Homework 8 - Continuous-time Markov chains

1) A single-server station. Potential customers arrive at a single-server station in accordance to a Poisson process with rate λ. However, if the arrival finds n customers already in the station, then she will enter the system with probability α_n. Assuming an exponential service rate μ, set this up as a birth and death process and determine the birth and death rates.

2) Model and analysis of a barbershop. A small barbershop, operated by a single barber has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables (RVs) with mean $1/4$ hour.

A) What is the average number of customers in the shop?

B) What is the proportion of potential customers that enter the shop?

C) If the barber could work twice as fast, how much more business would he do?

3) Discrete-time and continuous-time queueing models. Discuss the relationship between the M/M/1 queue and the situation described in problem 1 of homework 6. What similarities are there between arrival processes in these two examples? What about similarities in service-time distributions? Compute the stationary distribution of the Markov chain obtained in problem 1 of homework 6 under the assumption that $p < q$. Explain the significance of this assumption.

4) Comparing occupancies of two M/M/1 queues. Consider two separate M/M/1 queues. In the first (called "system 1") the arrival rate is λ_1, and the mean service time is μ^{-1}. In the second (called "system 2') the arrival rate is λ_2, and the mean service time is μ^{-1}. Suppose that $\lambda_1 < \lambda_2 < \mu$. Intuition tells us that, in some sense, there should be fewer customers in system 1 than in system 2. Provide some formal mathematical arguments to back this up.

5) An M/M/1 queue with on/off server switching. Consider the following variation of an M/M/1 queue with arrival rate λ, service rate μ, and $\rho = \lambda/\mu < 1$. The server is turned off whenever the system becomes empty, and the server remains off until there are k (k is a fixed number) customers in the system, at which time the server is turned on. The server then remains on until the system empties, at which time the process repeats. Model the system as a CTMC with states 0 (system empty), (i, B) [system busy, i.e., $i \in (0, k)$ customers present and server on], (i, I) [system idle, i.e., $i \in (0, k)$ customers present and server off], i ($i \geq k$ customers present). Write down the balance equations and verify that the limiting distribution is given by

$$
\begin{aligned}
P_0 &= (1-\rho)/k, \\
P_{i,I} &= (1-\rho)/k, \quad i \in (0,k), \\
P_{i,B} &= \rho(1-\rho^i)/k, \quad i \in (0,k), \\
P_i &= \rho^{i+1-k}(1-\rho^k)/k, \quad i \geq k.
\end{aligned}
$$

6) A hiring decision. The manager of a market can hire Mary or Alice. Mary, who gives service at an exponential rate of 20 customers per hour, can be hired at a rate of \$3 per hour. Alice, who gives service at an exponential rate of 30 customers per hour, can be hired at a rate of \$$C$ per hour. The manager estimates that on average each customer's time is worth \$1 per hour, and should be accounted for in the model. Assume customers arrive at a Poisson rate of 10 per hour. What is the average cost per hour if Mary is hired? If Alice is hired? Find C if the average cost per hour is the same for Mary and Alice.

7) Insurance cash flow. The purpose of this problem is to simulate and analyze the cash flow $X(t)$ of an insurance company whose initial capital is $X(t) = X_0$. For simplicity we will assume that all customers are identical in that they pay the same premiums, submit claims for the same amount of money and have the same risk. A model where customers have different types that determine their premiums, risk and claims, along with a probability distribution of customers across types is conceptually very similar but computationally more involved.

The company has a total of N customers that pay 1 (say thousand dollars) per year to insure an asset of value c (thousand dollars). Each of this customers has an associated risk r meaning that in any given one-year interval they submit a claim with probability r. The times at which premiums are paid, as well as the times at which claims are made are modeled as a Poisson process. This assumption is reasonable as long as N is large and users reinsure after a claim. Of the many definitions of Poisson processes the important one here is that the time T_p between premiums paid is exponentially distributed with parameter $\lambda = N$ premiums paid per year, i.e.,

$$T_p \sim \exp(\lambda) = \exp(N). \tag{1}$$

Likewise, the time between claims is exponentially distributed with parameter $\alpha = rN$ claims per year

$$T_c \sim \exp(\alpha) = \exp(rN). \tag{2}$$

Besides clients, the insurance company has also shareholders that expect dividends of value d to be paid at a rate of β payoffs per year. These responsible shareholders, however, expect this payoffs only if the company has sufficient reserve capital $X_r >> c$. Consequently, the time T_d elapsed between dividend payoffs is modeled as exponential with parameter β payoffs per year as long as the company's capital exceeds X_r, i.e.,

$$T_d \sim \exp(\beta), \qquad X(t) \geq X_r. \tag{3}$$

To complete the model let $X_{\min} = 0$ and $X_{\max}$ be the minimum and maximum capital the company is expected to hold. We assume that if $X(t) = 0$ no dividends or claims are paid, but the company is not bankrupt and may still receive premiums. If $X(t) = X_{\max}$ no more premiums are cashed but the company still pays claims and dividends. We also assume that if $X(t) < c$ and a claim is made then the company's capital transitions to $X(t) = 0$. These assumptions are unreasonable, but the idea is that $X_{\min} = 0$ and $X_{\max}$ are chosen so that the probability of seeing $X(t) = 0$ or $X(t) = X_{\max}$ is negligible.

