Solutions to Homework 8 - Continuous-Time Markov Chains

1) A single-server station. Potential customers arrive at a single-server station in accordance to a Poisson process with rate λ. However, if the arrival finds n customers already in the station, then she will enter the system with probability α_n. Assuming an exponential service rate μ, we will set this up as a birth and death process and determine the birth and death rates.

Specifically, if we let the state $\{X(t) : t \geq 0\}$ be the number of customers in the system at any given time instant t, then clearly this system (which is very similar to an $M/M/1$ queue) can be modeled as a birth and death process with state space $\mathcal{S} = \{0, 1, 2, \ldots\}$. From the problem description, whenever the system state is n then arrivals occur with exponential rate λ, but only a fraction α_n of them will actually enter the system. Thus, the effective arrival rate is $\alpha_n\lambda$. As in the M/M/1 queue, for all system states we have that customers will leave with exponential rate μ. Summarizing our previous discussion, then $\{X(t) : t \geq 0\}$ is a birth and death process with

$$\begin{aligned}
\mu_0 &= 0, \\
\mu_n &= \mu, \quad n \geq 1, \\
\lambda_n &= \alpha_n\lambda, \quad n \geq 0.
\end{aligned}$$

2) Model and analysis of a barbershop. A small barbershop, operated by a single barber has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables (RVs) with mean $1/4$ per hour.

A) Letting $\{X(t) : t \geq 0\}$ be the number of customers in the barbershop at any given time instant t, we can model the system as a birth and death process with finite state space $S = \{0, 1, 2\}$ and transition rates

$$\begin{aligned}
\mu_0 &= 0, \\
\mu_n &= 4, \quad n = 1 \text{ and } 2, \\
\lambda_n &= 3, \quad n = 0 \text{ and } 1, \\
\lambda_2 &= 0.
\end{aligned}$$

A state transition diagram for this simple CTMC model of the barbershop is shown in Fig. 1.

To compute the average number of customers in the barbershop, and we must first compute the limiting distribution (notice that the CTMC is ergodic)

$$P_j = \lim_{t\to\infty} P_{ij}(t), \quad j = 0, 1, 2.$$

To that end, state the balance equations

State	Rate out	Rate in
0	$3P_0$	$4P_1$
1	$7P_1$	$3P_0 + 4P_2$
2	$4P_2$	$3P_1$

which yield

$$\begin{aligned}
P_1 &= \frac{3}{4}P_0, \\
P_2 &= \left(\frac{3}{4}\right)^2 P_0.
\end{aligned}$$

To solve for P_0, recall that $\sum_{j=0}^{2} P_j = 1$ from where it follows that

$$P_0 = \frac{16}{37}, \quad P_1 = \frac{12}{37}, \quad P_2 = \frac{9}{37}.$$

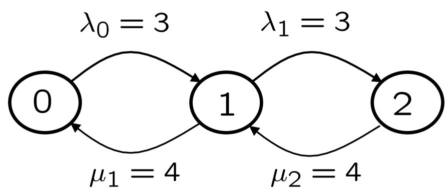

Fig. 1. State transition diagram for the CTMC model of the barbershop.

Given the previous stationary distribution, the long-run average number of customers in the shop is

$$\lim_{t \to \infty} \mathbb{E}[X(t)] = 0 \times \frac{16}{37} + 1 \times \frac{12}{37} + 2 \times \frac{9}{37} = \frac{30}{37}.$$

B) Exactly as in the model in problem **1)**, here arrivals occur at an exponential rate $\lambda = 3$ but customers will not be admitted to the system with probability $P_2 = 9/37$ (when the barbershop is full). In other words, the proportion of potential customers that enter the shop is

$$\alpha = 1 - \frac{9}{37} = \frac{28}{37} = 0.757.$$

C) The barber working twice as fast can be modeled as a new birth and death process with

$$\begin{aligned} \mu_0 &= 0, \\ \mu_n &= 8, \quad n = 1 \text{ and } 2, \\ \lambda_n &= 3, \quad n = 0 \text{ and } 1, \\ \lambda_2 &= 0. \end{aligned}$$

Repeating all the calculations in *A)* and *B)* yields the new limiting distribution

$$P_0 = \frac{64}{97}, \quad P_1 = \frac{24}{97}, \quad P_2 = \frac{9}{97}$$

so that the proportion of admitted customers increases to

$$\alpha^* = \frac{88}{97} = 0.907.$$

Because the arrival rate remains invariant, the barbershop is admitting $\frac{0.907-0.757}{0.757} \times 100 = 19.8\%$ more customers than in the original setting.

3) Discrete-time and continuous-time queueing models. The interarrival and service time distributions in the discrete-time queuing model described in problem 1 of homework 6 are geometric. The geometric distribution has the memoryless property and can be viewed as a discrete analog of the exponential distribution. Accordingly, the the queue described in problem 1 of homework 6 can be viewed as a discrete-time analog of the M/M/1 queue.

To obtain the stationary distribution $\boldsymbol{\pi}$ for the discrete-time queue, we need to solve

$$\boldsymbol{\pi} = \mathbf{P}^T \boldsymbol{\pi}, \quad \mathbf{1}^T \boldsymbol{\pi} = 1$$

where the matrix of transition probabilities obtained for that model was

$$\mathbf{P} = \begin{pmatrix} 1-p & p & 0 & 0 & \dots \\ (1-p)q & 1+2pq-p-q & p(1-q) & 0 & \dots \\ 0 & (1-p)q & 1+2pq-p-q & p(1-q) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

These equations can be written as

$$\pi_0 = (1-p)\pi_0 + (1-p)q\pi_1, \tag{1}$$
$$\pi_1 = p\pi_0 + (1+2pq-p-q)\pi_1 + (1-p)q\pi_2, \tag{2}$$
$$\pi_2 = p(1-q)\pi_1 + (1+2pq-p-q)\pi_2 + (1-p)q\pi_3,$$
$$\vdots$$
$$\pi_i = p(1-q)\pi_{i-1} + (1+2pq-p-q)\pi_i + (1-p)q\pi_{i+1}, \quad i \geq 2.$$

