ECE440 - Introduction to Random Processes

Midterm Exam

November 6, 2017

Instructions:

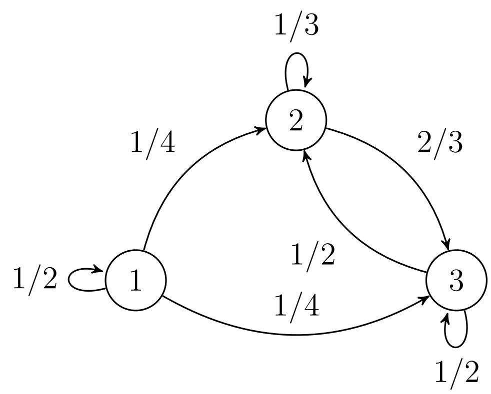
- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 101, extra point is a bonus point).
- Duration: 75 minutes.
- This exam has 10 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name: SOLUTIONS

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	28		5.	12	
2.	12		6.	18	
3.	8		7.	8	
4.	15				
			Total	101	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3\}$, state transition diagram



and initial distribution $\mathrm{P}(X_0 = 1) = 1$, $\mathrm{P}(X_0 = 2) = 0$ and $\mathrm{P}(X_0 = 3) = 0$. To spare you of pointless calculations, if needed you may use that the two-step transition probability matrix is

$$\mathbf{P}^2 = \begin{pmatrix} 1/4 & 1/3 & 5/12 \\ 0 & 4/9 & 5/9 \\ 0 & 5/12 & 7/12 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.33 & 0.42 \\ 0 & 0.44 & 0.56 \\ 0 & 0.42 & 0.58 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left(X_7 = 1 \,\middle|\, X_6 = 3, X_4 = 2\right) = ?$

0

From the Markov property it follows that

$$\mathrm{P}\left(X_7 = 1 \,\middle|\, X_6 = 3, X_4 = 2\right) = \mathrm{P}\left(X_7 = 1 \,\middle|\, X_6 = 3\right) = P_{31} = 0.$$

(b) (3 points) $\mathrm{P}(X_1 = 3, X_0 = 1) = ?$

$\frac{1}{4}$

From the definition of conditional probability one finds

$$\mathrm{P}(X_1 = 3, X_0 = 1) = \mathrm{P}\left(X_1 = 3 \,\middle|\, X_0 = 1\right)\mathrm{P}(X_0 = 1) = P_{13} \times 1 = \frac{1}{4}.$$

(c) (3 points) $\mathrm{P}(X_2 = 2) = ?$

$$\frac{1}{3}$$

From the law of total probability [conditioning on X_0, noting that $\mathrm{P}(X_0 = 1) = 1$], one has

$$\mathrm{P}(X_2 = 2) = \sum_{i=1}^{3} \mathrm{P}\left(X_2 = 2 \,\middle|\, X_0 = i\right) \mathrm{P}(X_0 = i) = \sum_{i=1}^{3} P_{i2}^2 \mathrm{P}(X_0 = i) = P_{12}^2 \times 1 = \frac{1}{3}.$$

(d) (4 points) $\mathbb{E}[X_2] = ?$

$$\frac{13}{6}$$

The unconditional pmf of X_2 is (note that $\mathrm{P}(X_0 = 1) = 1$)

$$\mathrm{P}(X_2 = j) = \sum_{i=1}^{3} \mathrm{P}\left(X_2 = j \,\middle|\, X_0 = i\right) \mathrm{P}(X_0 = i) = \sum_{i=1}^{3} P_{ij}^2 \mathrm{P}(X_0 = i) = P_{1j}^2, \quad j = 1, 2, 3.$$

Hence,

$$\mathrm{P}(X_2 = 1) = \frac{1}{4}, \quad \mathrm{P}(X_2 = 2) = \frac{1}{3}, \quad \text{and} \quad \mathrm{P}(X_2 = 3) = \frac{5}{12}$$

and so the expectation is $\mathbb{E}[X_2] = 1 \times \frac{1}{4} + 2 \times \frac{1}{3} + 3 \times \frac{5}{12} = \frac{13}{6}$.

(e) (4 points) $\mathbb{E}\left[X_3 \,\middle|\, X_1 = 2\right] = ?$

$$\frac{23}{9}$$

Using the definition of conditional expectation, one obtains

$$\begin{aligned}\mathbb{E}\left[X_3 \,\middle|\, X_1 = 2\right] &= \sum_{i=1}^{3} i \times \mathrm{P}\left(X_3 = i \,\middle|\, X_1 = 2\right) = \sum_{i=1}^{3} i \times P_{2i}^2 \\ &= 1 \times 0 + 2 \times \frac{4}{9} + 3 \times \frac{5}{9} = \frac{23}{9}.\end{aligned}$$

(f) (8 points) Compute the stationary distribution of $X_{\mathbb{N}}$. (Hint: one of the limiting probabilities requires no calculation.)

$$\boldsymbol{\pi} = \left[0,\ \frac{3}{7},\ \frac{4}{7}\right]^T$$

State 1 is transient and so $\pi_1 = 0$. We can thus focus on the ergodic component $\mathcal{E} = \{2, 3\}$ which has transition probability matrix

$$\mathbf{P}_{\mathcal{E}} = \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix}.$$

The unique stationary distribution $\boldsymbol{\pi}_{\mathcal{E}} = [\pi_2, \pi_3]^T$ of this reduced ergodic Markov chain satisfies

$$\begin{pmatrix} \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{pmatrix} \begin{pmatrix} \pi_2 \\ \pi_3 \end{pmatrix}, \quad \pi_2 + \pi_3 = 1.$$

Solving the linear system yields $\boldsymbol{\pi}_{\mathcal{E}} = [3/7, 4/7]^T$, which implies $\boldsymbol{\pi} = [0, 3/7, 4/7]^T$.

(g) (4 points) Consider multiple independent realizations of $X_{\mathbb{N}}$, all with the same initial distribution as specified earlier in this problem. Different realizations are indexed by i, so that $X_{n,i}$ denotes the state of the ith realization at time n. Calculate

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} X_{2,i}$$

and provide justification for the existence of the limit.

$$\frac{13}{6}$$

Because all realizations of the Markov chain are independent and initialized with the same distribution, then $X_{2,\mathbb{N}} = X_{2,1}, X_{2,2}, \ldots, X_{2,i}, \ldots$ is an i.i.d. sequence. By the strong law of large numbers the limit exists and is equal to

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} X_{2,i} = \mathbb{E}\left[X_2\right] = \frac{13}{6}, \quad \text{w.p. } 1.$$

2. (12 points) Recall that if a random variable Z is Poisson distributed with parameter λ, then

$$\mathrm{P}(Z = z) = \frac{e^{-\lambda}\lambda^z}{z!}, \quad \mathbb{E}[Z] = \mathrm{var}[Z] = \lambda, \quad \mathbb{E}\left[Z^2\right] = \mathrm{var}[Z] + (\mathbb{E}[Z])^2 = \lambda + \lambda^2.$$

Suppose that X is a non-negative discrete random variable with

$$\mathbb{E}[X] = \mu \ \text{ and } \ \mathrm{var}[X] = \mathbb{E}\left[X^2\right] - \mu^2 = \sigma^2.$$

Let Y be a random variable which, conditioned on $X = x$, has the Poisson distribution with parameter βx, that is

$$\mathrm{P}\left(Y = y \,\middle|\, X = x\right) = \frac{e^{-\beta x}(\beta x)^y}{y!}.$$

Compute $\mathrm{var}[Y]$ and write your result in terms of μ, σ^2, and β.

$\beta^2\sigma^2 + \beta\mu$

Conditioned on X, the distribution of Y is $Y \,|\, X \sim \text{Poisson}(\beta X)$. Hence, $\mathbb{E}_Y\left[Y \,\middle|\, X\right] = \beta X$ and $\mathrm{var}_Y\left[Y \,\middle|\, X\right] = \beta X$. Using the conditional variance formula we find

$$\begin{aligned}\mathrm{var}[Y] &= \mathbb{E}_X\left[\mathrm{var}_Y\left[Y \,\middle|\, X\right]\right] + \mathrm{var}_X\left[\mathbb{E}_Y\left[Y \,\middle|\, X\right]\right] \\ &= \mathbb{E}[\beta X] + \mathrm{var}[\beta X] \\ &= \beta\mathbb{E}[X] + \beta^2\mathrm{var}[X] = \beta\mu + \beta^2\sigma^2.\end{aligned}$$

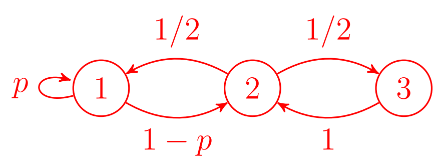
3. (8 points) Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} p & 1-p & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

For what values of $0 \le p \le 1$ is the Markov chain ergodic? Justify your answer.

$0 < p < 1$

The state transition diagram for the Markov chain is



For $p = 1$, state 1 is absorbing and $X_{\mathbb{N}}$ is not ergodic because it is reducible. For $p = 0$, all states have period 2 and likewise $X_{\mathbb{N}}$ is not ergodic. For all other values of $0 < p < 1$, the Markov chain is ergodic because it is irreducible, aperiodic (state 1 is aperiodic since $P_{11} = p > 0$), and positive recurrent (finite, irreducible Markov chain).

4. Suppose you toss a penny and a nickel. For both tosses assume that a "Head" outcome is mapped into 1 and a "Tail" into 0. Let X and Y be binary random variables recording the outcomes of the penny and nickel tosses, respectively. The joint probability mass function (pmf) of X and Y is given by

$$\begin{aligned} \mathrm{P}(X=0, Y=0) &= 3/8, \quad \mathrm{P}(X=0, Y=1) = 1/8 \\ \mathrm{P}(X=1, Y=0) &= 1/8, \quad \mathrm{P}(X=1, Y=1) = 3/8. \end{aligned}$$

(a) (5 points) Are both coins fair?

Yes

The marginal probability $\mathrm{P}(X=0)$ is given by

$$\mathrm{P}(X=0) = \sum_{y=0}^{1} \mathrm{P}(X=0, Y=y) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}.$$

Clearly, $\mathrm{P}(X=1) = 1 - \mathrm{P}(X=0) = \frac{1}{2}$. Likewise, we find that

$$\mathrm{P}(Y=0) = \sum_{x=0}^{1} \mathrm{P}(X=x, Y=0) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2} \Rightarrow \mathrm{P}(Y=1) = \frac{1}{2}.$$

The conclusion is that both coins are fair.

(b) (5 points) Are the coin tosses independent?

No

The coin tosses are dependent because

$$\mathrm{P}(X=0, Y=0) = \frac{3}{8} \neq \mathrm{P}(X=0) \times \mathrm{P}(Y=0) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

(c) (5 points) $\mathrm{P}\left(Y=1 \,\middle|\, X=0\right) = ?$

$\frac{1}{4}$

From the definition of conditional probability

$$\mathrm{P}\left(Y=1 \,\middle|\, X=0\right) = \frac{\mathrm{P}(Y=1, X=0)}{\mathrm{P}(X=0)} = \frac{1/8}{1/2} = \frac{1}{4}.$$

5. Let $Y \sim \mathcal{N}(\mu, \sigma^2)$ be a Normal distributed random variable with $\mathbb{E}[Y] = \mu$ and $\text{var}[Y] = \sigma^2$. Denote by Φ the complementary cumulative distribution function (ccdf) of a standard Normal distributed random variable, that is for $X \sim \mathcal{N}(0, 1)$ then

$$\Phi(x) = \mathrm{P}(X \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-x^2/2} dx.$$

(a) (4 points) Obtain an expression for $\mathrm{P}(Y \geq y)$ in terms of y, μ, σ and Φ. You can use (without proof) that $Z = aY + b$ is Normal distributed, for arbitrary scalar constants $a \neq 0$ and b.

$$\Phi\left(\frac{y - \mu}{\sigma}\right)$$

After centering and scaling the random variable Y, we obtain [note that $(Y - \mu)/\sigma \sim \mathcal{N}(0, 1)$]

$$\mathrm{P}(Y \geq y) = \mathrm{P}\left(\frac{Y - \mu}{\sigma} \geq \frac{y - \mu}{\sigma}\right) = \Phi\left(\frac{y - \mu}{\sigma}\right).$$

(b) (8 points) There are 1000 resistors in a box labeled 10 ohms. Due to manufacturing fluctuations, however, the resistance of the resistors are somewhat different. Assume that the resistance can be modeled as a random variable with a mean of 10 ohms and a variance of 1 ohm^2. If 100 resistors are chosen from the box and they are connected in series (so the resistances add together), what is the approximate probability that the total resistance will exceed 1030 ohms? Express your result in terms of Φ, and justify your approximation.

