ECE440 - Introduction to Random Processes

Midterm Exam

October 29, 2018

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 points.
- