ECE440 - Introduction to Random Processes

Midterm Exam

November 2, 2015

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 104, extra points are bonus points).
- Duration: 75 minutes.
- This exam has 11 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:__________SOLUTIONS__________

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	20		4.	20	
2.	12		5.	10	
3.	20		6.	22	
			Total	104	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/5 & 2/5 & 2/5 \end{pmatrix}$$

and initial distribution $\mathrm{P}(X_0 = 1) = 0$, $\mathrm{P}(X_0 = 2) = 0$, and $\mathrm{P}(X_0 = 3) = 1$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 4/15 & 1/5 & 8/15 \\ 2/15 & 13/30 & 13/30 \\ 16/75 & 13/50 & 79/150 \end{pmatrix} = \begin{pmatrix} 0.27 & 0.20 & 0.53 \\ 0.14 & 0.43 & 0.43 \\ 0.21 & 0.26 & 0.53 \end{pmatrix}.$$

(a) (3 points) $\mathrm{P}\left(X_7 = 2 \,\middle|\, X_5 = 2, X_4 = 1\right) =?$

$$\boxed{\frac{13}{30}}$$

From the Markov property it follows that

$$\mathrm{P}\left(X_7 = 2 \,\middle|\, X_5 = 2, X_4 = 1\right) = \mathrm{P}\left(X_7 = 2 \,\middle|\, X_5 = 2\right) = P^2_{22} = \frac{13}{30}.$$

(b) (4 points) $\mathrm{P}(X_2 = 1) =?$

$$\boxed{\frac{16}{75}}$$

From the law of total probability (conditioning on X_0, noting that $\mathrm{P}(X_0 = 3) = 1$), one has

$$\mathrm{P}(X_2 = 1) = \sum_{i=1}^{3} \mathrm{P}\left(X_2 = 1 \,\middle|\, X_0 = i\right) \mathrm{P}(X_0 = i) = \sum_{i=1}^{3} P^2_{i1} \mathrm{P}(X_0 = i) = P^2_{31} \times 1 = \frac{16}{75}.$$

(c) (5 points) $\mathbb{E}\left[X_3 \,\middle|\, X_1 = 1\right] =?$

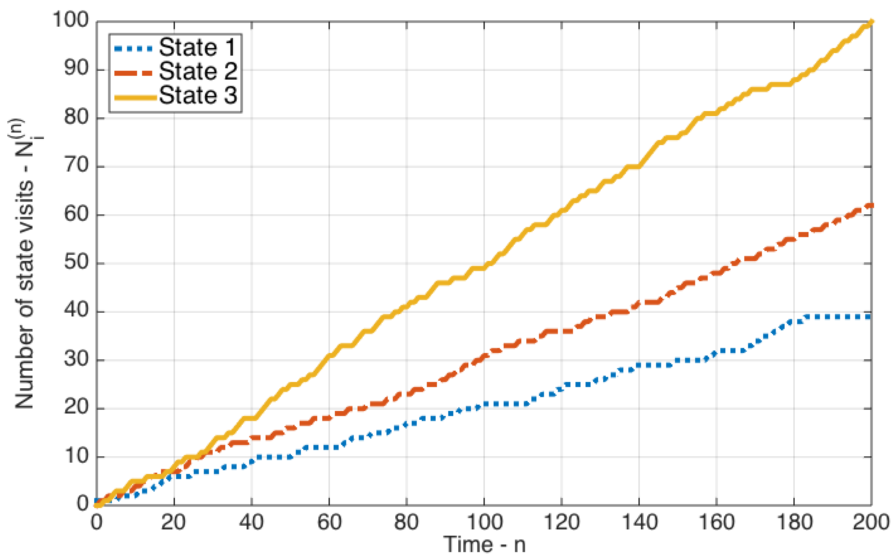
$$\boxed{\frac{34}{15}}$$

Using the definition of conditional expectation, one obtains

$$\begin{aligned}\mathbb{E}\left[X_3 \,\middle|\, X_1 = 1\right] &= \sum_{i=1}^{3} i \times \mathrm{P}\left(X_3 = i \,\middle|\, X_1 = 1\right) = \sum_{i=1}^{3} i \times P^2_{1i} \\ &= 1 \times \frac{4}{15} + 2 \times \frac{1}{5} + 3 \times \frac{8}{15} = \frac{34}{15}.\end{aligned}$$

(d) (8 points) After simulating a realization of this Markov chain for 200 time instants, below is a plot of the number of visits $N_i^{(n)}$ to each state $i = 1, 2, 3$ by time n, that is

$$N_i^{(n)} = \sum_{m=1}^{n} \mathbb{I}\{X_m = i\}, \quad i = 1, 2, 3 \text{ and } n = 1, \ldots, 200.$$



Use the information in the plot to estimate the stationary distribution of $X_{\mathbb{N}}$. Briefly explain. (You can use that the Markov chain $X_{\mathbb{N}}$ is ergodic, no need to prove it.)

$$\hat{\boldsymbol{\pi}} = \left[\frac{1}{5}, \frac{3}{10}, \frac{1}{2}\right]^T$$

Since $X_{\mathbb{N}}$ is ergodic, the long-run fraction of time the Markov chain is in state i converges to the stationary probability π_i, almost surely. Formally, this can be stated as the ergodic limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{X_m = i\} = \lim_{n\to\infty} \frac{1}{n} N_i^{(n)} = \pi_i, \qquad \text{a.s.}$$

From the plot, one can thus focus on the largest value of n which is 200, and approximate the limits π_i with the estimates $\hat{\pi}_i := N_i^{(n)}/n$, $i = 1, 2, 3$, namely

$$\pi_1 \approx \hat{\pi}_1 = \frac{N_1^{(200)}}{200} \approx \frac{40}{200} = \frac{1}{5},$$
$$\pi_2 \approx \hat{\pi}_2 = \frac{N_2^{(200)}}{200} \approx \frac{60}{200} = \frac{3}{10},$$
$$\pi_3 \approx \hat{\pi}_3 = \frac{N_3^{(200)}}{200} \approx \frac{100}{200} = \frac{1}{2}.$$

2. Let M be a positive integer and consider the sample space of possible outcomes $S = \{s_1, \ldots, s_M\}$. Let $\mathcal{F}$ be the collection of all subsets of S. Suppose that $p(s_1), \ldots, p(s_M)$ are nonnegative real numbers such that $\sum_{i=1}^{M} p(s_i) = 1$. For any subset $E \in \mathcal{F}$, define

$$\mathrm{P}(E) = \sum_{s \in E} p(s).$$

(a) (8 points) Show that $\mathrm{P} : \mathcal{F} \mapsto \mathbb{R}$ satisfies the three axioms of probability.

The first axiom is satisfied since for any subset $E \in \mathcal{F}$, one has that

$$\mathrm{P}(E) = \sum_{s \in E} p(s) \geq 0$$

because by assumption all $p(s_1), \ldots, p(s_M)$ are nonnegative.

The second axiom is satisfied since

$$\mathrm{P}(S) = \sum_{s \in S} p(s) = \sum_{i=1}^{M} p(s_i) = 1.$$

Finally, to check the third axiom we consider an arbitrary collection of disjoint events $E_1, \ldots, E_N$ (which has to be finite because S has finite cardinality M), and note that

$$\mathrm{P}\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{s \in \bigcup_{i=1}^{N} E_i} p(s) = \sum_{i=1}^{N} \sum_{s \in E_i} p(s) = \sum_{i=1}^{N} P(E_i)$$

where the second equality follows because events $E_1, \ldots, E_N$ are disjoint.

(b) (4 points) Determine the values $p(s_1), \ldots, p(s_M)$ so that $\mathrm{P} : \mathcal{F} \mapsto \mathbb{R}$ corresponds to the uniform probability distribution, that is

$$\mathrm{P}(E) = \sum_{s \in E} p(s) \equiv \frac{|E|}{|S|}$$

where $|\cdot|$ denotes the cardinality of a set. (Hint: What is $\mathrm{P}(\{s_i\})$, for $i = 1, \ldots, M$?)

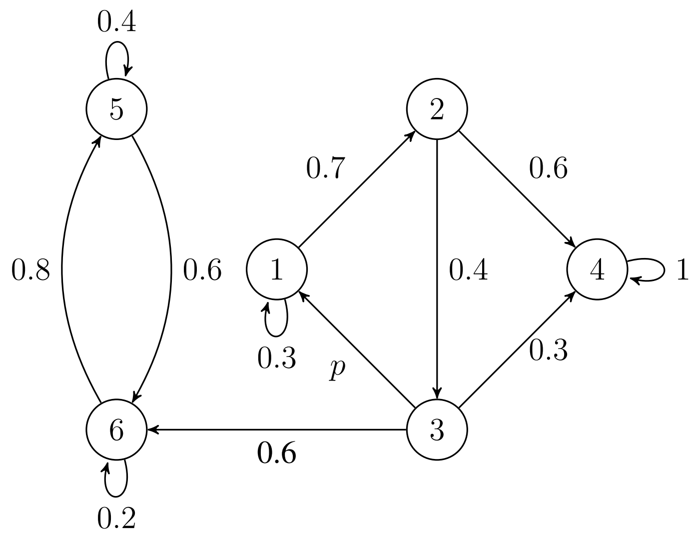
$$\boxed{p(s_1) = p(s_2) = \ldots = p(s_M) = \frac{1}{M}}$$

Notice that if $\mathrm{P} : \mathcal{F} \mapsto \mathbb{R}$ is the uniform distribution, then for all $i = 1, \ldots, M$

$$\mathrm{P}(\{s_i\}) = p(s_i) = \frac{|\{s_i\}|}{|S|} = \frac{1}{M}.$$

This implies that $p(s_1) = p(s_2) = \ldots = p(s_M) = \frac{1}{M}$ and hence all outcomes are equally likely.

3. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2, 3, 4, 5, 6\}$ and state transition diagram



(a) (3 points) What is the value of p? Explain.

0.1

The sum of the weights of those arrows coming out of any state should equal 1. Thus,

$$\sum_{i \in S} P_{3i} = p + 0.3 + 0.6 = 1 \quad \Rightarrow \quad p = 0.1.$$

(b) (3 points) Is state 5 aperiodic? Explain.

Yes

State 5 is aperiodic because $P_{55} = 0.4 \neq 0$, and so the period is $d_5 := \gcd\{n : P_{55}^n \neq 0\} = 1$.

(c) (8 points) Is the Markov chain ergodic? Explain.

No

State 4 is an absorbing state, hence it does not communicate with any other state. Accordingly, it is the sole member of the recurrent communication class $\mathcal{R}_1 = \{4\}$. This implies the total number of classes is strictly greater than 1, meaning $X_{\mathbb{N}}$ is neither irreducible, nor ergordic.

(d) (6 points) $\lim_{n\to\infty} \mathsf{P}\left(X_n = 1 \,\middle|\, X_0 = 6\right) =?$

0

Since $\mathcal{R}_2 = \{5, 6\}$ is a recurrent communication class, if the Markov chain is initialized in the class (cf. $X_0 = 6$), it will never leave $\mathcal{R}_2$. This readily implies that

$$\lim_{n\to\infty} \mathsf{P}\left(X_n = 1 \,\middle|\, X_0 = 6\right) = 0.$$

4. Suppose that the sample space of possible outcomes is $S = \{s_1, s_2, s_3, s_4\}$, and $\mathsf{P}(s_1) = 1/10$, $\mathsf{P}(s_2) = 1/10$, $\mathsf{P}(s_3) = 1/5$, and $\mathsf{P}(s_4) = 3/5$. Let X and Y be random variables such that

$$\begin{aligned} X(s_1) &= 1, \quad Y(s_1) = 2, \\ X(s_2) &= 2, \quad Y(s_2) = 2, \\ X(s_3) &= 3, \quad Y(s_3) = 4, \\ X(s_4) &= 4, \quad Y(s_4) = 4. \end{aligned}$$

(a) (5 points) $\mathsf{P}\left(X = 3 \,\middle|\, Y = 4\right) =?$

$\frac{1}{4}$

From the definition of conditional probability, one has

$$\mathsf{P}\left(X = 3 \,\middle|\, Y = 4\right) = \frac{\mathsf{P}(X = 3, Y = 4)}{\mathsf{P}(Y = 4)} = \frac{\mathsf{P}(s_3)}{\mathsf{P}(s_3) + \mathsf{P}(s_4)} = \frac{1}{4}.$$

(b) (5 points) $\mathsf{P}\left(X + Y \le 5 \,\middle|\, X \le 3\right) =?$

$\frac{1}{2}$

Likewise,

$$\mathsf{P}\left(X + Y \le 5 \,\middle|\, X \le 3\right) = \frac{\mathsf{P}(X + Y \le 5, X \le 3)}{\mathsf{P}(X \le 3)} = \frac{\mathsf{P}(s_1) + \mathsf{P}(s_2)}{\mathsf{P}(s_1) + \mathsf{P}(s_2) + \mathsf{P}(s_3)} = \frac{1}{2}.$$

