ECE440 - Introduction to Random Processes

Midterm Exam

November 2, 2015

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 104, extra points are bonus points).
- Duration: 75 minutes.
- This exam has 11 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:__________________________________

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	20		4.	20	
2.	12		5.	10	
3.	20		6.	22	
			Total	104	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/5 & 2/5 & 2/5 \end{pmatrix}$$

and initial distribution $\mathrm{P}(X_0 = 1) = 0$, $\mathrm{P}(X_0 = 2) = 0$, and $\mathrm{P}(X_0 = 3) = 1$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 4/15 & 1/5 & 8/15 \\ 2/15 & 13/30 & 13/30 \\ 16/75 & 13/50 & 79/150 \end{pmatrix} = \begin{pmatrix} 0.27 & 0.20 & 0.53 \\ 0.14 & 0.43 & 0.43 \\ 0.21 & 0.26 & 0.53 \end{pmatrix}.$$

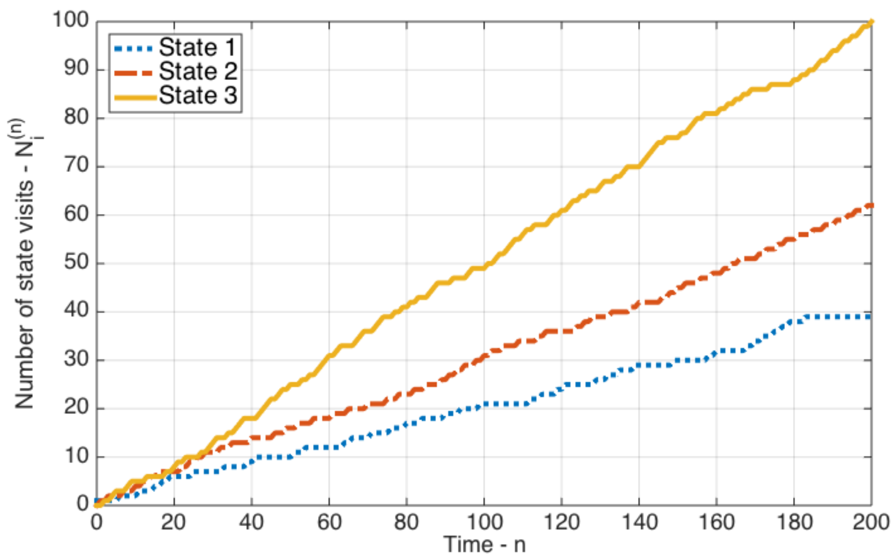
(a) (3 points) $\mathrm{P}\left(X_7 = 2 \,\middle|\, X_5 = 2, X_4 = 1\right) =?$

(b) (4 points) $\mathrm{P}(X_2 = 1) =?$

(c) (5 points) $\mathbb{E}\left[X_3 \,\middle|\, X_1 = 1\right] =?$

(d) (8 points) After simulating a realization of this Markov chain for 200 time instants, below is a plot of the number of visits $N_i^{(n)}$ to each state $i = 1, 2, 3$ by time n, that is

$$N_i^{(n)} = \sum_{m=1}^{n} \mathbb{I}\{X_m = i\}, \quad i = 1, 2, 3 \text{ and } n = 1, \ldots, 200.$$



Use the information in the plot to estimate the stationary distribution of $X_{\mathbb{N}}$. Briefly explain. (You can use that the Markov chain $X_{\mathbb{N}}$ is ergodic, no need to prove it.)

2. Let M be a positive integer and consider the sample space of possible outcomes $S = \{s_1, \ldots, s_M\}$. Let $\mathcal{F}$ be the collection of all subsets of S. Suppose that $p(s_1), \ldots, p(s_M)$ are nonnegative real numbers such that $\sum_{i=1}^{M} p(s_i) = 1$. For any subset $E \in \mathcal{F}$, define

