X_3 + X_2)^2\right], \quad \text{w.p. 1.}$$

To simplify the expression, expand the quadratic term and use linearity of expectation

$$\mathbb{E}\left[(X_3 + X_2)^2\right] = \mathbb{E}\left[X_3^2 + 2X_3 X_2 + X_2^2\right] = \mathbb{E}\left[X_3^2\right] + 2\mathbb{E}\left[X_3 X_2\right] + \mathbb{E}\left[X_2^2\right].$$

Because X_2 and X_3 are independent, then $2\mathbb{E}\left[X_3 X_2\right] = 2\mathbb{E}\left[X_3\right]\mathbb{E}\left[X_2\right] = 2\mu^2$. The second moments can be written in terms of the variance and squared means as $\mathbb{E}\left[X_3^2\right] = \mathbb{E}\left[X_2^2\right] = \sigma^2 + \mu^2$. All in all, the result is

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (X_{2i+1} + X_{2i})^2 = 2(\sigma^2 + \mu^2) + 2\mu^2 = 2\sigma^2 + 4\mu^2.$$

6 The black box. Suppose we have a black box that generates realizations from distribution function $F(x)$ that has mean μ and variance σ^2. One "experiment" consists of drawing a long sequence of i.i.d. realizations from the box. We will perform the experiment many times. In the $j-$th experiment, denote the $i-$th i.i.d. draw from the box by $X_{i,j}$. Finally we will suppose that m_1, m_2, n_1, n_2 are very large numbers with $m_2 \gg m_1$ and $n_2 \gg n_1$.

A First we define

$$Y_{j,k} = \frac{1}{k} \sum_{i=1}^{k} X_{i,j}$$

and want to argue whether Y_{j,m_1} and Y_{j,m_2} will be approximately equal. This quantities are sample averages of a sequence of i.i.d. RVs $X_{i,j}$, and by virtue of the strong law of large numbers (SLLN) we know that

$$\lim_{k\to\infty} Y_{j,k} = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} X_{i,j} = \mathbb{E}\left[X_{1,1}\right] = \mu, \quad \text{w.p. 1.}$$

So due to definition of limit and for very large m_1, m_2 we have that

$$Y_{j,m_1} \approx Y_{j,m_2} \approx \lim_{k\to\infty} Y_{j,k} = \mu$$

hence we expect them to be similar and equal to μ.

B   Now we define

$$Q_{j,k} = \frac{kY_{j,k} - k\mu}{\sqrt{k\sigma^2}}$$

and will argue whether Q_{j,m_1} and Q_{j,m_2} will be approximately equal. First we recall that $\mathbb{E}\left[Y_{j,k}\right] = \mu$ and $\text{var}\left[Y_{j,k}\right] = \sigma^2/k$. Accordingly, $\mathbb{E}\left[Q_{j,k}\right] = 0$ and $\text{var}\left[Q_{j,k}\right] = 1$. So by virtue of the central limit theorem (CLT) we know that the random sequence $Q_{j,k}$ will converge in distribution to a standard normal, i.e., for very large k then $Q_{j,k} \sim \mathcal{N}(0,1)$. This allows us to conclude that Q_{j,m_1} and Q_{j,m_2} will correspond to independent samples drawn from a distribution which is very close to the standard normal, and so there is no reason for them to be approximately equal.

C   Define the following sample average of indicator RVs

$$U_{j,k}(x) = \frac{1}{k}\sum_{i=1}^{k} \mathbb{I}\{X_{i,j} \leq x\}, \quad x \in \mathbb{R}.$$

Because the $X_{i,j}$ are i.i.d., then for a fixed $x \in \mathbb{R}$ the corresponding indicator RVs will also be i.i.d. The SLLN in this case states that for every $x \in \mathbb{R}$

$$\lim_{k\to\infty} U_{j,k}(x) = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \mathbb{I}\{X_{i,j} \leq x\} = \mathbb{E}\left[\mathbb{I}\{X_{1,1} \leq x\}\right] = P[X_{1,1} \leq x] = F(x), \quad \text{w.p. 1.}$$

So again $U_{j,k}(x)$ converges almost surely as $k \to \infty$, and thus for very large m_1, m_2 we have that

$$U_{j,m_1}(x) \approx U_{j,m_2}(x) \approx \lim_{k\to\infty} U_{j,k}(x) = F(x), \quad x \in \mathbb{R}$$

and we expect them to be similar and equal to $F(x)$.