(c) (5 points) Compute $\mathbb{E}[2X+Y] =?$

$\frac{51}{5}$

From linearity of expectation, it follows that $\mathbb{E}[2X+Y] = 2\mathbb{E}[X] + \mathbb{E}[Y]$ so we focus on computing $\mathbb{E}[X]$ and $\mathbb{E}[Y]$. These are immediately obtained from the definition of expected value, that is

$$\mathbb{E}[X] = \sum_{i=1}^{4} i \times \mathrm{P}(X=i) = 1 \times \mathrm{P}(s_1) + 2 \times \mathrm{P}(s_2) + 3 \times \mathrm{P}(s_3) + 4 \times \mathrm{P}(s_4) = \frac{33}{10},$$

$$\mathbb{E}[Y] = \sum_{i \in \{2,4\}} i \times \mathrm{P}(Y=i) = 2 \times [\mathrm{P}(s_1) + \mathrm{P}(s_2)] + 4 \times [\mathrm{P}(s_3) + \mathrm{P}(s_4)] = \frac{18}{5}.$$

All in all, the desired result is

$$\mathbb{E}[2X+Y] = 2\mathbb{E}[X] + \mathbb{E}[Y] = 2 \times \frac{33}{10} + \frac{18}{5} = \frac{51}{5}.$$

(d) (5 points) $\mathbb{E}\left[X \,\middle|\, Y\right](s_3) =?$

$\frac{15}{4}$

Recall that the conditional expectation $\mathbb{E}\left[X \,\middle|\, Y\right]$ is a function of the random variable Y, and Y itself (as any other random variable) is a function of the elements in the sample space S. Since $Y(s_3) = 4$, one has that $\mathbb{E}\left[X \,\middle|\, Y\right](s_3) = \mathbb{E}\left[X \,\middle|\, Y=4\right]$. Now, from the definition of conditional expectation

$$\mathbb{E}\left[X \,\middle|\, Y=4\right] = \sum_{i=1}^{4} i \times \mathrm{P}\left(X=i \,\middle|\, Y=4\right).$$

Proceeding as in part (a), the conditional pmf of interest is immediately obtained as

$$\mathrm{P}\left(X=1 \,\middle|\, Y=4\right) = 0, \quad \mathrm{P}\left(X=2 \,\middle|\, Y=4\right) = 0,$$

$$\mathrm{P}\left(X=3 \,\middle|\, Y=4\right) = \frac{1}{4}, \quad \mathrm{P}\left(X=4 \,\middle|\, Y=4\right) = \frac{3}{4}.$$

Putting all these pieces together, the final result is

$$\mathbb{E}\left[X \,\middle|\, Y=4\right] = 3 \times \frac{1}{4} + 4 \times \frac{3}{4} = \frac{15}{4}.$$

5. (10 points) Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S_X = \{1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/4 & 3/4 \\ 4/5 & 1/5 \end{pmatrix}.$$

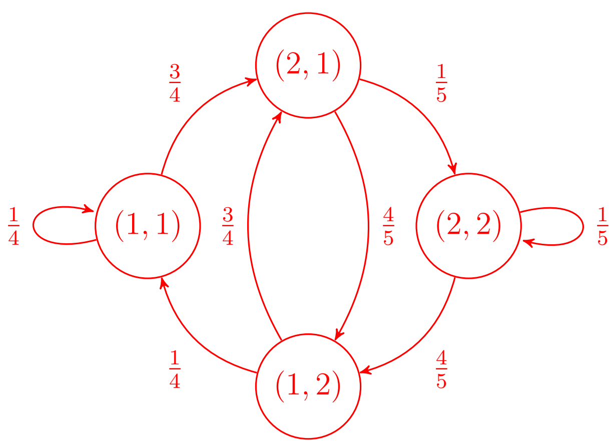
Define the state $Y_n := (X_n, X_{n-1})$ and the corresponding Markov chain $Y_{\mathbb{N}} = Y_1, Y_2, \ldots, Y_n, \ldots$ with state space $S_Y = \{(1,1), (1,2), (2,1), (2,2)\}$.

Draw the state transition diagram of $Y_{\mathbb{N}}$.

From the definition of the state $Y_n := (X_n, X_{n-1})$ it is key to notice that some transitions are impossible, for instance $(1,1) \to (1,2)$, or $(2,1) \to (2,1)$ just to name a couple. For the feasible transitions, the probabilities are readily obtained from the entries in $\mathbf{P}$. For example,

$$\mathrm{P}\left(Y_n = (2,1) \,\middle|\, Y_{n-1} = (1,1)\right) = \mathrm{P}\left(X_n = 2 \,\middle|\, X_{n-1} = 1\right) = P_{12} = \frac{3}{4}.$$

The complete state transition diagram of $Y_{\mathbb{N}}$ follows.