From (1) we find

$$\pi_1 = \frac{p}{q(1-p)}\pi_0$$

which can be used along (2) to obtain

$$\pi_2 = \frac{p^2(1-q)}{q^2(1-p)^2}\pi_0.$$

Since the structure of the balance equations repeats from now onwards, proceeding sequentially yields

$$\pi_i = \frac{p^i(1-q)^{i-1}}{q^i(1-p)^i}\pi_0, \quad i \geq 1. \tag{3}$$

To solve for π_0, we resort as usual to $\sum_{i=0}^{\infty}\pi_i = 1$. Using (3) and summing the geometric series after some manipulations yields

$$\begin{aligned} 1 &= \pi_0 + \sum_{i=1}^{\infty}\frac{p^i(1-q)^{i-1}}{q^i(1-p)^i}\pi_0 \\ &= \pi_0 + \pi_0\frac{p}{q(1-p)}\sum_{i=1}^{\infty}\left[\frac{p(1-q)}{q(1-p)}\right]^{i-1} \\ &= \pi_0 + \pi_0\frac{p}{q(1-p)}\left[\frac{1}{1-\frac{p(1-q)}{q(1-p)}}\right]. \end{aligned} \tag{4}$$

The last equality uses the fact that $\frac{p(1-q)}{q(1-p)} < 1$, which is true since $p < q$. The condition that $p < q$ ensures services can "catch up" with arrivals, and is an analog to $\lambda < \mu$ in the M/M/1 queue. Absent the stability condition, the queue will grow unbounded. Back to the pursuit of π_0, solving (4) yields

$$\pi_0 = 1 - \frac{p}{q}$$

so that the stationary distribution is [cf. (3)]

$$\pi_i = \left(1-\frac{p}{q}\right)\frac{p^i(1-q)^{i-1}}{q^i(1-p)^i}, \quad i \geq 1.$$

To make a final connection with the M/M/1 queue, suppose that p and q are both very small (reasonable when the duration of the discrete-time slots vanishes and we approach the continuous-time limit) and define the traffic intensity $\rho = p/q$. This way, $\pi_1 \approx (1-\rho)\rho^i$ as in the M/M/1 queue.

4) Comparing occupancies of two M/M/1 queues. Consider two separate M/M/1 queues. In the first (called "system 1") the arrival rate is λ_1, and the mean service time is μ^{-1}. In the second (called "system 2') the arrival rate is λ_2, and the mean service time is μ^{-1}. Suppose that $\lambda_1 < \lambda_2 < \mu$. In order to argue formally, that on the average, there will be less customers in system 1 than in system 2; we first note that given the assumptions both queues will be stable and the limiting distributions are well defined.

Furthermore, it follows that for $\rho_i := \lambda_i/\mu_i$, $i = 1, 2$, then $\rho_1 < \rho_2$. Finally we recall that for these M/M/1 queues, the average number of customers in the system is given by

$$\mathbb{E}[L_i] = \frac{\rho_i}{1-\rho_i}, \quad i = 1, 2. \tag{5}$$

Therefore, it immediately follows that $\mathbb{E}[L_1] < \mathbb{E}[L_2]$, which is what we wanted to show.

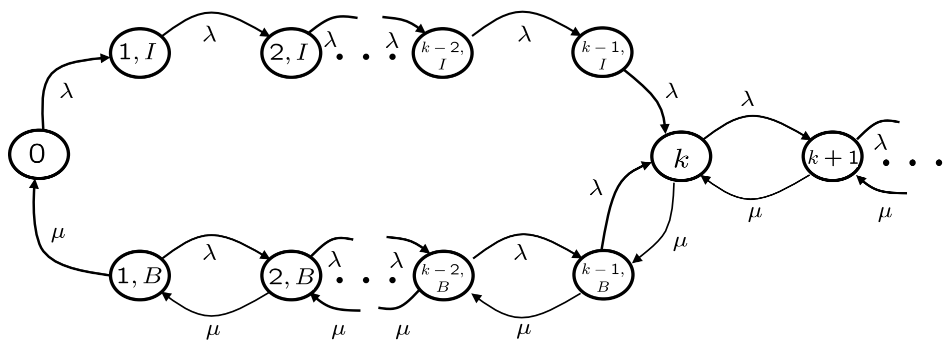

Fig. 2. State transition diagram for the CTMC model of an M/M/1 queue with on/off server switching.

5) An M/M/1 queue with on/off server switching. Consider the following variation of an M/M/1 queue with arrival rate λ, service rate μ, and traffic intensity $\rho = \lambda/\mu < 1$. The server is turned off whenever the system becomes empty, and the server remains off until there are k (k is a fixed number) customers in the system, at which time the server is turned on. The server then remains on until the system empties, at which time the process repeats. We can model the system as a CTMC with states 0 (system empty), (i, B) [system busy, i.e., $i \in (0, k)$ customers present and server on], (i, I) [system idle, i.e., $i \in (0, k)$ customers present and server off], i ($i \geq k$ customers present). From the given problem description, the state transition diagram describing the CTMC is given in Fig. 2.