The different events that may occur when the CTMC's state is $X(t)$ are summarized in the following table

Range reference	Capital range	Possible events
A	$X(t) = 0$	premium
B	$0 < X(t) < c$	premium, claim payed at $X(t)$, not c
C	$c \leq X(t) < X_r$	premium, claim
D	$X_r \leq X(t) < X_{\max}$	premium, claim, dividend
E	$X_{\max} = X(t)$	claim, dividend

The cash flow $X(t)$ of the insurance company is a CTMC. Your goal is to find the probability of the insurance company paying dividends. To get there, you are asked to build a CTMC model of $X(t)$, write code to simulate the $X(t)$ process and solve Kolmogorov's equations numerically to find the probability distribution of $X(t)$.

A) CTMC states. Assume that c and d are integers. What are the states of the CTMC, i.e., possible values x of $X(t)$?

B) Transition times out of given state. Given that the current state of the CTMC is $X(t) = x$, let T_x be the random time until the next transition out of x. If $X_r \leq x < X_{\max}$, i.e., range (D) in the table above, this transition occurs whenever a dividend, claim, or premium is paid. The probability distribution of T_x is exponential. Explain why and give the parameter ν_x. Repeat the question when the state is $X(t) = x$ with x in ranges (A), (B), (C), and (E).

C) Possible states going out of $X(t) = x$. Given that the current state of the CTMC is $X(t) = x$ with $X_r \leq x < X_{\max}$, i.e., range (D) and that a transition out of x occurs at time t. What are the possible states after the transition occurs? Repeat the question for x in ranges (A), (B), (C), and (E).

D) Transition probabilities. Given that the current state of the CTMC is $X(t) = x$ with $X_r \leq x < X_{\max}$, i.e., range (D) and that a transition out of x occurs at time t. What are the transition probabilities associated with the states of part *C*? Explain. Repeat the question for x in ranges (A), (B), (C), and (E).

E) System simulation. Write a function to simulate the cash flow of the insurance company. Inputs to the function are the initial capital X_0; rates, λ, α and β; claim and dividend costs c and d; capital thresholds, X_r, and $X_{\max}$; and a maximum amount of time $T_{\max}$. The output should be a vector of times t at which state transitions occurred and a vector of states $X(t)$ associated with such times. To guide you in writing this code consider the following:

1. Given the current state $X(t)$ you first need to draw the time of the next event. This time is exponential with the parameter you computed in part *B*. Note that the parameter is different for different transition times.
2. Given that a transition occurs at the time drawn, there are a few possible destination states. These states are the ones you found in part *C*.
3. The transitions into the aforementioned states occur with the probabilities of part *D*. Draw the state for the transition.
4. Update the state $X(t)$ according to the transition drawn.

Run your code for $X_0 = 200$; number of clients $N = 200$, risk $r = 4\%$, dividends payed quarterly; claim and dividend costs $c = 20$ and $d = 30$; capital thresholds, $X_r = 200$, and $X_{\max} = 300$; and maximum amount of time $T_{\max} = 5$ years. Plot the evolution of the cash flow $X(t)$ over a period of 5 years.

F) Kolmogorov's forward equation. Kolmogorov's forward equations are a set of coupled linear differential equations whose solution is the transition probability function $P_{xy}(t)$. The function $P_{xy}(t)$ determines the probability of transitioning from state x to state y between times s and $s+t$. Write down the transition rates q_{xy} from state x into state y for the five different capital ranges (A)-(E). As before, it is a good idea to start with range (D). Use these transition rates to write Kolmogorov's forward equation for the five capital ranges (A)-(E).

G) Kolmogorov's backward equation. Kolmogorov's backward equations are an alternative set of coupled linear differential equations whose solution is the transition probability function $P_{xy}(t)$. Write Kolmogorov's backward equation for the five capital ranges (A)-(E).

H) Solution of Kolmogorov's equations. Group the functions $P_{xy}(t)$ in a matrix function $\mathbf{P}(t)$ and define the vector $\mathbf{p}(t)$ whose x-th element is the probability $\mathrm{P}[X(t) = x]$ of $X(t)$ being x. Build a matrix $\mathbf{R}$ with off-diagonal elements $r_{xy} = q_{xy}$ and diagonal elements $r_{xx} = -\nu_x$. The solution of Kolmogorov's (forward and backward) equations is given by the matrix exponential $\mathbf{P}(t) = e^{\mathbf{R}t}$. The probability $\mathbf{p}(t)$ is then obtained as $\mathbf{p}(t) = \mathbf{P}^T(t)\mathbf{p}(0)$. For the same parameters of part *E*, use the matrix exponential to find the probability distribution $\mathbf{p}(t)$ between years 0 and 5 in quarterly increments. Show all of these quarterly probabilities in one plot. The matrix exponential in Matlab is obtained through the function $e^{\mathbf{A}}$ =`expm`($\mathbf{A}$).

I) Probability of paying dividends. Use your results in part *H* to approximately determine the probability of the insurance company paying dividends in quarters 1 through 20. This is a typical engineering question, there is more than one possible correct answer.