Solutions to Homework 7 - Continuous-Time Markov Chains

1) Machine repair times. The time T required to repair a machine is exponentially distributed with mean $1/2$ (hours). From this information we can conclude that $T \sim \exp(2)$. The probability that the repair time exceeds $1/2$ hour is given by the formula for the ccdf (tail) of the exponential distribution, namely

$$\mathrm{P}\left[T > 1/2\right] = e^{-2\times\frac{1}{2}} = e^{-1} = 0.3679.$$

The probability that a repair takes at least 12.5 hours given that its duration exceeds 12 hours is readily obtained from the memoryless property of the exponential distribution. In fact,

$$\mathrm{P}\left[T > 12.5 \,\middle|\, T > 12\right] = \mathrm{P}\left[T > 1/2\right] = e^{-1} = 0.3679.$$

2) The post office. We consider a post office which is run by two clerks. When Mr. Smith enters the system he discovers that Mr. Jones is being served by one of the clerks and Mr. Brown is being served by the other. Mr. Smith's service will begin as soon as either Jones or Brown leaves, and we also assume that the amount of time that clerk $i = 1, 2$ spends with a customer is exponentially distributed with rate λ, and that the service times are independent of each other.

In this exercise we want to compute the probability that Mr. Smith is not the last customer to leave the office. On first terms this can be formulated as

$$\begin{aligned}\mathrm{P}\left[\{\text{Smith is not last}\}\right] &= \mathrm{P}\left[\{\text{Jones finishes after Smith}\}\right] + \mathrm{P}\left[\{\text{Brown finishes after Smith}\}\right]\\ &= \mathrm{P}\left[\{\text{Jones finishes after Brown}\}\right]\mathrm{P}\left[\{\text{ Jones finishes after Smith} \,\middle|\, \text{Brown is gone}\}\right]\\ &\quad + \mathrm{P}\left[\{\text{Brown finishes after Jones}\}\right]\mathrm{P}\left[\{\text{Brown finishes after Smith} \,\middle|\, \text{Jones is gone}\}\right].\end{aligned}$$

To determine the above probabilities, introduce the following RVs

T_1 : Service time of Jones,
T_2 : Service time of Brown,
T_3 : Service time of Smith given that Jones is gone,
T_4 : Remaining service time of Brown given that Jones is gone,
T_5 : Service time of Smith given that Brown is gone,
T_6 : Remaining service time of Jones given that Brown is gone.

It is thus possible to reformulate the desired probability as

$$\mathrm{P}\left[\{\text{Smith is not last}\}\right] = \mathrm{P}\left[T_2 < T_1\right]\mathrm{P}\left[T_5 < T_6\right] + \mathrm{P}\left[T_1 < T_2\right]\mathrm{P}\left[T_3 < T_4\right].$$

From the problem description it follows that $T_1 \sim \exp(\lambda_1)$ and $T_2 \sim \exp(\lambda_2)$. Likewise, $T_3 \sim \exp(\lambda_1)$ and $T_5 \sim \exp(\lambda_2)$. The key observation is that by the memoryless property of the exponential distribution we can also conclude that $T_4 \sim \exp(\lambda_2)$ and $T_6 \sim \exp(\lambda_1)$. Also, in general for $X \sim \exp(\lambda_1)$ and $Y \sim \exp(\lambda_2)$ then $\mathrm{P}\left[X < Y\right] = \frac{\lambda_1}{\lambda_1+\lambda_2}$. Using this result in our previous setup yields

$$P[\{\text{Smith is not last}\}] = \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2 + \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2$$

as desired.