To determine the limiting distribution, write down the balance equations that readily follow from the previous diagram, namely

State	Rate out	Rate in
0	λP_0	$\mu P_{1,B}$
$1, I$	$\lambda P_{1,I}$	λP_0
$\vdots$	$\vdots$	$\vdots$
i, I	$\lambda P_{i,I}$	$\lambda P_{i-1,I}$
$\vdots$	$\vdots$	$\vdots$
$k-1, I$	$\lambda P_{k-1,I}$	$\lambda P_{k-2,I}$
$1, B$	$(\lambda+\mu)P_{1,B}$	$\mu P_{2,B}$
$\vdots$	$\vdots$	$\vdots$
i, B	$(\lambda+\mu)P_{i,B}$	$\lambda P_{i-1,B} + \mu P_{i+1,B}$
$\vdots$	$\vdots$	$\vdots$
$k-1, B$	$(\lambda+\mu)P_{k-1,B}$	$\lambda P_{k-2,B} + \mu P_k$
k	$(\lambda+\mu)P_k$	$\lambda(P_{k-1,I} + P_{k-1,B}) + \mu P_{k+1}$
$\vdots$	$\vdots$	$\vdots$
i	$(\lambda+\mu)P_i$	$\lambda P_{i-1} + \mu P_{i+1}$
$\vdots$	$\vdots$	$\vdots$

From the balance equations involving the idle states, it follows that

$$P_{i,I} = P_0, \quad i \in (0, k)$$

From the balance equations involving the busy states, we obtain that the limiting probabilities satisfy the following second-order recursion for $i + 1 \in (2, k)$

$$P_{i+1,B} = (\rho+1)P_{i,B} - \rho P_{i-1,B}$$

with the specific initial conditions $P_{1,B} = \rho P_0$ and $P_{2,B} = (1+\rho)\rho P_0$. Using standard techniques for solving second-order difference equations with constant coefficients, we finally obtain that

$$P_{i,B} = \frac{\rho}{\rho - 1}\left[\rho^i - 1\right] P_0, \quad i \in (0, k).$$

From the balance equation for state $(k-1,B)$, it follows from the above expression for $P_{i,B}$ that

$$P_k = (1+\rho)P_{k-1,B} - \rho P_{k-2,B} = \frac{\rho}{\rho-1}\left[\rho^k - 1\right] P_0.$$

Likewise, from the balance equation for state k and using our previous results for P_k, $P_{k-1,I}$ and $P_{k-1,B}$; we find that

$$P_{k+1} = (\rho+1)P_k - \rho(P_{k-1,I} + P_{k-1,B}) = \frac{\rho^2}{\rho-1}\left[\rho^k - 1\right] P_0 = \rho P_k.$$

Finally, for $i > k$ the limiting probabilities satisfy again the second-order recursion given by

$$P_{i+1} = (\rho+1)P_i - \rho P_{i-1}$$

with initial conditions $P_k = \frac{\rho}{\rho-1}\left[\rho^k - 1\right] P_0$ and $P_{k+1} = \rho P_k$. Solving the difference equation (exactly as we did before for the busy states) we find for $i \geq k$ that

$$P_i = \frac{\rho}{\rho-1}\left[\rho^k - 1\right] \rho^{i-k} P_0.$$

At this point we have been able to express all limiting probabilities as a function of P_0. Using all limiting probabilities should sum up to one, we can solve for P_0 after summing the corresponding geometric sums

$$\begin{aligned}
1 &= \left[1 + (k-1) + \sum_{j=1}^{k} \frac{\rho}{\rho-1}\left[\rho^j - 1\right] + \sum_{j=k+1}^{\infty} \frac{\rho}{\rho-1}\left[\rho^k - 1\right]\rho^{j-k}\right] P_0 \\
&= \left[k + \frac{\rho}{(\rho-1)}\frac{\left[\rho - \rho^{k+1}\right]}{(1-\rho)} - \frac{\rho}{\rho-1}k + \frac{\rho^{1-k}}{(\rho-1)}\frac{\rho^{k+1}}{(1-\rho)}\left[\rho^k - 1\right]\right] P_0 \\
&= \left[k - \frac{\rho}{\rho-1}k\right] P_0 = \frac{k}{1-\rho} P_0.
\end{aligned}$$

From the last equality, it follows that $P_0 = \frac{1-\rho}{k}$. Substituting this value of P_0 in the previous expressions for the limiting probabilities, we arrive at the desired result

$$\begin{aligned}
P_0 &= (1-\rho)/k, \\
P_{i,I} &= (1-\rho)/k, \quad i \in (0,K), \\
P_{i,B} &= \rho(1-\rho^i)/k, \quad i \in (0,K), \\
P_i &= \rho^{i+1-k}(1-\rho^k)/k, \quad i \geq k.
\end{aligned}$$

6) A hiring decision. The manager of a market can hire Mary or Alice. Mary, who gives service at an exponential rate of 20 customers per hour, can be hired at a rate of \$3 per hour. Alice, who gives service at an exponential rate of 30 customers per hour, can be hired at a rate of \$$C$ per hour. The manager estimates that on average each customer's time is worth \$1 per hour, and should be accounted for in the model. Assume customers arrive at a Poisson rate of 10 per hour.

First we want to compute what is the average cost per hour if Mary is hired. Given the model description, the system can be modeled as an M/M/1 system with traffic intensity of $\rho_{\text{Mary}} = 1/2$. We also note that the average number of customers in the system is given by

$$\mathbb{E}\left[L_{\text{Mary}}\right] = \frac{\rho_{\text{Mary}}}{1-\rho_{\text{Mary}}} = 1$$

Therefore the expected cost is

$$\mathbb{E}\left[\text{Cost}_{\text{Mary}}\right] = \$3 + \$1 \times \mathbb{E}\left[L_{\text{Mary}}\right] = \$4$$

On the other hand, when Alice is the server then the traffic intensity is $\rho_{\text{Alice}} = 1/3$ and the resulting average number of customers in the system is $\mathbb{E}\left[L_{\text{Alice}}\right] = 1/2$. So the expected cost is

$$\mathbb{E}\left[\text{Cost}_{\text{Alice}}\right] = \$C + \$1 \times \mathbb{E}\left[L_{\text{Alice}}\right] = \$C + \frac{1}{2}$$

From the previous calculations, it follows that for $\mathbb{E}\left[\text{Cost}_{\text{Mary}}\right] = \mathbb{E}\left[\text{Cost}_{\text{Alice}}\right]$ to hold true, we need $C = \$7/2$.

