ECE440 - Introduction to Random Processes

Midterm Exam

November 6, 2017

Instructions:

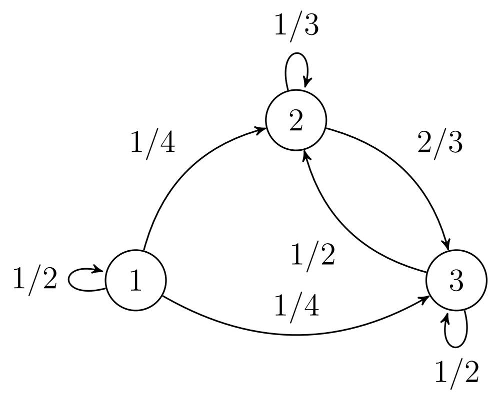
- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 101, extra point is a bonus point).
- Duration: 75 minutes.
- This exam has 10 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:__________________________________

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	28		5.	12	
2.	12		6.	18	
3.	8		7.	8	
4.	15				
			Total	101	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3\}$, state transition diagram



and initial distribution $\mathrm{P}(X_0 = 1) = 1$, $\mathrm{P}(X_0 = 2) = 0$ and $\mathrm{P}(X_0 = 3) = 0$. To spare you of pointless calculations, if needed you may use that the two-step transition probability matrix is

$$\mathbf{P}^2 = \begin{pmatrix} 1/4 & 1/3 & 5/12 \\ 0 & 4/9 & 5/9 \\ 0 & 5/12 & 7/12 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.33 & 0.42 \\ 0 & 0.44 & 0.56 \\ 0 & 0.42 & 0.58 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left(X_7 = 1 \,\middle|\, X_6 = 3, X_4 = 2\right) =?$

(b) (3 points) $\mathrm{P}(X_1 = 3, X_0 = 1) =?$

(c) (3 points) $\mathrm{P}\left(X_2 = 2\right) = ?$

(d) (4 points) $\mathbb{E}\left[X_2\right] = ?$

(e) (4 points) $\mathbb{E}\left[X_3 \,\middle|\, X_1 = 2\right] = ?$

(f) (8 points) Compute the stationary distribution of $X_{\mathbb{N}}$. (Hint: one of the limiting probabilities requires no calculation.)

(g) (4 points) Consider multiple independent realizations of $X_{\mathbb{N}}$, all with the same initial distribution as specified earlier in this problem. Different realizations are indexed by i, so that $X_{n,i}$ denotes the state of the ith realization at time n. Calculate

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} X_{2,i}$$

and provide justification for the existence of the limit.

2. (12 points) Recall that if a random variable Z is Poisson distributed with parameter λ, then

$$\mathrm{P}\left(Z=z\right)=\frac{e^{-\lambda}\lambda^{z}}{z!},\quad \mathbb{E}\left[Z\right]=\mathrm{var}\left[Z\right]=\lambda,\quad \mathbb{E}\left[Z^2\right]=\mathrm{var}\left[Z\right]+\left(\mathbb{E}\left[Z\right]\right)^2=\lambda+\lambda^2.$$

Suppose that X is a non-negative discrete random variable with

$$\mathbb{E}\left[X\right]=\mu \ \text{ and } \ \mathrm{var}\left[X\right]=\mathbb{E}\left[X^2\right]-\mu^2=\sigma^2.$$

Let Y be a random variable which, conditioned on $X=x$, has the Poisson distribution with parameter βx, that is

$$\mathrm{P}\left(Y=y \,\middle|\, X=x\right)=\frac{e^{-\beta x}(\beta x)^{y}}{y!}.$$

Compute $\mathrm{var}\left[Y\right]$ and write your result in terms of μ, σ^2, and β.

3. (8 points) Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} p & 1-p & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

For what values of $0 \leq p \leq 1$ is the Markov chain ergodic? Justify your answer.

4. Suppose you toss a penny and a nickel. For both tosses assume that a "Head" outcome is mapped into 1 and a "Tail" into 0. Let X and Y be binary random variables recording the outcomes of the penny and nickel tosses, respectively. The joint probability mass function (pmf) of X and Y is given by

$$\mathrm{P}\left(X=0, Y=0\right) = 3/8, \quad \mathrm{P}\left(X=0, Y=1\right) = 1/8$$
$$\mathrm{P}\left(X=1, Y=0\right) = 1/8, \quad \mathrm{P}\left(X=1, Y=1\right) = 3/8.$$

(a) (5 points) Are both coins fair?

(b) (5 points) Are the coin tosses independent?

(c) (5 points) $\mathrm{P}\left(Y = 1 \,\middle|\, X = 0\right) =?$

5. Let $Y \sim \mathcal{N}(\mu, \sigma^2)$ be a Normal distributed random variable with $\mathbb{E}[Y] = \mu$ and $\text{var}[Y] = \sigma^2$. Denote by Φ the complementary cumulative distribution function (ccdf) of a standard Normal distributed random variable, that is for $X \sim \mathcal{N}(0,1)$ then

$$\Phi(x) = \mathrm{P}(X \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-x^2/2} dx.$$

(a) (4 points) Obtain an expression for $\mathrm{P}(Y \geq y)$ in terms of y, μ, σ and Φ. You can use (without proof) that $Z = aY + b$ is Normal distributed, for arbitrary scalar constants $a \neq 0$ and b.

(b) (8 points) There are 1000 resistors in a box labeled 10 ohms. Due to manufacturing fluctuations, however, the resistance of the resistors are somewhat different. Assume that the resistance can be modeled as a random variable with a mean of 10 ohms and a variance of 1 ohm^2. If 100 resistors are chosen from the box and they are connected in series (so the resistances add together), what is the approximate probability that the total resistance will exceed 1030 ohms? Express your result in terms of Φ, and justify your approximation.

6. Consider a sequence of i.i.d. random variables $Y_{\mathbb{N}} = Y_1, Y_2, \ldots, Y_n, \ldots$ such that $\mathrm{P}(Y_1 = 0) = 0.2$, $\mathrm{P}(Y_1 = 1) = 0.4$, and $\mathrm{P}(Y_1 = 2) = 0.4$. Let $X_0 = 0$ and define

$$X_n = \max\{Y_1, \ldots, Y_n\}, \quad n \geq 1.$$

(a) (14 points) Show that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain, specify its state space and determine the transition probability matrix.

(b) (4 points) Is the Markov chain irreducible? Explain.

7. (8 points) Let X be a uniform random variable on $\{-1, 0, 1\}$, meaning $P(X = -1) = P(X = 0) = P(X = 1) = 1/3$. Let $Y = X^2$. Are X and Y uncorrelated?