Homework 9 - Gaussian processes

---

1) Conditional Gaussian density. Consider a two-dimensional *standard* Gaussian random vector $\mathbf{X} = [U, V]^T$, i.e., the joint pdf of $\mathbf{X}$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi\sqrt{\det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

where the mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{C}$ are

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

The off-diagonal entry of $\mathbf{C}$ is known as the correlation coefficient ρ, which satisfies $-1 < \rho < 1$.

A) Show that the joint pdf $f_{\mathbf{X}}(\mathbf{x})$ of $\mathbf{X} = [U, V]^T$ can be written as

$$f_{UV}(u, v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right).$$

Under what condition are U and V uncorrelated? Under what condition are U and V independent?

B) Determine the conditional pdf $f_{U \mid V}(u \mid v)$. Calculate $\mathbb{E}\left[U \mid V = v\right]$ and $\text{var}\left[U \mid V = v\right]$.

2) Jointly Gaussian random variables. A random vector $\mathbf{X} = [X_1, \ldots, X_n]^T$ is said to be Gaussian if every linear combination of the entries of $\mathbf{X}$, i.e., $\sum_{i=1}^n a_i X_i$ is a scalar Gaussian random variable (RV). Equivalent terminology is that $X_1, \ldots, X_n$ are jointly Gaussian RVs.

A) Show that if $\mathbf{X}$ is a Gaussian random vector, then every subvector of $\mathbf{X}$ (containing a subset of its entries) is also a Gaussian random vector. In particular, show that each $X_1, \ldots, X_n$ is a Gaussian RV.

B) Show that an affine transformation of a Gaussian random vector is Gaussian, i.e., for arbitrary (deterministic) matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and column vector $\mathbf{b} \in \mathbf{R}^m$, show that $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$ is a Gaussian random vector. Suppose that $\mathbf{X}$ has mean vector $\boldsymbol{\mu}_X$ and covariance matrix $\mathbf{C}_X$. Determine the mean vector $\boldsymbol{\mu}_Y$ and covariance matrix $\mathbf{C}_Y$ of $\mathbf{Y}$, and write down the pdf $f_{\mathbf{Y}}(\mathbf{y})$.

C) Show that if $Z_1, \ldots, Z_n$ are mutually independent Gaussian RVs, then they are also jointly Gaussian. [Hint: You may want to follow an inductive argument as in problem 4 of homework 7, and use (without need of a proof) that the convolution of two Gaussian pdfs is also a Gaussian pdf.]

3) Decorrelation of a Gaussian random vector. A random vector $\mathbf{X} = [X_1, \ldots, X_n]^T$ has zero mean and covariance matrix $\mathbf{C}_X$. Show that there exists a unitary matrix $\mathbf{U} \in \mathbf{R}^{n\times n}$ (meaning $\mathbf{U}^T\mathbf{U} = \mathbf{U}\mathbf{U}^T = \mathbf{I}$) such that the random vector $\mathbf{Y} := \mathbf{UX}$ has uncorrelated entries $Y_1, \ldots, Y_n$, i.e., the covariance matrix $\mathbf{C}_Y$ of $\mathbf{Y}$ is diagonal. For the chosen $\mathbf{U}$, argue that if $\mathbf{X}$ is a Gaussian random vector then $Y_1, \ldots, Y_n$ are mutually independent and jointly Gaussian.

4) Difference of two independent Brownian motion processes. Let $X_1(t)$ and $X_2(t)$ be two Brownian motion processes defined for $t \geq 0$, with respective variance parameters σ_1^2 and σ_2^2. Suppose that $X_1(t)$ and $X_2(t)$ are independent. Let the random process $D(t)$ be defined as $D(t) = X_1(t) - X_2(t)$, $t \geq 0$.

A) What is $D(t)$'s autocorrelation function $R_D(t_1, t_2)$ for $t_1, t_2 \geq 0$?

B) What is the pdf of $D(t)$ for $t \geq 0$?

5) A Brownian motion processes is a Martingale. A random process $X(t)$ defined for $t \geq 0$ is said to be a Martingale process if

$$\mathbb{E}\left[X(t) \,\middle|\, X(u), 0 \leq u \leq s\right] = X(s), \quad s < t.$$

Show that a Brownian motion process $X(t)$ is a Martingale. (Hint: For $s < t$ write $X(t) = X(s) + [X(t) - X(s)]$, use linearity of expectation and the independent increments property of $X(t)$.)