7) Insurance cash flow.

A) CTMC states. Since we assume that c, d and $X_{\max}$ are integers, while the premiums that the customers pay are worth 1, every integer between 0 and $X_{\max}$ is achievable. Accordingly, given our assumptions every cash flow consists of an integer value of money. Thus, the CTMC cannot reach any non-integer state. We are obviously assuming that the initial amount of cash $X(0)$ is also an integer. Consequently, the state space of the CTMC consists of every nonnegative integer number between 0 and $X_{\max}$.

B) Transition times out of given state. We now compute the probability distribution of the transition times out of an arbitrary state $x \in \{0, 1, \ldots, X_{\max}\}$. It is immediate that this probability depends on the range to which the current state x belongs; see the range reference table in the problem statement. For example, if $x = 0$, i.e. x is in range (A), the only possible event triggering a transition out of this state is the payment of a premium. Recall that the time between premiums T_p is exponentially distributed with parameter $\lambda = N$. Thus, the time until a transition out of $x = 0$ is given by $T_x = T_p$.

When $X(t) = x$ and x is in range (B), the transition time out of state x is also exponential because it is the minimum among the times for the next claim (T_c) or premium (T_p) being paid, and both T_c and T_p are exponentially distributed. Therefore

$$T_x = \min(T_p, T_c)$$

meaning that we transition out of x whenever the first of the two events occurs. To obtain the distribution of T_x, notice that for a transition *not* to occur by time t, i.e. $T_x > t$, none of the two possible events must have occurred. Since both events are independent, it follows that

$$\mathrm{P}\left[T_X > t\right] = \mathrm{P}\left[T_p > t\right] \mathrm{P}\left[T_c > t\right] = e^{-\lambda t} e^{-\alpha t} = e^{-(\lambda+\alpha)t}.$$

We conclude that the cumulative distribution function of T_x is $e^{-(\lambda+\alpha)t}$, therefore it is exponentially distributed with rate $\nu_x = (\lambda + \alpha)$. Following the same reasoning, we may derive the probability distribution of the transition times out of any state by carefully identifying those events that can occur in each state range. All of them are exponentially distributed with different parameters ν_x. The results are summarized in the following table:

Range	Possible events	Rate ν_x of exponential time T_x
A	premium	λ
B	premium, claim paid at $X(t)$	$\lambda + \alpha$
C	premium, claim	$\lambda + \alpha$
D	premium, claim, dividend	$\lambda + \alpha + \beta$
E	claim, dividend	$\alpha + \beta$

C) Possible states going out of $X(t) = x$. Given that $X(t) = x$, where x is in range (D), the possible states after a transition out of x occurs are

$$\begin{aligned} x &\to x+1 \quad \text{(a premium is paid)}, \\ x &\to x-c \quad \text{(a claim is paid)}, \\ x &\to x-d \quad \text{(a dividend is paid)}. \end{aligned}$$

The possible states going out of $X(t) = x$ for each other range are summarized in the following table:

Range	Possible events	Possible states out of $X(t) = x$
A	premium	1
B	premium, claim paid at $X(t)$	$x + 1$, 0
C	premium, claim	$x + 1$, $x - c$
D	premium, claim, dividend	$x + 1$, $x - c$, $x - d$
E	claim, dividend	$x - c$, $x - d$

D) Transition probabilities When x is in range (D), the transition probabilities from state x to y, P_{xy}, for each possible state y out of x as determined in part *C* are

$$P_{x,x+1} = \mathrm{P}\left[T_p = \min(T_p, T_c, T_d)\right] = \frac{\lambda}{\lambda + \alpha + \beta},$$
$$P_{x,x-c} = \mathrm{P}\left[T_c = \min(T_p, T_c, T_d)\right] = \frac{\alpha}{\lambda + \alpha + \beta},$$
$$P_{x,x-d} = \mathrm{P}\left[T_d = \min(T_p, T_c, T_d)\right] = \frac{\beta}{\lambda + \alpha + \beta}.$$

In calculating the above transition probabilities, we have used that T_p, T_c and T_d are independent, exponentially-distributed RVs. The transition probabilities out of state $X(t) = x$ for each other range are summarized in the following table:

Range	Event	State y out of x	Transition probabilities, P_{xy}
A	premium	1	$\frac{\lambda}{\lambda} = 1$
B	premium	$x+1$	$\frac{\lambda}{\lambda+\alpha}$
B	claim	0	$\frac{\alpha}{\lambda+\alpha}$
C	premium	$x+1$	$\frac{\lambda}{\lambda+\alpha}$
C	claim	$x-c$	$\frac{\alpha}{\lambda+\alpha}$
D	premium	$x+1$	$\frac{\lambda}{\lambda+\alpha+\beta}$
D	claim	$x-c$	$\frac{\alpha}{\lambda+\alpha+\beta}$
D	dividend	$x-d$	$\frac{\beta}{\lambda+\alpha+\beta}$
E	claim	$x-c$	$\frac{\alpha}{\alpha+\beta}$
E	dividend	$x-d$	$\frac{\beta}{\alpha+\beta}$

