ECE440 - Introduction to Random Processes

Midterm Exam

November 7, 2016

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 103, extra points are bonus points).
- Duration: 75 minutes.
- This exam has 12 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:_____SOLUTIONS_____

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	22		4.	14	
2.	14		5.	15	
3.	20		6.	18	
			Total	103	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/5 & 4/5 \\ 2/5 & 3/5 \end{pmatrix}$$

and initial distribution $P(X_0 = 1) = 3/4$ and $P(X_0 = 2) = 1/4$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 9/25 & 16/25 \\ 8/25 & 17/25 \end{pmatrix} = \begin{pmatrix} 0.36 & 0.64 \\ 0.32 & 0.68 \end{pmatrix}.$$

(a) (2 points) $P\left(X_4 = 2 \,\middle|\, X_2 = 1, X_1 = 2, X_0 = 2\right) =?$

$$\boxed{\frac{16}{25}}$$

From the Markov property it follows that

$$P\left(X_4 = 2 \,\middle|\, X_2 = 1, X_1 = 2, X_0 = 2\right) = P\left(X_4 = 2 \,\middle|\, X_2 = 1\right) = P_{12}^2 = \frac{16}{25}.$$

(b) (3 points) $P\left(X_2 = 1, X_0 = 1\right) =?$

$$\boxed{\frac{27}{100}}$$

From the definition of conditional probability one finds

$$P(X_2 = 1, X_0 = 1) = P\left(X_2 = 1 \,\middle|\, X_0 = 1\right) P(X_0 = 1) = P_{11}^2 \times \frac{3}{4} = \frac{27}{100}$$

(c) (3 points) $P(X_2 = 2) =?$

$$\boxed{\frac{13}{20}}$$

From the law of total probability (conditioning on X_0, noting that $P(X_0 = 3) = 1$), one has

$$P(X_2 = 2) = \sum_{i=1}^{2} P\left(X_2 = 2 \,\middle|\, X_0 = i\right) P(X_0 = i) = P_{12}^2 \times P(X_0 = 1) + P_{22}^2 \times P(X_0 = 2) = \frac{13}{20}.$$

(d) (10 points) Compute the stationary distribution of $X_{\mathbb{N}}$.

$$\boldsymbol{\pi} = \left[\frac{1}{3}, \frac{2}{3}\right]^T$$

The unique stationary distribution $\boldsymbol{\pi} = [\pi_1, \pi_2]^T$ (the Markov chain is ergodic) satisfies

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 1/5 & 2/5 \\ 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \quad \pi_1 + \pi_2 = 1.$$

Solving the linear system yields $\boldsymbol{\pi} = [1/3, 2/3]^T$.

(e) (4 points) Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{X_i = 2\}$$

and provide justification for the existence of the limit.

$$\frac{2}{3}$$

Since the Markov chain is ergodic, the long-run fraction of time spent in state 2 is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{X_i = 2\} = \pi_2 = \frac{2}{3}, \quad \text{a. s.}$$

2. Suppose X and Y are random variables with joint probability mass function given by

$$\begin{aligned}
P(X = 1, Y = 2) &= 1/6, \\
P(X = 1, Y = 3) &= 1/2, \\
P(X = 2, Y = 2) &= p, \\
P(X = 2, Y = 3) &= 1/6.
\end{aligned}$$

(a) (2 points) What is the value of p? Explain.

$$\frac{1}{6}$$

From the axioms, the probability of the universe (i.e., the sure event) is $P(S) = 1$. Hence

$$1 = P(S) = \sum_{y=2}^{3} \sum_{x=1}^{2} P(X = x, Y = y) = p + \frac{5}{6} \Rightarrow p = \frac{1}{6}.$$

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(b) (3 points) $\mathrm{P}\,(Y = 3) =$?

$$\frac{2}{3}$$

The marginal probability $\mathrm{P}\,(Y = 3)$ is given by

$$\mathrm{P}\,(Y = 3) = \sum_{x=1}^{2} \mathrm{P}\,(X = x, Y = 3) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}.$$

(c) (4 points) $\mathrm{P}\left(X = 1\,\middle|\,Y = 3\right) =$?