3) Conditional expectation of exponential random variables. Suppose T is an exponentially-distributed RV with parameter λ. The goal is to calculate $\mathbb{E}\left[T \,\middle|\, T < t\right]$, $t > 0$, by relying on the identity

$$\mathbb{E}\left[T\right] = \mathbb{E}\left[T \,\middle|\, T < t\right]\mathrm{P}\left[T < t\right] + \mathbb{E}\left[T \,\middle|\, T > t\right]\mathrm{P}\left[T > t\right]. \tag{1}$$

Because T is exponentially distributed with rate λ, we first note that $\mathbb{E}[T] = 1/\lambda$, while $\mathrm{P}[X < t] = 1 - e^{-\lambda t}$ and $\mathrm{P}[X > t] = e^{-\lambda t}$. Furthermore, because of the memoryless property of the exponential distribution, then

$$\mathbb{E}\left[T \,\middle|\, T > t\right] = t + \mathbb{E}[T] = t + \frac{1}{\lambda}.$$

Substitution of all these intermediate results in (1) yields

$$\frac{1}{\lambda} = \mathbb{E}\left[T \,\middle|\, T < t\right]\left(1 - e^{-\lambda t}\right) + \left(t + \frac{1}{\lambda}\right) e^{-\lambda t} \quad \Rightarrow \quad \mathbb{E}\left[T \,\middle|\, T < t\right] = \frac{1}{\lambda} - \frac{t}{e^{\lambda t} - 1}.$$

4) Sum of i.i.d. exponential random variables. Suppose that $T_1, \ldots, T_n$ are i.i.d. exponential RVs with parameter λ. In this problem, we will show that the sum $T = \sum_{i=1}^{n} T_i$ has the gamma distribution with parameters n and λ; i.e. T has pdf

$$f_T(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t \geq 0. \tag{2}$$

To that end, we will argue by mathematical induction.

A) Base case: The claim holds true for $n = 1$, since (2) simplifies to

$$f_T(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$

which is the density of an exponential RV with rate λ.

B) Inductive step: Suppose the claim is true for $n-1$, i.e., $T^{(n-1)} := T_1 + \ldots + T_{n-1}$ has the gamma distribution with parameters $n-1$ and λ. Then we need to show it also holds for n. To that end, notice that $T = T^{(n-1)} + T_n$, where $T^{(n-1)}$ and T_n are independent. Hence, the pdf of T is given by the convolution

$$\begin{aligned} f_T(t) &= \int_0^t f_{T^{(n-1)}}(x) f_{T_n}(t-x)dx \\ &= \int_0^t \lambda e^{-\lambda x} \frac{(\lambda x)^{n-2}}{(n-2)!} \lambda e^{-\lambda(t-x)} dx \\ &= \lambda e^{-\lambda t} \frac{\lambda^{n-1}}{(n-2)!} \int_0^t x^{n-2} dx \\ &= \lambda e^{-\lambda t} \frac{\lambda^{n-1}}{(n-2)!} \times \frac{t^{n-1}}{n-1} = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t \geq 0 \end{aligned}$$

which is identical to (2), completing the proof.

5) Sum of a random number of exponential random variables. Suppose $T_{\mathbb{N}} = T_1, T_2, \ldots$ is an i.i.d. sequence of exponential RVs, each with parameter λ. Suppose N is a geometric RV with parameter p. Assume that N and $T_{\mathbb{N}}$ are independent, and let

$$X = \sum_{n=1}^{N} T_n.$$

Upon conditioning on $N = n$, we can write

$$\begin{aligned} \mathrm{P}[X < t] &= \sum_{n=1}^{\infty} \mathrm{P}\left[\sum_{i=1}^{N} T_i < t \,\middle|\, N = n\right] \mathrm{P}[N = n] \\ &= \sum_{n=1}^{\infty} \mathrm{P}\left[\sum_{i=1}^{n} T_i < t\right] \mathrm{P}[N = n] \end{aligned}$$

where the last equality follows from the independence of N and $T_{\mathbb{N}}$. Next, notice that $\sum_{i=1}^{n} T_i$ has the gamma distribution with parameters n and λ, and substitute the pmf expression $\mathrm{P}[N=n]=p(1-p)^{n-1}$ to obtain

$$\begin{aligned}\mathrm{P}[X<t] &= \sum_{n=1}^{\infty}\mathrm{P}\left[\sum_{i=1}^{n}T_i<t\right]\mathrm{P}[N=n]\\ &= \sum_{n=1}^{\infty}p(1-p)^{n-1}\int_0^t \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}dt\\ &= \int_0^t \lambda p e^{-\lambda t}\left[\sum_{n=1}^{\infty}\frac{[(1-p)\lambda t]^{n-1}}{(n-1)!}\right]dt\\ &= \int_0^t \lambda p e^{-\lambda t}e^{(1-p)\lambda t}dt = \int_0^t \lambda p e^{-\lambda p t}dt.\end{aligned}$$

The conclusion is that X is exponentially distributed with parameter λp. In obtaining the fourth equality above, we used the Taylor series expansion of the exponential function $e^{(1-p)\lambda t}$.