E) System simulation. Two different Matlab functions that simulate the stochastic system are provided next. Please notice the nuances, as each of them offers a different interpretation of the CTMC.

```
% Method 1: Following the suggested procedure in Homework 8, Problem 7-E
function [X,T]=cashflow1(X_0,lambda,alpha,beta,c,d,X_r,X_max,T_max)
index=1;
X(index)=X_0;
T(index)=0;
while T(index)<T_max
    x=X(index);
    if x==0 %only premium is possible
        tau=exprnd(1/lambda);
        T(index+1)=T(index)+tau;
        X(index+1)=x+1;
    elseif 0<x && x<c %premium, claim payed at X(t) not c
        tau=exprnd(1/(lambda+alpha));
        T(index+1)=T(index)+tau;
        u=rand;
        if u<(lambda/(lambda+alpha))   % premium
            X(index+1)=x+1;
        else            % claim
            X(index+1)=0;
        end
    elseif c<=x && x<X_r %premium, claim
        tau=exprnd(1/(lambda+alpha));
        T(index+1)=T(index)+tau;
        u=rand;
        if u<(lambda/(lambda+alpha))   %
            X(index+1)=x+1;
        else
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            X(index+1)=x-c;
        end
    elseif X_r<=x && x<X_max %premium, claim, dividend
        tau=exprnd(1/(lambda+alpha+beta));
        T(index+1)=T(index)+tau;
        u=rand;
        if u<(lambda/(lambda+alpha+beta))  % premium
            X(index+1)=X(index)+1;
        elseif u<((lambda+alpha)/(lambda+alpha+beta))  % claim
            X(index+1)=X(index)-c;
        else    % dividend
            X(index+1)=X(index)-d;
        end
    elseif x==X_max % claim, dividend
        tau=exprnd(1/(alpha+beta));
        T(index+1)=T(index)+tau;
        u=rand;
        if u<(alpha/(lambda+alpha))   % claim
            X(index+1)=x-c;
        else          % dividend
            X(index+1)=x-d;
        end
    else
        disp('Out Of Range')
        break
    end
    index=index+1;
end

end
```

An alternative second approach is also provided for completeness.

```
% Method 2: Based on the "alarm clock" interpretation
function [X,T]=cashflow2(X_0,lambda,alpha,beta,c,d,X_r,X_max,T_max)
index=1;
X(index)=X_0;
T(index)=0;
while T(index)<T_max
    x=X(index);
    if x==0 %only premium is possible
        t_premium=exprnd(1/lambda);
        T(index+1)=T(index)+t_premium;
        X(index+1)=x+1;
    elseif 0<x && x<c %premium, claim payed at X(t) not c
        t_premium=exprnd(1/lambda);
        t_claim=exprnd(1/alpha);
        T(index+1)=T(index)+min(t_premium,t_claim);
        X(index+1)=x+1*(t_premium<t_claim)-x*(t_premium>t_claim);
    elseif c<=x && x<X_r %premium, claim
        t_premium=exprnd(1/lambda);
        t_claim=exprnd(1/alpha);
        T(index+1)=T(index)+min(t_premium,t_claim);
        X(index+1)=x+1*(t_premium<t_claim)-c*(t_premium>t_claim);
```

```
    elseif X_r<=x && x<X_max %premium, claim, dividend
        t_premium=exprnd(1/lambda);
        t_claim=exprnd(1/alpha);
        t_dividend=exprnd(1/beta);
        [t_min,I]=min([t_premium,t_claim,t_dividend]);
        T(index+1)=T(index)+t_min;
        X(index+1)=x+(1:3==I)*[1;-c;-d];
    elseif x==X_max % claim, dividend
        t_claim=exprnd(1/alpha);
        t_dividend=exprnd(1/beta);
        [t_min,I]=min([t_claim,t_dividend]);
        T(index+1)=T(index)+t_min;
        X(index+1)=x+(1:2==I)*[-c;-d];
    else
        disp('Out Of Range')
        break
    end
    index=index+1;
end

end
```

See below a Matlab script that calls one of the functions and plots the results. The code is run for an initial capital of $X_0 = 200$, number of clients $N = 200$, risk $r = 4\%$, dividends payed quarterly; claim and dividend costs $c = 20$ and $d = 30$; capital thresholds, $X_r = 200$, and $X_{\max} = 300$; and maximum amount of time $T_{\max} = 5$ years.

```
clc; clear all; close all;
X_0=200;
N=200;
r=0.04;

lambda=N;
alpha=r*N;
beta=4;

X_r=200;
X_max=300;

T_max=5;
d=30;
c=20;

[X,t]=cashflow2(X_0,lambda,alpha,beta,c,d,X_r,X_max,T_max);

% Plotting the results
hold on
grid on
xlabel('time','Fontsize',14)
ylabel('Cash Level','Fontsize',14)
title('(a sample) Evolution of Cash Level over 5 Years','Fontsize',14)
axis([0 5 0 310])
stairs(t,X,'Linewidth',2,'Color','r');
```

Fig. 3 depicts a sample realization of the described CTMC. For the simulated 5 years, the company did not go

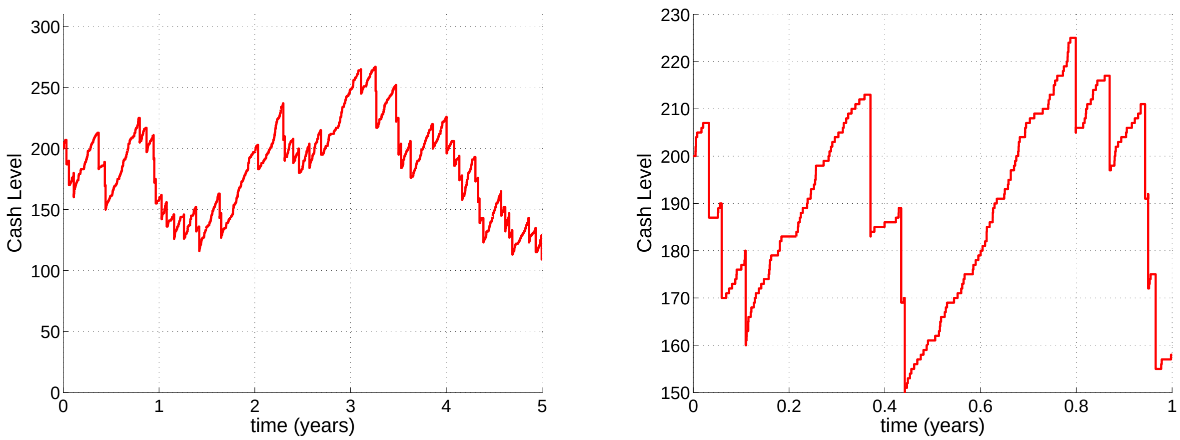

Fig. 3. A sample evolution of the cash flow of the insurance company for 5 years (left) and zoomed representation of the first year (right). (Part *E*)

bankrupt or reach the maximum cash threshold. The small increases in cash are due to premiums received while the drops in cash are given either by claims or dividends paid. For instance, the drop of 30 units of cash just before 0.4 years occurred due to the payment of dividends whereas the next decrease of 20 units is attributable to the payment of a claim.