6. Suppose customers can arrive to a service station at times $n = 0, 1, 2, \ldots$ In any given period, independent of everything else, there is one arrival with probability p, and there is no arrival with probability $1 - p$. Customers are served one-at-a-time on a first-come-first-served basis. If at the time of an arrival, there are no customers present, then the arriving customer immediately enters service. Otherwise, the arrival joins the back of the queue.

Assume that service times are i.i.d. geometric random variables (each with parameter q) that are independent of the arrival process. So, $\mathrm{P}(\text{Service time} = \ell) = (1-q)^{\ell-1}q$, for $\ell = 1, 2, \ldots$ Note that a customer who enters service in time n can complete service, at the earliest, in time $n+1$ (in which case her service time is 1). Upon a service completion, the just-served customer will depart the system with probability α, or will immediately rejoin the back of the queue with probability $1 - \alpha$.

In a time period n, events happen in the following order: (i) arrivals, if any, occur; (ii) service completions followed by departures or rejoins, if any, occur; and (iii) service begins on the next customer if there are customers present in the system.

Let X_n denote the number of customers at the station at the end of time period n. Note that X_n includes both customers waiting as well as any customer being served, and that the random process $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$.

(a) (4 points) Determine the transition probabilities P_{0j} for all $j \geq 0$.

$$P_{00} = 1 - p,\ P_{01} = p,\ P_{0j} = 0,\ j > 1$$

If the present state is $X_n = 0$ (empty station), then there are only two possible transitions:

1) If an arrival does not occur at time instant $n+1$ then $X_{n+1} = 0$. Note that because $X_n = 0$, then it is also impossible that some user exits the system or rejoins the queue after being served because the system was empty. Then we conclude that $P_{00} = 1 - p$.
2) If an arrival occurs at time instant $n + 1$ then $X_{n+1} = 1$. Then we conclude that $P_{01} = p$.

All other transition probabilities from state 0 are null, namely $P_{0j} = 0,\ j > 1$.

(b) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j = i$.

$$P_{ii} = pq\alpha + (1-p)(1-q\alpha),\ i > 0$$

Suppose the present state is $X_n = i,\ i > 0$. If an arrival does not occur at time instant $n + 1$ and the user being served does not complete service, then $X_{n+1} = i$. This will happen with probability $(1-p)(1-q)$. Furthermore, if an arrival does not occur at time instant $n+1$ and the user being served completes service but rejoins the back of the queue, then also $X_{n+1} = i$. This will happen with probability $(1-p)q(1-\alpha)$. Finally, if an arrival occurs at time instant $n + 1$ and the user currently being served exits the system after completing service, then $X_{n+1} = i$ as well. This will happen with probability $pq\alpha$. The conclusion is that $P_{ii} = pq\alpha + (1-p)(1-q\alpha)$.

(c) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j > i$.

$$P_{i,i+1} = p(1 - q\alpha),\ P_{ij} = 0,\ i > 0 \text{ and } j > i + 1$$

Suppose the present state is $X_n = i$, $i > 0$. If an arrival occurs at time instant $n + 1$ and the user being served does not complete service, then $X_{n+1} = i + 1$. This will happen with probability $p(1 - q)$. Furthermore, if an arrival occurs at time instant $n + 1$ and the user being served completes service but rejoins the back of the queue, then also $X_{n+1} = i + 1$. This will happen with probability $pq(1 - \alpha)$. The conclusion is that $P_{i,i+1} = p(1 - q\alpha)$.

All other transition probabilities from state $i > 0$ to $j > i + 1$ are null.

(d) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $0 \leq j < i$.

$$P_{i,i-1} = (1 - p)q\alpha,\ P_{ij} = 0,\ i > 0 \text{ and } 0 \leq j < i - 1$$

Suppose the present state is $X_n = i$, $i > 0$. If an arrival does not occur at time instant $n + 1$ and the user exits the system after being served, then $X_{n+1} = i - 1$. This will happen with probability $(1 - p)q\alpha$. The conclusion is that $P_{i,i-1} = (1 - p)q\alpha$.

All other transition probabilities from state $i > 0$ to $0 \leq j < i - 1$ are null.