$$\frac{3}{4}$$

From the definition of conditional probability

$$\mathrm{P}\left(X = 1\,\middle|\,Y = 3\right) = \frac{\mathrm{P}\,(X = 1, Y = 3)}{\mathrm{P}\,(Y = 3)} = \frac{1/2}{2/3} = \frac{3}{4}.$$

(d) (5 points) $\mathbb{E}\left[X\,\middle|\,Y = 3\right] =$?

$$\frac{5}{4}$$

The conditional pmf of X given $Y = 3$ is

$$\mathrm{P}\left(X = 1\,\middle|\,Y = 3\right) = \frac{3}{4},$$
$$\mathrm{P}\left(X = 2\,\middle|\,Y = 3\right) = \frac{1}{4}.$$

Hence, the conditional expectation is $\mathbb{E}\left[X\,\middle|\,Y = 3\right] = 1 \times \frac{3}{4} + 2 \times \frac{1}{4} = \frac{5}{4}$.

3. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0 & 0 & 0.8 & 0 \\ 0.3 & 0.5 & 0.2 & 0 & 0 \\ 0.6 & 0.3 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) (8 points) Draw the corresponding state transition diagram.

The state transition diagram is
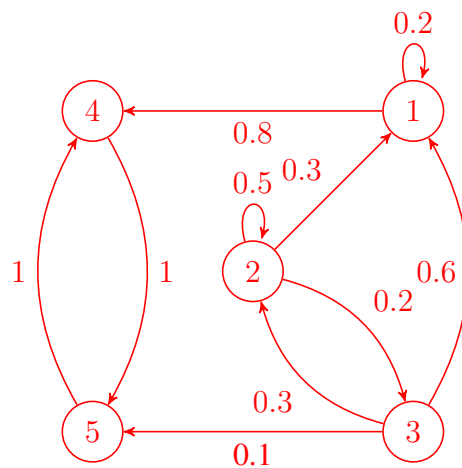
(b) (3 points) Is state 4 aperiodic? Explain.

No, state 4 has period 2

From the state transition diagram, it is apparent that $P_{44}^{2n+1} = 0$ and $P_{44}^{2n} = 1$ so $\gcd\{2, 4, \ldots\} = 2$. The conclusion is that state 4 has period 2.

(c) (4 points) $\lim_{n \to \infty} \mathbf{P}\left(X_n = 3 \mid X_0 = 1\right) = ?$

0

Since state 3 belongs to one of the transient classes $\mathcal{T}_2 = \{2, 3\}$, visits to this state eventually stop (almost surely). Accordingly, the limiting probability

$$\lim_{n \to \infty} \mathbf{P}\left(X_n = 3 \mid X_0 = 1\right) = \lim_{n \to \infty} P_{13}^n = 0.$$

It also suffices to see that state 3 is not accessible from 1 to argue the limit is 0.

5

(d) (5 points) Explain in a few sentences why, in the long run, you would expect to find the Markov chain in state 5 half of the time.

The Markov chain has three communication classes, one recurrent $\mathcal{R} = \{4, 5\}$ and the others $\mathcal{T}_1 = \{1\}$ and $\mathcal{T}_2 = \{1, 2, 3\}$ are transient. So regardless of the initial condition, in the long run the process will end up in $\mathcal{R}$ and stay there forever. Once the Markov chain hits $\mathcal{R}$, the sequence of state visits will be $\ldots 4, 5, 4, 5, 4, 5, \ldots$, which implies the process will be in state 5 half of the time.

4. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables with mean $\mathbb{E}[X_1] = 5$ and variance $\text{var}[X_1] = 1$. Consider the following random variables

$$S_n := \sum_{i=1}^{n} X_i,$$

$$\bar{X}_n := \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{S_n}{n},$$

$$Z_n := \frac{\sum_{i=1}^{n} X_i - 5n}{\sqrt{n}} = \frac{S_n - 5n}{\sqrt{n}}.$$

Suppose that $n = 100$ is large enough so that limiting behaviors become apparent.