F) Kolmogorov's forward equation. We are now interested in finding $P_{xy}(t)$, i..e, the probability of transitioning from state x to state y between times s and $s+t$ for every starting time s. To do so, we must solve Kolmogorov's equations. The transition rates q_{xy} from state x into state y are parameters of these equations and and are given by $q_{xy} = \nu_x P_{xy}$, where ν_x is the rate of transition out of state x, and P_{xy} is the probability of transitioning from state x into state y. For example, when the current state x is in the capital range (B) and we want to compute the transition rate to $y = x+1$ (premium payment), we have that

$$q_{x\,x+1} = \nu_x P_{x\,x+1} = (\lambda + \alpha)\,\frac{\lambda}{\lambda + \alpha} = \lambda$$

where ν_x was obtained from the table of transition rates in part *B*, and the transition probability $P_{x\,x+1}$ is tabulated in part *D*. If we repeat the above computation for all combinations of origin and destination states, we obtain the transition rates summarized in the following table:

Range	Event	State y out of x	Transition rates q_{xy}
A	premium	1	λ
B	premium	$x+1$	λ
B	claim	0	α
C	premium	$x+1$	λ
C	claim	$x-c$	α
D	premium	$x+1$	λ
D	claim	$x-c$	α
D	dividend	$x-d$	β
E	claim	$x-c$	α
E	dividend	$x-d$	β

Using the aforementioned transition rates, we can write Kolmogorov's forward equations

$$\frac{\partial P_{xy}(t)}{\partial t} = P'_{xy}(t) = \sum_{k=0,k\neq y}^{\infty} q_{ky} P_{xk}(t) - \nu_y P_{xy}(t)$$

for each range. Henceforth, we will drop the time dependency in the transition probability functions for notational simplicity. For transitioning into range (A), i.e., $y = 0$:

$$P'_{xy} = \alpha \sum_{k=1}^{c} P_{xk} - \lambda P_{xy}.$$

For transitioning into range (B), i.e., $0 < y < c$:

$$P'_{xy} = \lambda P_{x,y-1} + \alpha P_{x,y+c} - (\lambda + \alpha) P_{xy}.$$

For transitioning into range (C), i.e., $c \leq y < X_r$:

- When $y < X_r - d$, the transition into y cannot be the result of dividend payment since, after paying dividends, the smallest possible amount of cash if $X_r - d$:

$$P'_{xy} = \lambda P_{x,y-1} + \alpha P_{x,y+c} - (\lambda + \alpha) P_{xy}.$$

- When $y \geq X_r - d$, transition into y could have occurred due to a dividend payment:

$$P'_{xy} = \lambda P_{x,y-1} + \alpha P_{x,y+c} + \beta P_{x,y+d} - (\lambda + \alpha) P_{xy}.$$

For transitioning into range (D), i.e., $X_r \leq y \leq X_{\max}$:

- When $y \leq \min(X_{\max} - c, X_{\max} - d)$, the transition could have been a result of a claim payment, premium payment, or dividend payment:

$$P'_{xy} = \lambda P_{x,y-1} + \alpha P_{x,y+c} + \beta P_{x,y+d} - (\lambda + \alpha + \beta) P_{xy}.$$

- When $y > \max(X_{\max} - c, X_{\max} - d)$, the transition could only have been a result of a premium payment:

$$P'_{xy} = \lambda P_{x,y-1} - (\lambda + \alpha + \beta) P_{xy}.$$

- When $c < d$ and $X_{\max} - d < y \leq X_{\max} - c$, the transition could have been a result of a claim payment or a premium payment, but not a dividend payment:

$$P'_{xy} = \lambda P_{x,y-1} + \alpha P_{x,y+c} - (\lambda + \alpha + \beta) P_{xy}.$$

- When $d < c$ and $X_{\max} - c < y \leq X_{\max} - d$, the transition could have been a result of a dividend payment or a premium payment, but not a claim payment:

$$P'_{xy} = \lambda P_{x,y-1} + \beta P_{x,y+d} - (\lambda + \alpha + \beta) P_{xy}.$$

For transitioning into range (E), i.e., $y = X_{\max}$:

$$P'_{xy} = \lambda P_{x,y-1} - (\alpha + \beta) P_{xy}.$$

G) Kolmogorov's backward equation. Similarly, we may state the Kolmogorov's backward equations

$$\frac{\partial P_{xy}(t)}{\partial t} = P'_{xy}(t) = \sum_{k=0, k \neq x}^{\infty} q_{xk} P_{ky}(t) - \nu_x P_{xy}(t).$$

This is an alternative set of coupled linear differential equations whose solutions is also the transition probability function $P_{xy}(t)$. For transitioning out of range (A), i.e., $x = 0$, the only possible state y is $y = 1$:

$$P'_{xy} = \lambda P_{x+1,y} - P_{x,y}.$$

For transitioning out of range (B), i.e., $0 < x < c$:

$$P'_{xy} = \lambda P_{x+1,y} + \alpha P_{0,y} - (\lambda + \alpha) P_{x,y}.$$

For transitioning out of range (C), i.e., $c \leq x < X_r$:

$$P'_{xy} = \lambda P_{x+1,y} + \alpha P_{x-c,y} - (\lambda + \alpha) P_{x,y}.$$

For transitioning out of range (D), i.e., $X_r \leq x < X_{\max}$:

$$P'_{xy} = \lambda P_{x+1,y} + \alpha P_{x-c,y} + \beta P_{x-d,y} - (\lambda + \alpha + \beta) P_{x,y}.$$

