ECE440 - Introduction to Random Processes

Midterm Exam

November 5, 2014

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 104, extra points are bonus points).
- Duration: 75 minutes.
- This exam has 9 numbered pages, check now that all pages are present.
- Show all your work, and write your final answers in the boxes when provided.

Name: SOLUTIONS

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	20		5.	20	
2.	8		6.	16	
3.	8		7.	18	
4.	14				
			Total	104	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix}.$$

To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^3 = \begin{pmatrix} 19/32 & 13/32 \\ 39/64 & 25/64 \end{pmatrix} = \begin{pmatrix} 0.59 & 0.41 \\ 0.61 & 0.39 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left[X_4 = 2 \,\middle|\, X_3 = 1, X_2 = 2, X_1 = 1\right] = ?$

$$\boxed{\frac{1}{2}}$$

From the Markov property it follows that

$$\mathrm{P}\left[X_4 = 2 \,\middle|\, X_3 = 1, X_2 = 2, X_1 = 1\right] = \mathrm{P}\left[X_4 = 2 \,\middle|\, X_3 = 1\right] = P_{12} = \frac{1}{2}.$$

(b) (2 points) $\mathrm{P}\left[X_5 = 1 \,\middle|\, X_2 = 1, X_0 = 1\right] = ?$

$$\boxed{\frac{19}{32}}$$

Likewise, $\mathrm{P}\left[X_5 = 1 \,\middle|\, X_2 = 1, X_0 = 1\right] = \mathrm{P}\left[X_5 = 1 \,\middle|\, X_2 = 1\right] = P^3_{11} = \frac{19}{32}$.

(c) (2 points) $\mathrm{P}\left[X_3 = 1 \,\middle|\, X_3 = 2, X_2 = 1, X_1 = 1, X_0 = 2\right] = ?$

$$\boxed{0}$$

Given $X_3 = 2$, then $X_3 = 1$ is impossible so $\mathrm{P}\left[X_3 = 1 \,\middle|\, X_3 = 2, X_2 = 1, X_1 = 1, X_0 = 2\right] = 0$.

(d) (6 points) $\mathbb{E}\left[X_7 \,\middle|\, X_4 = 2\right] = ?$

$$\boxed{\frac{89}{64}}$$

The conditional pmf of X_7 given $X_4 = 2$ is

$$\mathrm{P}\left[X_7 = 1 \,\middle|\, X_4 = 2\right] = P^3_{21} = \frac{39}{64},$$
$$\mathrm{P}\left[X_7 = 2 \,\middle|\, X_4 = 2\right] = P^3_{22} = \frac{25}{64}.$$

Hence, the conditional expectation is $\mathbb{E}\left[X_7 \,\middle|\, X_4 = 2\right] = 1 \times \frac{39}{64} + 2 \times \frac{25}{64} = \frac{89}{64}$.

(e) (8 points) Let $N = \min\{n > 0 : X_n = 2\}$. $\mathbb{E}\left[N \,\middle|\, X_0 = 1\right] =$?

$$\boxed{2}$$

The random variable N indicates the first (strictly positive) time instant the Markov chain visits state 2. Notice that conditioned on $X_0 = 1$, then N is a Geometric random variable with parameter $p := P_{12} = 1/2$. Accordingly,

$$\mathbb{E}\left[N \,\middle|\, X_0 = 1\right] = \frac{1}{P_{12}} = 2.$$

2. (8 points) Consider a probability space $(S, \mathcal{F}, \mathrm{P}[.])$. Suppose that D and E are events in $\mathcal{F}$ such that $\mathrm{P}[D] = 3/5$ and $\mathrm{P}[E] = 4/5$. From this information, is it possible to tell if D and E are mutually exclusive? Explain.

$$\boxed{\text{They are not mutually exclusive}}$$

If D and E were mutually exclusive, then

$$\mathrm{P}[D \cup E] = \mathrm{P}[D] + \mathrm{P}[E] = \frac{3}{5} + \frac{4}{5} = \frac{7}{5} > 1$$

which violates the axioms of probability.

3. (8 points) Suppose that a random variable X is Poisson-distributed with parameter $\lambda > 0$; i.e., $\mathrm{P}[X = x] = e^{-\lambda}\lambda^x/x!$ for $x = 0, 1, 2, \ldots$ Define the random variable $Y = qX$, where q is a number such that $0 < q < 1$. Is Y Poisson distributed? Justify your answer.

$$\boxed{Y \text{ is not Poisson distributed}}$$

For $0 < q < 1$, the random variable $Y = qX$ takes on non-integer values with positive probability. Hence, Y is not Poisson distributed.

4. Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & a & b & 0 & 0 \\ c & 0 & d & 0 & 0 \\ e & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{pmatrix}$$

where $a, b, c, d, e, f, g, h > 0$.

(a) (8 points) Is the Markov chain irreducible? Explain.

$\boxed{\text{No}}$

The Markov is not irreducible because it has two communicating classes, namely $\mathcal{R} = \{1, 2, 3\}$ and $\mathcal{T} = \{4, 5\}$.

(b) (6 points) Is state 4 transient? Explain.

$\boxed{\text{Yes}}$

Given that the Markov chain is in state 4, there is a positive probability ($f > 0$) of transitioning to state 3 and never visiting state 4 again. Hence, state 4 is transient.

5. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables, where $\mathrm{P}[X_1 = 1] = 1/4$, $\mathrm{P}[X_1 = 2] = 1/3$, $\mathrm{P}[X_1 = 3] = 5/12$.