(a) (12 points) Sketch the probability density functions (pdfs) of S_{100}, $\bar{X}_{100}$, and Z_{100}, superimposing the three plots in the set of axes provided below. Only rough, qualitative depictions are required, focusing on the notable values where the pdfs are centered, and their relative widths and heights. Justify your answer and label your plots.
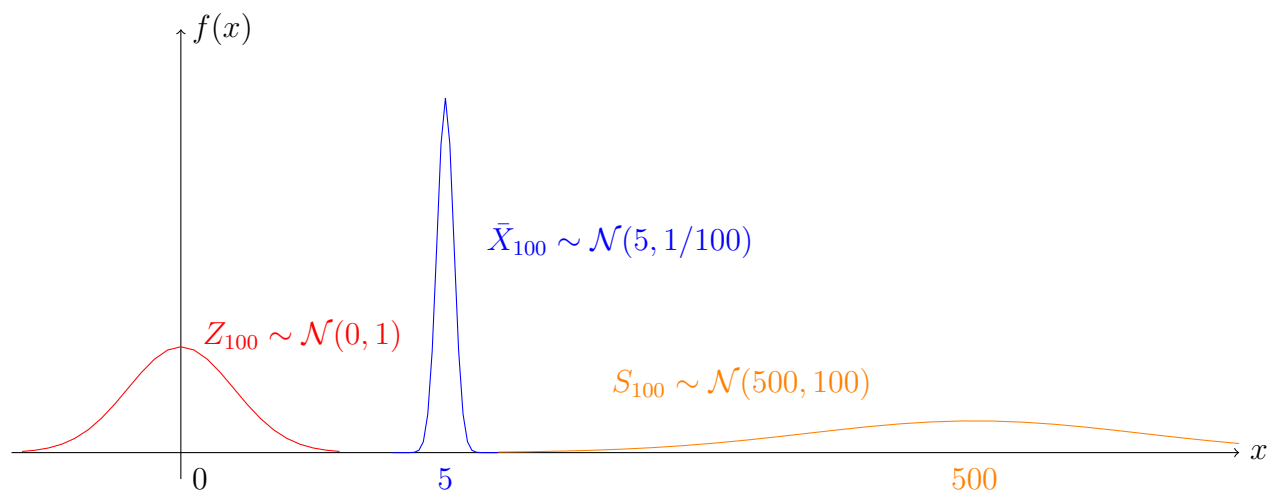
From the Central Limit Theorem it follows that (approximately)

$$S_{100} \sim \mathcal{N}(500, 100),$$
$$\bar{X}_{100} \sim \mathcal{N}(5, 1/100),$$
$$Z_{100} \sim \mathcal{N}(0, 1).$$

The corresponding sketches of the pdfs are depicted above (not to scale).

(b) (2 points) Will your plots fundamentally change if the common distribution of the random variables $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ differs from that of (a), while the mean and variance remain the same (that is, one still has $\mathbb{E}[X_1] = 5$ and $\text{var}[X_1] = 1$)? Explain.

No

The Central Limit Theorem asserts that the limiting distribution of sums of i.i.d. random variables will be Normal, regardless of the distribution of the summands (this universality make the result truly remarkable). So even if we consider a different distribution from the one in (a), the plots will remain roughly the same provided the values of $\mathbb{E}[X_1]$ and $\text{var}[X_1]$ are kept unchanged.

5. Old McDonald had a farm, but now he runs a monster-sighting business in Loch Ness, Scotland. Every day, he is unable to run the boat tour due to bad weather with probability p, independently of all other days. McDonald works every day except the bad-weather days, which he takes as holiday.

Let Y be the number of consecutive days McDonald has to work between bad-weather days. Let X be the total number of customers who go on McDondald's boat trip in this period of Y days. Conditioned on Y, the distribution of X is $X \mid Y \sim \text{Poisson}(\lambda Y)$, meaning the conditional probability mass function is given by

$$P\left(X = x \mid Y = y\right) = \frac{(\lambda y)^x e^{-\lambda y}}{x!}.$$

(a) (6 points) $\mathbb{E}[Y] =$? [Hint: Argue that the random variable $Z := Y + 1 \sim \text{Geometric}(p)$]

$$\frac{1-p}{p}$$

Consider each potential workday as an independent trial with a binary outcome. View actual workdays as "failures" happening with probability $1 - p$, while "successes" correspond to bad-weather days that occur with probability p. If Y is the number of consecutive workdays between bad-weather days, then $Z := Y + 1 \sim \text{Geometric}(p)$ (counting Y consecutive trial failures before one success). Since $\mathbb{E}[Z] = 1/p$, using the linearity of expectation we find

$$\mathbb{E}[Z] = \mathbb{E}[Y] + 1 = \frac{1}{p} \Rightarrow \mathbb{E}[Y] = \frac{1-p}{p}.$$