6) Poisson process. Let $N(t)$ be a Poisson process with rate λ, and let S_i denote the i-th arrival time.

To determine $\mathbb{E}[S_4]$, recall that the fourth arrival time equals the sum of the first four interarrival times $T_1,\ldots,T_4$. Hence, from the linearity of expectation

$$\mathbb{E}[S_4]=\mathbb{E}\left[\sum_{i=1}^{4}T_i\right]=\sum_{i=1}^{4}\mathbb{E}[T_i]=\frac{4}{\lambda}$$

where in obtaining the last equality we used that interarrival times are i.i.d. exponential RVs with mean $1/\lambda$. Next, since interarrival times are memoryless, we can write

$$\mathbb{E}\left[S_4 \,\middle|\, N(1)=2\right]=\mathbb{E}\left[S_4 \,\middle|\, S_2<1, S_3>1\right]=1+\mathbb{E}[T_3+T_4]=1+\frac{2}{\lambda}.$$

Finally, from the independent increments property of the Poisson process, it holds that $\mathbb{E}\left[N(4)-N(2)\,\middle|\,N(1)=3\right]=\mathbb{E}[N(4)-N(2)]$. Since increments are also stationary, the counts $N(4)-N(2)$ have the same distribution as $N(2)$, namely Poisson with parameter 2λ. All in all, the desired conditional expectation is

$$\mathbb{E}\left[N(4)-N(2)\,\middle|\,N(1)=3\right]=2\lambda.$$

7) Arrivals to a subway station.

A) See below a Matlab script to generate the (Poisson) arrival process. The parameters chosen for the simulation are $T=10$ minutes, $\lambda=1$ customer per minute, and $n=10^3$ subintervals.

```
clc; clear all; close all
T=10;       %minutes
lambda= 1; %customers per minute;
nr_experiments=10^4;
n=1000;

h=T/n;
p = lambda*h;

% Generate arrivals for all times and experiments
arrival = binornd(1,p,n,nr_experiments);

% Compare with Poisson pmfs
x=0:30;
pdf_approx = hist(sum(arrival),x)/nr_experiments;
bar(x,pdf_approx,'r')
hold on
```