$$\Phi(3)$$

The resistors in the box have resistance R_i, which are modeled as i.i.d. random variables with $\mathbb{E}[R_i] = 10$ and $\text{var}[R_i] = 1$, for $i = 1, \ldots, 1000$. By virtue of the Central Limit Theorem, the total resistance $\bar{R}$ of the series connection has (approximately) a Normal distribution

$$\bar{R} = \sum_{i=1}^{100} R_i \sim \mathcal{N}(1000, 100).$$

We used that $\mathbb{E}\left[\bar{R}\right] = 100 \times \mathbb{E}[R_1] = 1000$ and $\text{var}\left[\bar{R}\right] = 100 \times \text{var}[R_1] = 100$ (since the R_i are independent). From the result in part (a), we immediately obtain

$$\mathrm{P}\left(\bar{R} \geq 1030\right) = \Phi\left(\frac{1030 - 1000}{\sqrt{100}}\right) = \Phi(3).$$

6. Consider a sequence of i.i.d. random variables $Y_{\mathbb{N}} = Y_1, Y_2, \ldots, Y_n, \ldots$ such that $\mathrm{P}(Y_1 = 0) = 0.2$, $\mathrm{P}(Y_1 = 1) = 0.4$, and $\mathrm{P}(Y_1 = 2) = 0.4$. Let $X_0 = 0$ and define

$$X_n = \max\{Y_1, \ldots, Y_n\}, \quad n \geq 1.$$

(a) (14 points) Show that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain, specify its state space and determine the transition probability matrix.

Since $X_0 = 0$, notice we can write for all $n \geq 1$

$$X_n = \max\{Y_1, \ldots, Y_n\} = \max\{X_{n-1}, Y_n\}.$$

So $X_{\mathbb{N}}$ is a Markov chain because we have expressed the state as $X_n = f(X_{n-1}, Y_n)$, where $Y_{\mathbb{N}}$ is an i.i.d. process independent of the initial condition. Moreover, since $Y_n \in \{0, 1, 2\}$ then the state space of $X_{\mathbb{N}}$ is $S = \{0, 1, 2\}$.

Regarding transition probabilities $P_{ij} = \mathrm{P}\left(X_n = j \,\middle|\, X_{n-1} = i\right)$, for $i = 0$ we have

$$P_{0j} = \mathrm{P}\left(\max\{X_{n-1}, Y_n\} = j \,\middle|\, X_{n-1} = 0\right) = \mathrm{P}(Y_n = j), \quad j = 0, 1, 2.$$

For $i = 1$, then

$$P_{1j} = \mathrm{P}\left(\max\{X_{n-1}, Y_n\} = j \,\middle|\, X_{n-1} = 0\right) = \begin{cases} 0, & j = 0, \\ \mathrm{P}(Y_n = 0) + \mathrm{P}(Y_n = 1), & j = 1, \\ \mathrm{P}(Y_n = 2), & j = 2 \end{cases}.$$

Finally, for $i = 2$ the transition probabilities are

$$P_{2j} = \mathrm{P}\left(\max\{X_{n-1}, Y_n\} = j \,\middle|\, X_{n-1} = 2\right) = \mathbb{I}\{j = 2\}, \quad j = 0, 1, 2.$$

In summary, the transition probability matrix is

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) (4 points) Is the Markov chain irreducible? Explain.

$X_{\mathbb{N}}$ is not irreducible.

State 2 is an absorbing state, hence it does not communicate with any other state. Accordingly, it is the sole member of the recurrent communication class $\mathcal{R} = \{2\}$. This implies the total number of classes is strictly greater than 1, meaning $X_{\mathbb{N}}$ is not irreducible.

7. (8 points) Let X be a uniform random variable on $\{-1, 0, 1\}$, meaning $\mathrm{P}(X = -1) = \mathrm{P}(X = 0) = \mathrm{P}(X = 1) = 1/3$. Let $Y = X^2$. Are X and Y uncorrelated?

Yes

The covariance of X and $Y = X^2$ is given by

$$\mathrm{cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \mathbb{E}\left[X^3\right] - \mathbb{E}[X]\,\mathbb{E}\left[X^2\right].$$

Computing the first and third moments of X we obtain

$$\mathbb{E}[X] = -1 \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3} = 0,$$
$$\mathbb{E}\left[X^3\right] = (-1)^3 \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3} = 0,$$

which is enough to conclude that $\mathrm{cov}[X, Y] = 0$.