For transitioning out of range (E), i.e., $x = X_{\max}$:

$$P'_{xy} = \alpha P_{x-c,y} + \beta P_{x-d,y} - (\alpha+\beta) P_{x,y}.$$

H) Solution of Kolmogorov's equations A Matlab function is provided to construct the matrix $\mathbf{R}$ such that Kolmogorov's forward equations are represented by $\dot{\mathbf{P}} = R\mathbf{P}$.

```
function [R]=Kolmogrov_F(lambda,alpha,beta,c,d,X_r,X_max)

R=zeros(X_max+1); % initialization

% Range A:
R(1,1)=-lambda;
R(1,2:c+1)=alpha;

% Range B:
for i=2:c
    R(i,i-1)=lambda;
    R(i,i+c)=alpha;
    R(i,i)=-(lambda+alpha);
end

% Range C-1:
for i=c+1:X_r-d
    R(i,i-1)=lambda;
    R(i,i+c)=alpha;
    R(i,i)=-(lambda+alpha);
end

% Range C_2:
for i=X_r-d+1:X_r
    R(i,i-1)=lambda;
    R(i,i+c)=alpha;
    R(i,i+d)=beta;
    R(i,i)=-(lambda+alpha);
end

% Range D_1:
for i=X_r+1:X_max-d+1
    R(i,i-1)=lambda;
    R(i,i+c)=alpha;
    R(i,i+d)=beta;
    R(i,i)=-(lambda+alpha+beta);
end

% Range D_2:
for i=X_max-d+2:X_max-c+1
    R(i,i-1)=lambda;
    R(i,i+c)=alpha;
    R(i,i)=-(lambda+alpha+beta);
end

% Range D_3:
for i=X_max-c+2:X_max
```

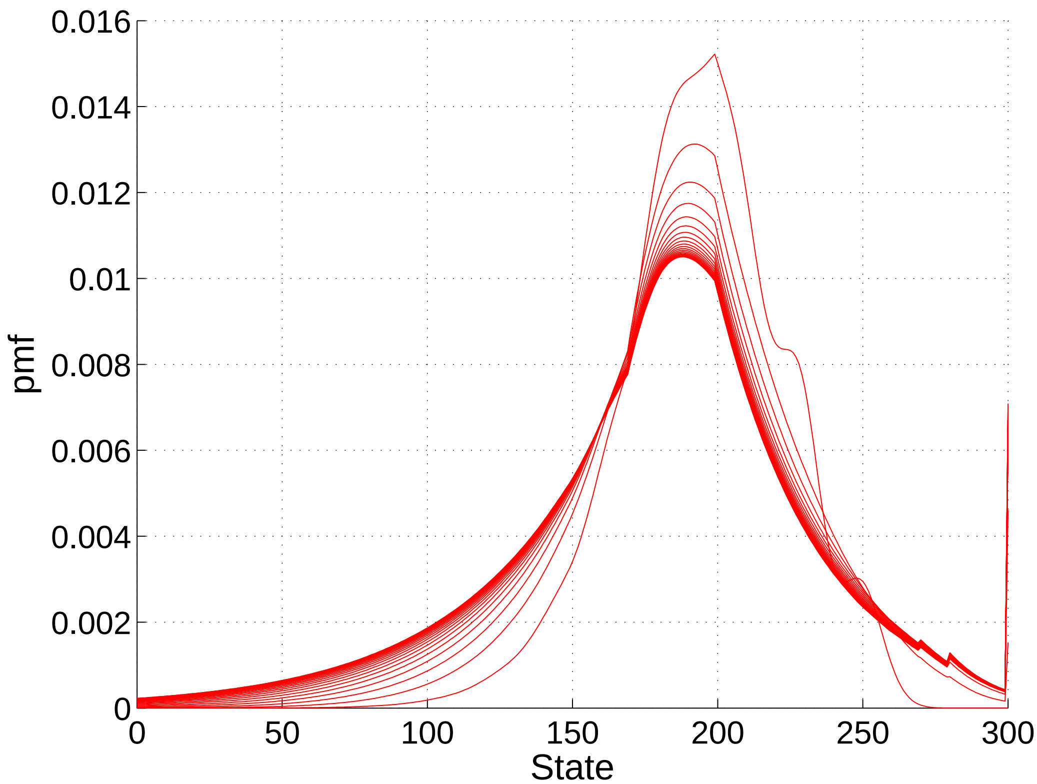

Fig. 4. The probability distribution $\mathbf{p}(t)$ for the times corresponding to the end of each quarter for the 5 years analyzed.

```
    R(i,i-1)=lambda;
    R(i,i)=-(lambda+alpha+beta);
end

% Range E:
R(X_max+1, X_max)=lambda;
R(X_max+1, X_max+1)=-(alpha+beta);
```

For the same parameters used in part *E*, we use the matrix exponential to find the probability distribution $\mathbf{p}(t)$ between years 0 and 5 in quarterly increments; see Fig. 4. The Matlab script that calls the function to construct $\mathbf{R}$, calculates $\mathbf{P}(t) = e^{\mathbf{R}(t)}$ and plots the obtained probability distribution $\mathbf{p}(t)$ is given next. In the figure we present a number of pmfs for the states in the CTMC. Each of these pmfs correspond to different quarters of the period studied. The mass function corresponding to the end of the first quarter, has its peak at the state 200 since this is the initial amount of capital. As time goes by, the initial information is lost and the pmf smooths out.

```
R=Kolmogrov_F(lambda,alpha,beta,c,d,X_r,X_max);
p0=zeros(X_max+1,1);
p0(X_0+1,1)=1;
T=0:0.25:5;

figure
hold on
xlabel('X','Fontsize',14)
ylabel('pmf','Fontsize',14)
```

```
title('pmf of the states between 0 and 5 over quarterly intervals',...
...'Fontsize',14)
axis([0 300 0 0.016])
for t=T
    pmf=expm(R.*t)*p0;
    plot(0:X_max,pmf,'r')
end
```