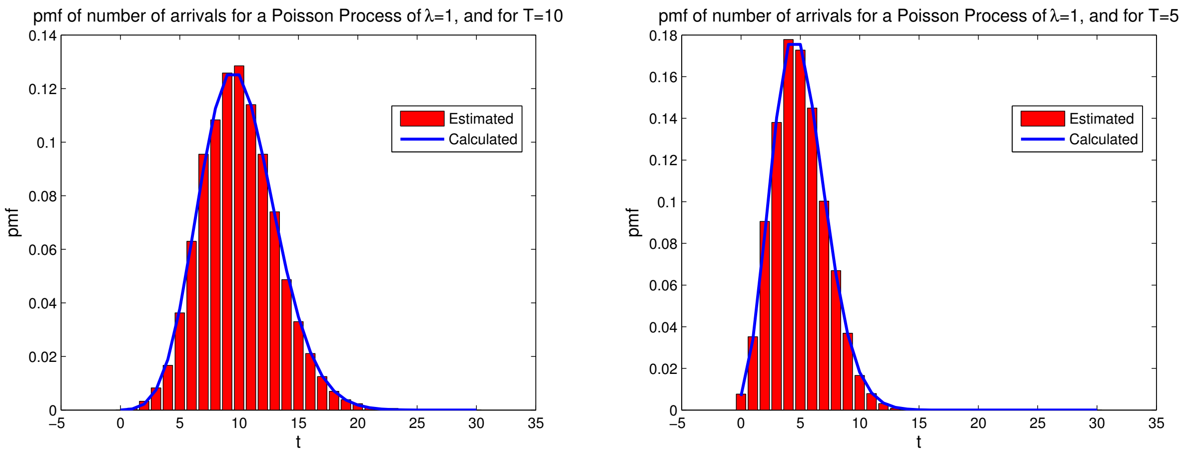

Fig. 1. Comparison between the histogram of $N(T)$ obtained from 10^4 experiments with the Poisson pmf with parameter λT for $T = 10$ minutes, $\lambda = 1$ customer per minute, and $n = 10^3$ subintervals (left). Comparison of the histogram of $N(T)/2$ with the Poisson pmf with parameter $\lambda T/2$ (right). (Part *A*)

```
plot(x,poisspdf(x,lambda*T),'b','Linewidth', 2)
xlabel('t','Fontsize',12)
ylabel('pmf','Fontsize',12)
title('pmf of number of arrivals for a Poisson Process of \lambda=1, ...
        ...and for T=10','Fontsize',12)
legend('Estimated','Calculated','Location','Best')

figure
pdf_approx = hist(sum(arrival(1:n/2,:)),x)/nr_experiments;
bar(x,pdf_approx,'r')
hold on
plot(x,poisspdf(x,lambda*T/2),'b','Linewidth', 2)
xlabel('t','Fontsize',12)
ylabel('pmf','Fontsize',12)
title('pmf of number of arrivals for a Poisson Process of \lambda=1, ...
        ...and for T=5','Fontsize',12)
legend('Estimated','Calculated','Location','Best')
```

The results are shown in Fig. 1, where we first compare the histogram of $N(T)$ obtained from 10^4 experiments with the Poisson pmf with parameter λT; see Fig. 1 (left). We also compare in Fig. 1 (right) the histogram of $N(T)/2$ with the Poisson pmf with parameter $\lambda T/2$. A good fit is obtained as expected.

B) The number of customers arriving to the subway station by time t is given by

$$N(t) = \sum_{i=1}^{t/h} N_i(h) = \sum_{i=1}^{n} N_i(h) \tag{3}$$

where $N_i(h)$ is the number of customers that arrive in the i-th time interval of duration h. For any fixed t, if h is sufficiently small (equivalently the number of subintervals n is large) then the $N_i(h)$ can be approximated as i.i.d. Bernoulli random variables with parameter $p = \lambda h$. Hence, from (3) it follows that $N(t)$ is binomial distributed with parameters $n = t/h$ and $p = \lambda h$. The product of the two parameters of $N(t)$ is

$$n \times p = \left(\frac{t}{h}\right) \times \lambda h = \lambda t$$

so if we let $n \to \infty$, then $p = \lambda h \to 0$ but their product remains constant at λt. Invoking the law of rare events this latter observation implies that $N(t)$ has a Poisson distribution with parameter λt as we wanted to show.

C) The Matlab script used to compute a cumulative histogram of the first arrival time S_1 is shown below. The cumulative histogram naturally offers an estimate of S_1's cdf, which is also compared with the cdf of an exponential random variable with parameter $\lambda = 1$. A good fit is observed as shown in Fig. 2.

```
clc; clear all; close all
T=10;       %minutes
lambda= 1; %customers per minute;
nr_experiments=10^4;
n=1000;

h=T/n;
p = lambda*h;

% Generate arrivals for all times and experiments
arrival = binornd(1,p,n,nr_experiments);

% Compute time of first arrival : Method 1
first_arrival_times=n*ones(1,nr_experiments);
nr_arrived=cumsum(arrival);
for i=1:nr_experiments
    temp=find(nr_arrived(:,i),1);
    if ~isempty(temp)
        first_arrival_times(1,i)=temp;
    end
end
hist_firs_arrival_times=hist(first_arrival_times,1:n);

% Compute time of first arrival : Method 2
time=0;
experiment=1;
time_histogram = zeros(n,1);
while (experiment <= nr_experiments) && (time < n)
    time = time+1;
    if arrival(time, experiment)
        time_histogram(time)=time_histogram(time)+1;
        experiment = experiment+1;
        time=0;
    end
end

%Compare with exponential pdf
figure
plot((1:n)*h,hist_firs_arrival_times/nr_experiments/h,'r')
hold on
plot((1:n)*h,exppdf((1:n)*h,lambda),'b','Linewidth', 2)
xlabel('t','Fontsize',12)
ylabel('pdf','Fontsize',12)
title('pdf of first arrival time for a Poisson Process of \lambda=1, ...
        ...Method 1','Fontsize',12)
legend('Estimated','Calculated','Location','Best')

figure
plot((1:n)*h,time_histogram/nr_experiments/h,'r')
hold on
```