(a) (10 points) Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{X_i \geq 2\}$$

and provide justification for the existence of the limit.

$$\boxed{\frac{3}{4}}$$

Because $X_{\mathbb{N}}$ is i.i.d., then $Z_{\mathbb{N}} = \mathbb{I}\{X_1 \geq 2\}, \mathbb{I}\{X_2 \geq 2\}, \ldots, \mathbb{I}\{X_n \geq 2\}, \ldots$ is also i.i.d. By the strong law of large numbers the limit exists and is equal to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{X_i \geq 2\} = \mathbb{E}[\mathbb{I}\{X_i \geq 2\}] = \mathrm{P}[X_1 \geq 2] = 1 - \mathrm{P}[X_1 = 1] = \frac{3}{4}, \quad \text{w.p. } 1.$$

(b) (10 points) Specify the distribution of the random variable Y, defined as

$$Y = \sum_{i=1}^{7} \mathbb{I}\{X_i \leq 1\}.$$

$\boxed{\text{Binomial with parameters } (7, 1/4)}$

For $1 \leq i \leq 7$, the summands $\mathbb{I}\{X_i \leq 1\}$ are i.i.d. Bernoulli-distributed random variables with parameter $p = \mathrm{P}[X_1 \leq 1] = \mathrm{P}[X_1 = 1] = 1/4$. Because the sum of n i.i.d. Bernoulli random variables with paramter p is Binomial distributed with paramters (n, p), then it follows that Y is Binomial with parameters $(7, 1/4)$.

6. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/4 & 3/4 \\ 4/5 & 1/5 \end{pmatrix}.$$

(a) (12 points) Compute the stationary distribution of $X_{\mathbb{N}}$.

$$\boxed{\boldsymbol{\pi} = \left[\frac{16}{31}, \frac{15}{31}\right]^T}$$

The unique stationary distribution $\boldsymbol{\pi} = [\pi_1, \pi_2]^T$ (the Markov chain is ergodic) satisfies

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 1/4 & 4/5 \\ 3/4 & 1/5 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \quad \pi_1 + \pi_2 = 1.$$

Solving the linear system yields $\boldsymbol{\pi} = [16/31, 15/31]^T$.

(b) (4 points) Suppose that X_0 has the distribution obtained in part (a). $\mathsf{P}[X_3 = 1] =$?

$$\boxed{\frac{16}{31}}$$

If the initial distribution is $\boldsymbol{\pi}$, then $\boldsymbol{\pi}$ will be the distribution for all subsequent time instants $n \geq 1$. Hence, $\mathsf{P}[X_3 = 1] = \pi_1 = 16/31$.

7. (18 points) Suppose that X_n is the amount of inventory in a store at the end of the time period n, and that D_n is the amount of demand that arrives during period n. Suppose that $D_{\mathbb{N}} = D_1, D_2, \ldots, D_n, \ldots$ is an i.i.d. sequence (idependent of X_0) of non-negative integer-valued random variables, each with probability mass function $q(\cdot)$; i.e.,

$$\mathsf{P}[D_1 = i] = q(i), \quad i = 0, 1, 2, \ldots$$

At the start of each time period, we receive a shipment of 5 units of inventory to the store. Demand that cannot be met is assumed to go away unsatisfied. Under the preceding assumptions, $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$. For $n \geq 0$, the inventory level at the end of period $n+1$ is determined by

$$X_{n+1} = \max\{0, X_n + 5 - D_{n+1}\}.$$

Notice that $\max\{0, a\} = a$ if $a \geq 0$, and $\max\{0, a\} = 0$ if $a < 0$. Hence, the above expression enforces the physical constraint that $X_{n+1} \geq 0$ always, and $X_{n+1} = 0$ when the demand D_{n+1} exceeds the available inventory $X_n + 5$.

(a) (6 points) Determine the transition probabilities P_{ij} for all $i \geq 0$ and $j = 1, \ldots, i+5$.

$$\boxed{P_{ij} = q(i + 5 - j)}$$

For $j > 0$, one has the transition probabilities are given by

$$\begin{aligned} \mathsf{P}\left[X_{n+1} = j \,\middle|\, X_n = i\right] &= \mathsf{P}\left[X_n + 5 - D_{n+1} = j \,\middle|\, X_n = i\right] = \mathsf{P}\left[i + 5 - D_{n+1} = j \,\middle|\, X_n = i\right] \\ &= \mathsf{P}[i + 5 - D_{n+1} = j] = \mathsf{P}[D_{n+1} = i + 5 - j] = q(i + 5 - j). \end{aligned}$$

Notice that for $j = 1, \dots, i+5$, then $i + 5 - j \geq 0$ and hence $P_{ij} = q(i + 5 - j)$.

(b) (6 points) Determine the transition probabilities P_{ij} for all $i \geq 0$ and $j \geq i + 6$.

$$\boxed{P_{ij} = 0}$$

For $j \geq i + 6$, then $i + 5 - j < 0$ and hence $P_{ij} = q(i + 5 - j) = 0$ because the D_n are non-negative integer-valued random variables.

(c) (6 points) Determine the transition probabilities P_{i0} for all $i \geq 0$.

$$\boxed{P_{i0} = 1 - \textstyle\sum_{k=0}^{i+4} q(k)}$$

For $j = 0$, one has the transition probabilities are given by

$$\begin{aligned} \mathbf{P}\left[X_{n+1} = 0 \,\middle|\, X_n = i\right] &= \mathbf{P}\left[D_{n+1} \geq X_n + 5 \,\middle|\, X_n = i\right] = \mathbf{P}\left[D_{n+1} \geq i + 5 \,\middle|\, X_n = i\right] \\ &= \mathbf{P}\left[D_{n+1} \geq i + 5\right] = \sum_{k=i+5}^{\infty} q(k). \end{aligned}$$

Hence, $P_{i0} = \sum_{k=i+5}^{\infty} q(k)$ which is of course also equal to $P_{i0} = 1 - \sum_{k=0}^{i+4} q(k)$.