D   Finally we define

$$V_{l,k}(x) = \frac{1}{l}\sum_{j=1}^{l} \mathbb{I}\{Q_{j,k} \leq x\}, \quad x \in \mathbb{R}$$

and want to compare the quantities $V_{m_1,n_1}(x)$ and $V_{m_1,n_2}(x)$. First we note that due to our conclusions in Part B, for very large m_1 and due to the CLT, the random variables Q_{j,m_1} will be approximately standard normal distributed. Also for different values of j, they will be independent. So having realized this fact, then this problem reduces to the one in Part C, for $X_{i,j} \sim \mathcal{N}(0,1)$ instead of $X_{i,j} \sim F$.

From the SLLN we expect convergence of the sequences and thus for very large m_1, n_1, n_2 we have that

$$V_{m_1,n_1}(x) \approx V_{m_1,n_2}(x) \approx \lim_{l\to\infty} V_{m_1,l}(x) = \Phi(x), \quad x \in \mathbb{R}$$

and we expect them to be similar and equal to the standard normal cdf $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-u^2} du$.

7   Decision making.

A   *Simulate individual experiment.* Next, you can find three versions of the required Matlab function to simulate an individual experiment, each one revealing a different programming technique.

Version 1:

```
Version 1, using a while loop

function [accepted_rank,time_of_acceptance]= car_sale_v1(J,K,L)

offers=randperm(J);
rejected_offers=offers(1:K);
sorted_rejected_offers=sort(rejected_offers);
Lth_best_amongst_rejected=sorted_rejected_offers(1,L);

i=K+1;
while (offers(i)>Lth_best_amongst_rejected && i<J)
    i=i+1;
end % weird while!

accepted_rank=offers(i);
time_of_acceptance=i;

end
```

Version 2:

```
Version 2, using a for loop

function [accepted_rank,time_of_acceptance]= car_sale_v2(J,K,L)

offers=randperm(J);
rejected_offers=offers(1:K);
sorted_rejected_offers=sort(rejected_offers);
Lth_best_amongst_rejected=sorted_rejected_offers(1,L);

for i=K+1:J
    if (offers(i)<Lth_best_amongst_rejected)
        accepted_rank=offers(i);
        time_of_acceptance=i;
        return;
    end
end
accepted_rank=offers(J);
time_of_acceptance=J;

end
```

Version 3:

```
Version 3, using "find" and no loop

function [accepted_rank,time_of_acceptance]= car_sale_v3(J,K,L)

offers=randperm(J);
rejected_offers=offers(1:K);
sorted_rejected_offers=sort(rejected_offers);
Lth_best_amongst_rejected=sorted_rejected_offers(1,L);
```

```
i=find(offers(K+1:end)<Lth_best_amongst_rejected,1);

if isempty(i)
    time_of_acceptance=K+i;
    accepted_rank=offers(time_of_acceptance);
else
    time_of_acceptance=J;
    accepted_rank=offers(J);
end

end
```

B   Probability distribution of the rank of the selected offer. Here, we will estimate the pmf for the rank of the selected offer by running the simulations multiple times. To run the simulations, we fix $J = 50$, $K = 30$ and vary $L = 1, 2, 5$. Provided are two different versions, one that calculates relative frequencies directly as estimates of the probabilities, and the second one that uses Matlab's histogram functionality. Note specifically how one calls the function developed for Part *A*.

Version 1:

```
% plots the pmf of the accepted ranks
% J,K,L: refer to HW3 for explanation
% N: number of repetitions of the experiment
% Version 1: counting the frequency of each rank

function []=pmf_of_ranks_v1(J,K,L,N)

frequencies=zeros(1,J); % initialization of the vector of frequencies of
                        % each rank

for i=1:N
    [accepted_rank,ignore_the_time]=car_sale_v1(J,K,L);
    frequencies(1,accepted_rank)=frequencies(1,accepted_rank)+1;
end

pmf_vector=frequencies/N;

bar(1:J,pmf_vector,'r')
xlabel('X','FontSize',14)
ylabel('pmf','FontSize',14)
title(['N=',num2str(N)],'FontSize',14,'FontWeight','b')
axis([0,51,0,0.4])

end
```