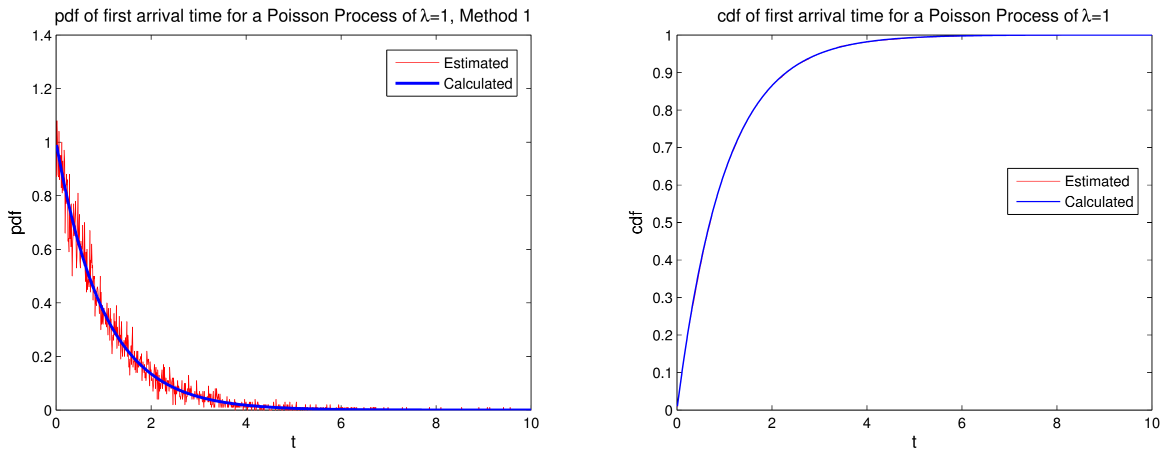

Fig. 2. Comparison between the pdf (left) and cdf (right) of an exponential random variable with parameter $\lambda = 1$, and the corresponding values based on the estimated cumulative histograms. (Part *C*)

```
plot((1:n)*h,exppdf((1:n)*h,lambda),'b','Linewidth', 2)
xlabel('t','Fontsize',12)
ylabel('pdf','Fontsize',12)
title('pdf of first arrival time for a Poisson Process of \lambda=1, ...
        ...Method 2','Fontsize',12)
legend('Estimated','Calculated','Location','Best')

%Compare with exponential cdf
figure
plot((1:n)*h,cumsum(time_histogram/nr_experiments),'r')
hold on
plot((1:n)*h,expcdf((1:n)*h,lambda),'b','Linewidth', 1)
xlabel('t','Fontsize',12)
ylabel('cdf','Fontsize',12)
title('cdf of first arrival time for a Poisson Process of \lambda=1',...
       ...'Fontsize',12)
legend('Estimated','Calculated','Location','Best')
```

D) To establish that the first arrival time is exponentially distributed with parameter λ (thus corroborating the observations from the simulation results), we calculate the ccdf of S_1 and show it is given by $\mathrm{P}[S_1 > t] = e^{-\lambda t}$. To this end, the key observation is that we have $S_1 > t$ if and only if there are no arrivals by time t, meaning $\mathrm{P}[S_1 > t] = \mathrm{P}[N(t) = 0]$. But since $N(t)$ is Poisson distributed with parameter λt (as shown in Part *B*) , then

$$\mathrm{P}[N(t) = k] = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

from where it follows that $\mathrm{P}[S_1 > t] = \mathrm{P}[N(t) = 0] = e^{-\lambda t}$.