6) Conditional density of a Brownian motion process with drift. Let $X(t)$ for $t \geq 0$ be a Brownian motion process with drift parameter μ and variance parameter σ^2, i.e., $X(t)$ is Gaussian distributed with mean μt and variance $\sigma^2 t$. Specify the conditional distribution of $X(t)$ given that $X(s) = c$, for $s < t$.

7) White Gaussian noise. White Gaussian noise (WGN) is probably the most common stochastic model used in engineering applications. The idea is to model a random process $X(t)$ for which individual values are normally distributed and values $X(t_1)$ and $X(t_2)$ for different times t_1, t_2 are independent. It is not difficult to believe that this is a very simple model. It simply represents the drawing of independent normal RVs at different times. In the first part of this problem you will develop a model of WGN. In the second part you will use WGN to model somewhat more complex systems. The goal is to observe that while WGN is very simple, it can be used to model complex stochastic systems.

For this problem we need a few preliminary definitions. Start by defining a Gaussian process as one for which the probability distribution of the linear combination of values $a_1X(t_1) + a_2X(t_2) + \ldots, a_nX(t_n)$ is normally distributed for arbitrary times t_i, coefficients a_i and number of terms n. Further define the mean value function as

$$\mu(t) = \mathbb{E}\left[X(t)\right]$$

the power of the process as

$$P(t) = \mathbb{E}\left[X^2(t)\right]$$

and the autocorrelation function as

$$R_X(t_1, t_2) = \mathbb{E}\left[X(t_1)X(t_2)\right].$$

Gaussian processes have the appealing property that they are completely determined by specifying their mean value and autocorrelation functions. All definitions here apply to continuous-time and discrete-time processes.

For this problem we also need to define the delta function $\delta(t)$. Intuitively, $\delta(t)$ represents a "function" that is 0 everywhere but infinite at 0, i.e., $\delta(0) = \infty$ and $\delta(t) = 0$ for $t \neq 0$. Of course, this is not a valid definition of a function, but a formalization of the idea can be considered as the definition of the generalized function $\delta(t)$. The definition that we adopt for the $\delta(t)$ is through the integral of the product $f(t)\delta(t)$. Such integral satisfies

$$\int_a^b f(t)\delta(t)dt = \begin{cases} f(0) & \text{if } a < 0 < b, \\ 0 & \text{otherwise} \end{cases}.$$

With this preliminary definitions we can define WGN as a Gaussian process $W(t)$ with zero mean and delta autocorrelation, i.e.,

$$\mu_W(t) = 0, \qquad R_W(t_1, t_2) = \sigma^2\delta(t_1 - t_2).$$

Notice that since the autocorrelation function of $W(t)$ is not really a function, WGN *cannot* model any real physical phenomena. Nonetheless it is a convenient abstraction to generate processes that *can* model real physical phenomena.

A) Independent values. Show that values $W(t_1)$ and $W(t_2)$ at different times $t_1 \neq t_2$ are independent.

B) Integral of WGN. Define the process $X(t)$ as the integral of $W(t)$ between 0 and t, i.e.,

$$X(t) = \int_0^t W(u)\,du.$$

For historical reasons the random process $X(t)$ is known as Brownian motion or Wiener process. Show that the process $X(t)$ is Gaussian and compute the mean and autocorrelation functions of $X(t)$. What is the probability of $X(t) > a$ for arbitrary a and $t > 0$?

C) Discrete-time representation of WGN. Define the discrete-time process $W_h(n)$ as the integral of $W(t)$ between times nh and $(n+1)h$, i.e.,

$$W_h(n) = \int_{nh}^{(n+1)h} W(u)\,du$$

Compute the mean value function $\mu_{W_h}(n)$ and the autocorrelation function $R_{W_h}(n_1, n_2)$ for the discrete-time process $W_h(n)$.

D) Simulation of a Brownian motion process. Use the result of part *C* to perform a discrete-time simulation $X_h(n) = \sum_{i=0}^{n} W_h(i)$ of the Brownian motion process $X(t)$ of part *B*. In this simulation you must have the probability distribution of the discrete-time process $X_h(n)$ to be the same as the probability distribution of the continuous-time process $X(t)$. Run your simulation for $h = 0.01$, $\sigma^2 = 1$, and maximum amount of time $t_{\max} = 10$. Plot the resulting sample path of $X(t)$ for $0 \le t \le t_{\max}$.