I) Probability of paying dividends. Finally, we use the solution of Kolmogorov's equation to estimate the probability that the insurance company pays dividends in any given quarter. As previously discussed, dividends can only be paid when the current state x is such that $X_r \leq x \leq X_{\max}$, i.e. when x is in range (D) or (E). When x is in this range, the expected number of events – either premiums, claims or dividends – is approximately $\lambda+\alpha+\beta = 212$. Notice that this is an approximation since we ignore that when x is exactly $X_{\max}$ it is impossible to receive payment for a premium. We may divide this number by four to find the expected number of events for each quarter, which is 53. For a given quarter spent entirely in $X_r \leq x \leq X_{\max}$, the probability that at least one dividend is paid can be computed as

$$\begin{aligned}
\mathrm{P}\left[\text{at least one dividend paid}\right] &= 1-\mathrm{P}\left[\text{no dividends paid}\right] \\
&= 1-\mathrm{P}\left[\text{all events are premium or claim payments}\right] \\
&= 1-\mathrm{P}\left[\text{single event is premium or claim payment}\right]^{53} \\
&= 1-\left(\frac{\lambda+\alpha}{\lambda+\alpha+\beta}\right)^{53} = 1-\left(\frac{208}{212}\right)^{53} \approx 0.64.
\end{aligned}$$

The probability that the company pays dividends at least once is approximately 0.64 for a quarter spent in the interval $X_r \leq x \leq X_{\max}$. Thus, we may estimate the probability of paying dividends in a given quarter as 0.64 times the probability that $X(t) \geq X_r$ for that quarter, which is achieved by summing up the pmf values from X_r to $X_{\max}$ output by the Kolmogorov's equation. For example, for the first quarter of the second year, i.e. quarter number 5, the probability of $X(t) \geq X_r$ is 0.372, thus the probability of paying dividends is 0.372 times 0.64 = 0.238. We repeat this procedure for every quarter and plot the result in Fig. 5. The corresponding Matlab script follows.

```
R=Kolmogrov_F(lambda,alpha,beta,c,d,X_r,X_max);
p0=zeros(X_max+1,1);
p0(X_0+1,1)=1;
T=0.25:0.25:5;
prob=zeros(20,1);

figure
hold on
xlabel('time (years)','Fontsize',18)
ylabel('Prob. of dividend','Fontsize',18)
axis([0 5 0.2 0.30])
for t=T
        pmf=expm(R.*t)*p0;
        prob(t/0.25) = sum(pmf(201:end))*0.64;
end
set(gca, 'fontsize', 16)
stairs(T,prob,'Linewidth',2,'Color','r');
grid on
```

It is evident that the probability of paying dividends decreases with time since it is harder for the company to maintain the cash level over $X_r = 200$. In the long run, the probability of paying dividends in a given quarter tends to around 0.21.

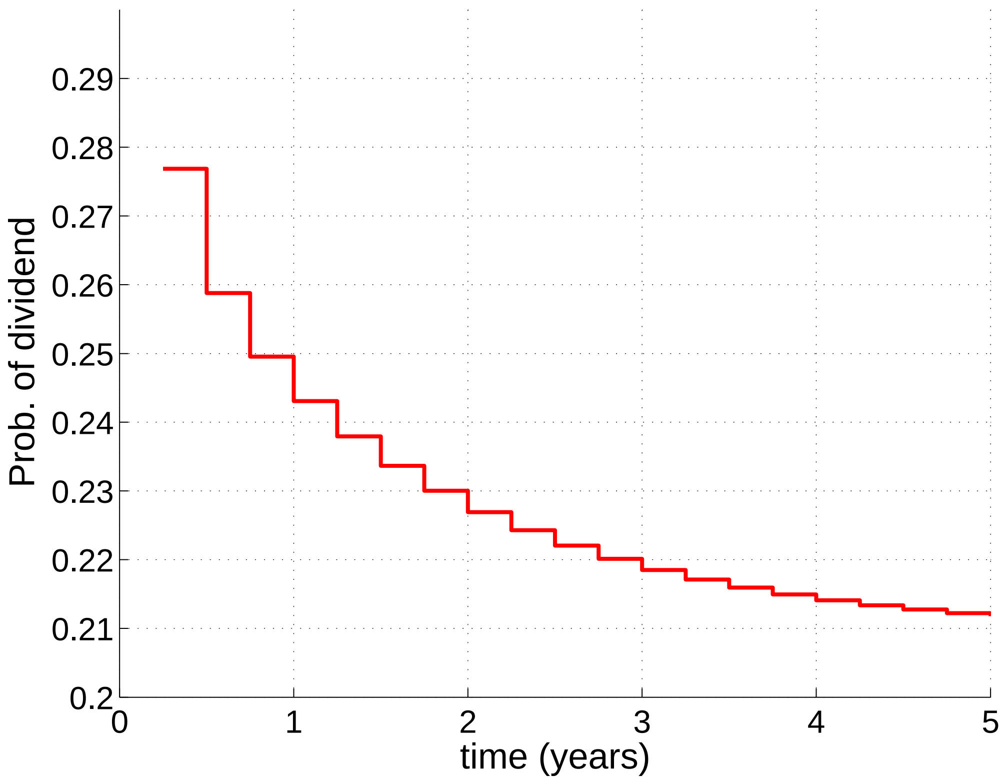

Fig. 5. Approximate probability of paying dividends over quarterly intervals. The probability decreases with time since it is less likely for the company to be over the X_r cash threshold.