Version 2:

```
% plots the pmf of the accepted ranks
% J,K,L: refer to HW3 for explanation
% N: number of repetitions of the experiment
% Version 2: using "hist" to do the task of counting

function []=pmf_of_ranks_v2(J,K,L,N)
```

```
accepted_ranks=zeros(1,N); % initialization of the vector of the ranks of
                           % the selected offers of the experiments!

for i=1:N
    [accepted_ranks(i),ignore_the_time]=car_sale_v1(J,K,L);
end

[frequencies, bins_locations]=hist(accepted_ranks,J);

pmf_vector=frequencies/N;

bar(1:J,pmf_vector,'r')
xlabel('X','FontSize',14)
ylabel('pmf','FontSize',14)
title(['N=',num2str(N)],'FontSize',14,'FontWeight','b')
axis([0,51,0,0.4])

end
```

The function is called in a separate .*m*-file:

```
close all
clear all
clc

J=50; K=30; L=1; % refer to HW3 for explanation
% number of repetitions of the experiment
i=1;
for N=10.^[2 3 4 5]
    figure(1)
    subplot(2,2,i)
    pmf_of_ranks_v1(J,K,L,N)
    i=i+1;
end
```

The plots for $L = 1$ are depicted in Fig. 1. As you will see when plotting $L = 2, 5$, the first L ranks are approximately equiprobable and the behavior for ranks $L + 1$ to $J = 50$ is similar. When $N = 100$, the pmf does not have a smooth shape, thus the number of experiments is increased from 10^2 to 10^5 with order-of-magnitude steps. It is apparent from Fig. 1 that the difference from $N = 10^4$ to $N = 10^5$ is insignificant, so we will use $N = 10^4$ in Part C.

As a general remark, observe that the proposed policy for selling the car performs quite favorably, since the likelihood of accepting the offer from the best bidder is quite high (more than 0.3). Moreover, the likelihood of accepting one of the top three offers is more than 0.4, and $P(X = j)$, $j > 4$ is less than 0.015.

C Probability of selecting the best offer ($L = 1$). From Part B we observed that $N = 10^4$ yields acceptable results for calculating the required probabilities. Here we will implement a function to determine the probability of selecting the best offer, as a function of the number of rejected offers K, which is varied between 1 and $J - 1$. As before, different implementations are provided.

Version 1:

```
% plots the probability of the best rank versus K for a given J, L
% J,K,L: refer to HW3 for explanation
% Version 1: adaptation of version 1 of Part B (counting frequency)
```
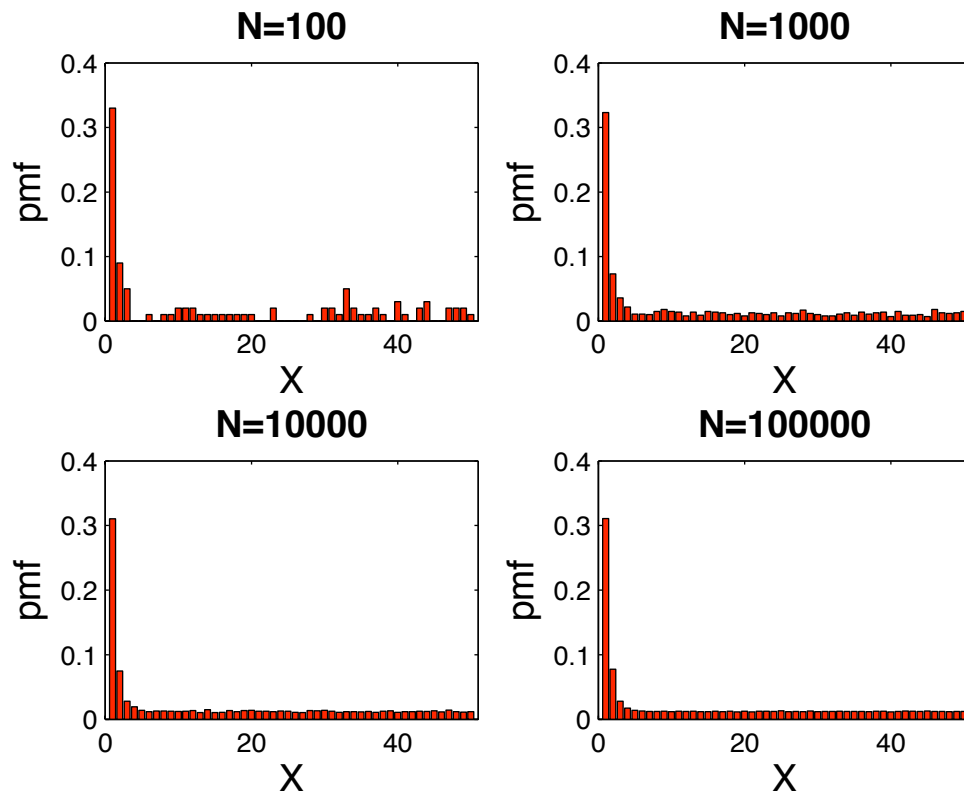
Fig. 1. Estimated pmf of accepted ranks for $J = 50$, $K = 30$, $L = 1$, and varying $N = 10^2, \ldots, 10^5$. (Part *B*)

```
function []=prob_1_versus_K_v1(J,L)

N=10000;

K_vector=L:J-1;
prob_of_rank_1=zeros(1,J-L);

K_index=1;
for K=K_vector
    frequency_of_rank_1=0; % initialization of the number of time accepted
                     % rank has been 1, i.e., the best offer was accepted
    for i=1:N
        [accepted_rank,ignore_the_time]=car_sale_v1(J,K,L);
        frequency_of_rank_1=frequency_of_rank_1 + (accepted_rank==1); %counter
    end
    prob_of_rank_1(1,K_index)=frequency_of_rank_1/N;
    K_index=K_index+1;
end

bar(K_vector,prob_of_rank_1) %create figure
    xlabel('K','FontSize',14)
    ylabel('P(X)=1','FontSize',14)
    title(['P(X)=1 for different K, J=',num2str(J),...
        ', L=', num2str(L),', N=',num2str(N)],'FontSize',...
```

```
        14,'FontWeight','b')
    axis([0,J,0,0.4])
end
```

Version 2:

```
% plots the probability of the best rank versus K for a given J, L
% J,K,L: refer to HW3 for explanation
% Version 2: adaptation of version 2 of Part B (using hist() function)

function []=prob_1_versus_K_v2(J,L)

N=10000;

K_vector=L:J-1;
prob_of_rank_1=zeros(1,J-L);

K_index=1;
for K=K_vector
    accepted_ranks=zeros(1,N); % initialization of the vector of the ranks of
                              % the selected offers of the experiments!
    for i=1:N
        [accepted_ranks(i),ignore_the_time]=car_sale_v1(J,K,L);
    end
    [frequencies, bins_locations]=hist(accepted_ranks,J);
    pmf_vector=frequencies/N;
    prob_of_rank_1(1,K_index)=pmf_vector(1,1);
    K_index=K_index+1;
end

bar(K_vector,prob_of_rank_1) %create figure
    xlabel('K','FontSize',14)
    ylabel('P(X)=1','FontSize',14)
    title(['P(X)=1 for different K, J=',num2str(J),...
        ', L=', num2str(L),', N=',num2str(N)],'FontSize',...
        14,'FontWeight','b')
    axis([0,J,0,0.4])
end
```