(b) (4 points) $\mathbb{E}\left[X \mid Y = y\right] =$?

$$\lambda y$$

Conditioned on Y, the distribution of X is $X \mid Y \sim \text{Poisson}(\lambda Y)$. Hence $\mathbb{E}\left[X \mid Y = y\right] = \lambda y$.

(c) (5 points) $\mathbb{E}[X] = ?$

$$\frac{\lambda(1-p)}{p}$$

Applying iterated expectations after conditioning on $Y = y$ yields

$$\mathbb{E}[X] = \sum_{y=0}^{\infty} \mathbb{E}[X \mid Y = y] \, \mathrm{P}(Y = y) = \sum_{y=0}^{\infty} \lambda y \mathrm{P}(Y = y) = \lambda \mathbb{E}[Y].$$

In (a) we obtained $\mathbb{E}[Y] = (1-p)/p$ so the mean of X is $\mathbb{E}[X] = \lambda(1-p)/p$.

6. Suppose that X_n is the amount of inventory in a store at the beginning of the time period n. At the beginning of each period, the inventory decreases by one unit provided the inventory level is positive, and otherwise the inventory remains at 0 until the end of the period. At the end of period n, the inventory is replenished by an amount R_n, where $R_{\mathbb{N}} = R_0, R_1, \ldots, R_n, \ldots$ is an i.i.d. sequence (independent of X_0) of non-negative integer-valued random variables, each with probability mass function $q(\cdot)$; i.e.,

$$\mathrm{P}(R_0 = i) = q(i), \quad i = 0, 1, 2, \ldots$$

Under the preceding assumptions, $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$. For $n \geq 0$, the inventory level at the beginning of period $n + 1$ is given by

$$X_{n+1} = \begin{cases} X_n - 1 + R_n, & X_n > 0 \\ R_n, & X_n = 0 \end{cases}.$$

(a) (6 points) Determine the transition probabilities P_{ij} for $i = 0$ and all $j \geq 0$. Show that $P_{0j} = P_{1j}$ for all $j \geq 0$.

$$P_{0j} = P_{1j} = q(j)$$

For $i = 0$, the transition probabilities are given by (note X_n and R_n are independent)

$$\mathrm{P}(X_{n+1} = j \mid X_n = 0) = \mathrm{P}(R_n = j \mid X_n = 0) = \mathrm{P}(R_n = j) = q(j).$$

Notice that if $X_n = 1$, one also has $X_{n+1} = R_n$. This implies $P_{0j} = P_{1j}$ for all $j \geq 0$.

(b) (6 points) Determine the transition probabilities P_{ij} for all $i \geq 2$ and $j \geq i - 1$.

$$P_{ij} = q(j + 1 - i)$$

For $i \geq 2$, the transition probabilities are given by

$$\mathbf{P}\left(X_{n+1} = j \mid X_n = i\right) = \mathbf{P}\left(X_n - 1 + R_n = j \mid X_n = i\right) = \mathbf{P}\left(i - 1 + R_n = j \mid X_n = i\right)$$
$$= \mathbf{P}\left(R_n = j + 1 - i\right) = q(j + 1 - i).$$

Notice that for $j \geq i - 1$, then $j + 1 - i \geq 0$ and hence $P_{ij} = q(j + 1 - i)$

(c) (6 points) Determine the transition probabilities P_{ij} for all $i \geq 2$ and $0 \leq j < i - 1$.

$$P_{ij} = 0$$

For $0 \leq j < i - 1$, then $j + 1 - i < 0$ and hence $P_{ij} = q(j + 1 - i) = 0$ because the R_n are non-negative integer-valued random variables.