$$\mathrm{P}(E) = \sum_{s \in E} p(s).$$

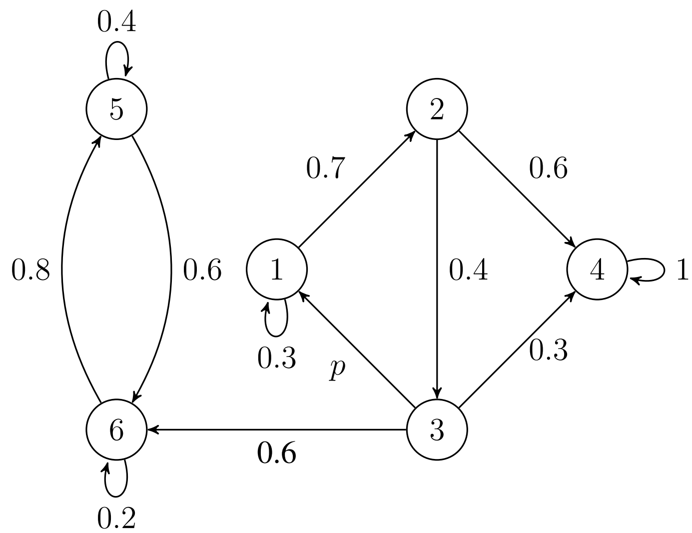
(a) (8 points) Show that $\mathrm{P} : \mathcal{F} \mapsto \mathbb{R}$ satisfies the three axioms of probability.

(b) (4 points) Determine the values $p(s_1), \ldots, p(s_M)$ so that $\mathrm{P} : \mathcal{F} \mapsto \mathbb{R}$ corresponds to the uniform probability distribution, that is

$$\mathrm{P}(E) = \sum_{s \in E} p(s) \equiv \frac{|E|}{|S|}$$

where $|\cdot|$ denotes the cardinality of a set. (Hint: What is $\mathrm{P}(\{s_i\})$, for $i = 1, \ldots, M$?)

3. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2, 3, 4, 5, 6\}$ and state transition diagram



(a) (3 points) What is the value of p? Explain.

(b) (3 points) Is state 5 aperiodic? Explain.

(c) (8 points) Is the Markov chain ergodic? Explain.

(d) (6 points) $\lim_{n\to\infty} \mathrm{P}\left(X_n = 1 \,\middle|\, X_0 = 6\right) =?$

4. Suppose that the sample space of possible outcomes is $S = \{s_1, s_2, s_3, s_4\}$, and $\mathrm{P}(s_1) = 1/10$, $\mathrm{P}(s_2) = 1/10$, $\mathrm{P}(s_3) = 1/5$, and $\mathrm{P}(s_4) = 3/5$. Let X and Y be random variables such that

$$\begin{aligned} X(s_1) &= 1, \quad Y(s_1) = 2, \\ X(s_2) &= 2, \quad Y(s_2) = 2, \\ X(s_3) &= 3, \quad Y(s_3) = 4, \\ X(s_4) &= 4, \quad Y(s_4) = 4. \end{aligned}$$

(a) (5 points) $\mathrm{P}\left(X = 3 \,\middle|\, Y = 4\right) = ?$

(b) (5 points) $\mathrm{P}\left(X + Y \leq 5 \,\middle|\, X \leq 3\right) = ?$

(c) (5 points) Compute $\mathbb{E}\left[2X + Y\right] =?$

(d) (5 points) $\mathbb{E}\left[X \,\middle|\, Y\right](s_3) =?$

5. (10 points) Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S_X = \{1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/4 & 3/4 \\ 4/5 & 1/5 \end{pmatrix}.$$

Define the state $Y_n := (X_n, X_{n-1})$ and the corresponding Markov chain $Y_{\mathbb{N}} = Y_1, Y_2, \ldots, Y_n, \ldots$ with state space $S_Y = \{(1,1), (1,2), (2,1), (2,2)\}$.

Draw the state transition diagram of $Y_{\mathbb{N}}$.

6. Suppose customers can arrive to a service station at times $n = 0, 1, 2, \ldots$ In any given period, independent of everything else, there is one arrival with probability p, and there is no arrival with probability $1 - p$. Customers are served one-at-a-time on a first-come-first-served basis. If at the time of an arrival, there are no customers present, then the arriving customer immediately enters service. Otherwise, the arrival joins the back of the queue.

Assume that service times are i.i.d. geometric random variables (each with parameter q) that are independent of the arrival process. So, $\mathrm{P}(\text{Service time} = \ell) = (1-q)^{\ell-1}q$, for $\ell = 1, 2, \ldots$ Note that a customer who enters service in time n can complete service, at the earliest, in time $n+1$ (in which case her service time is 1). Upon a service completion, the just-served customer will depart the system with probability α, or will immediately rejoin the back of the queue with probability $1 - \alpha$.

In a time period n, events happen in the following order: (i) arrivals, if any, occur; (ii) service completions followed by departures or rejoins, if any, occur; and (iii) service begins on the next customer if there are customers present in the system.

Let X_n denote the number of customers at the station at the end of time period n. Note that X_n includes both customers waiting as well as any customer being served, and that the random process $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$.

(a) (4 points) Determine the transition probabilities P_{0j} for all $j \geq 0$.

(b) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j = i$.

(c) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j > i$.

(d) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $0 \leq j < i$.