The script to execute the simulations follows:

```
close all
clear all
clc

J=50; K=30; % refer to HW3 for explanation
% number of repetitions of the experiment

for L=[1 2 5]
    figure
    prob_1_versus_K_v2(J,L)
end
```
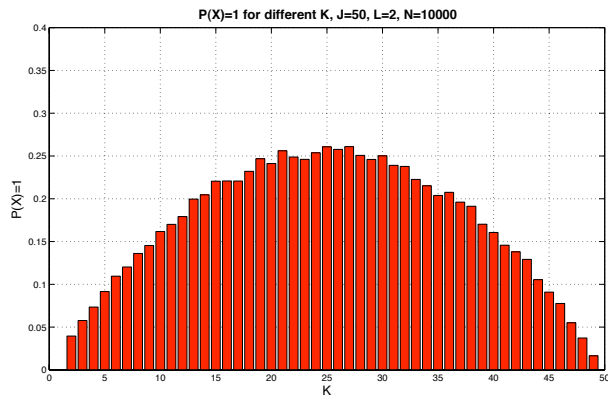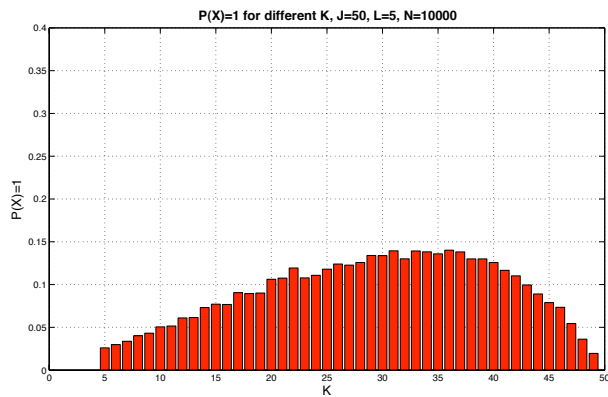
The results are depicted in Fig. 2. First, we observe that by selecting $L = 1$ and $K \approx 18$ we achieve a likelihood of accepting the best offer as high as 0.37. Also, for a wide range of $10 < K < 30$ the chance of accepting the best offer is at least one-third. Increasing L from 1 hurts the probability of accepting the best offer. For larger L, K better be appropriately larger too.
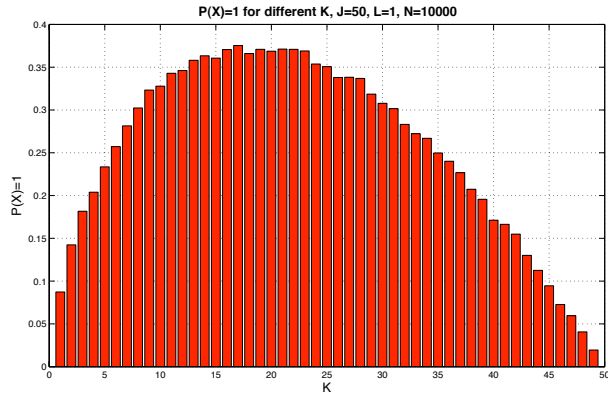
(a) $L = 1$

(b) $L = 2$

(c) $L = 5$

Fig. 2. Probability of accepting the best offer for $J = 50$ as a function of K, varying $L = 1, 2, 5$. Here, $N = 10^4$ is the number of repeated experiments performed to calculate the desired probability. Notice that the scale of the y-axis ranges from 0 to 0.4.

D Probability of selecting last offer (L=1). One accepts the last offer if the best offer is among the first K offers we discard ($X_0 = 1$), or, the second best offer is among the first K offers we discard and the best offer is the last offer (hence $X_0 = 2$ and $X_J = 1$). In terms of events this can be formally expressed as

$$\mathrm{P}\left[X = X_J\right] = \mathrm{P}\left[\{X_0 = 1\} \cup \{\{X_0 = 2\} \cap \{X_J = 1\}\}\right]$$
$$= \mathrm{P}\left[\{X_0 = 1\}\right] + \mathrm{P}\left[\{X_0 = 2\} \cap \{X_J = 1\}\right]$$

where the last equality follows readily since the events $\{X_0 = 1\}$ and $\{\{X_0 = 2\} \cap \{X_J = 1\}\}$ are disjoint. The first probability can be calculated by noticing that $\{X_0 = 1\}$ corresponds to the first offer being amongst the first K slots, out of J possible equiprobable slots. This implies

$$\mathrm{P}\left[\{X_0 = 1\}\right] = \frac{K}{J}.$$

To calculate the second relevant probability, notice that $\{\{X_0 = 2\} \cap \{X_J = 1\}\}$ is equivalent to the best offer being at the last slot, and, (i.e., fixing that) the second best offer being amongst the first K of the remaining $J - 1$ slots. Hence, from the definition of conditional probability

$$\mathrm{P}\left[\{X_0 = 2\} \cap \{X_J = 1\}\right] = \mathrm{P}\left[X_0 = 2, X_J = 1\right]$$
$$= \mathrm{P}\left[X_0 = 2 \mid X_J = 1\right] \mathrm{P}\left[X_J = 1\right]$$
$$= \frac{K}{J - 1} \times \frac{1}{J}.$$

Summing up both results yields the desired probability of selecting the last offer when $L = 1$, namely

$$\mathrm{P}\left[X = X_J\right] = \frac{K}{J} + \frac{1}{J} \times \frac{K}{J - 1}.$$

E Probability of selecting best offer ($L = 1$). To calculate $\mathrm{P}\left[X = 1\right]$ we will condition on $X_n = 1$, that is the probability that the n-th being the best possible offer (something that happens with probability $\mathrm{P}\left[X_n = 1\right] = 1/J$). From the law of total probability we have

$$\mathrm{P}\left[X = 1\right] = \sum_{n=1}^{J} \mathrm{P}\left[X = 1 | X_n = 1\right] \mathrm{P}\left[X_n = 1\right] = \sum_{n=1}^{J} \mathrm{P}\left[X = 1 | X_n = 1\right] \frac{1}{J}$$

and what is left is the calculation of the conditional probabilities $\mathrm{P}\left[X = 1 | X_n = 1\right]$, $1 \leq n \leq J$. To this end, notice that $\mathrm{P}\left[X = 1 | X_n = 1\right] = 0$ for $1 \leq n \leq K$ because if the first offer is among the first K it would be discarded. We can then adjust the index of summation and start from $n = K + 1$, that is

$$\mathrm{P}\left[X = 1\right] = \frac{1}{J} \sum_{n=K+1}^{J} \mathrm{P}\left[X = 1 | X_n = 1\right].$$

To calculate $\mathrm{P}\left[X = 1 | X_n = 1\right]$ for $K + 1 \leq n \leq J$, notice that the best offer will be chosen at time n only if X_0 is not beaten by those offers made during times $K + 1, \dots, n - 1$. In other words, the best offer among those made between times $1, \dots n - 1$ must have occured during the first K slots, something that happens with probability $K/(n - 1)$. All in all, $\mathrm{P}\left[X = 1 | X_n = 1\right] = K/(n - 1)$, $K + 1 \leq n \leq J$ and the desired result follows

$$\mathrm{P}\left[X = 1\right] = \frac{K}{J} \sum_{n=K+1}^{J} \frac{1}{n - 1}.$$

F Optimal number of rejected offers K (L=1). The optimal K^* can be determined by maximizing $\mathrm{P}\left[X = 1\right]$, after resorting to the suggested approximation

$$\sum_{n=K+1}^{J} \frac{1}{n - 1} \approx \int_{K}^{J-1} \frac{1}{x} \, dx = \ln\left(\frac{J - 1}{K}\right) \approx \ln\left(\frac{J}{K}\right).$$

The last approximation follows by considering $J \gg 1$, which is acceptable for $J = 50$ in our experiments. The approximate expression for the probability of selecting the best offer is

$$\mathrm{P}[X = 1] = \frac{K}{J} \sum_{n=K+1}^{J} \frac{1}{n-1} \approx \frac{K}{J} \ln\left(\frac{J}{K}\right).$$

To find the maximizing K^*, we null the derivative of $\mathrm{P}[X = 1]$ with respect to K, and solve for K^* (this actually neglects that K is integer-valued, but we will live with that here). Using the product rule for the derivatives as well as the chain rule, we find

$$\frac{d}{dK}\mathrm{P}[X = 1] \approx \frac{d}{dK}\left[\frac{K}{J} \ln\left(\frac{J}{K}\right)\right]$$

$$= \frac{1}{J} \times \ln\left(\frac{J}{K}\right) - \frac{K}{J} \times \left(\frac{K}{J} \times \frac{J}{K^2}\right)$$

$$= \frac{1}{J} \times \ln\left(\frac{J}{K}\right) - \frac{1}{J}.$$

Finding the root yields the desired result,

$$\frac{d}{dK}\mathrm{P}[X = 1] = 0 \Leftrightarrow \ln\left(\frac{J}{K^*}\right) = 1 \Rightarrow K^* \approx J/e.$$

For $J = 50$, this number is about 18 corroborating what we observed in Fig. 2(a). Now, the maximum probability is approximately

$$\mathrm{P}^*[X = 1] \approx \frac{K^*}{J} \ln\left(\frac{J}{K^*}\right) = \frac{J/e}{J} \times \ln \frac{J}{J/e} = 1/e \approx 0.37$$

This is a remarkably high probability, which matches the simulated results in Fig. 2(a) as well.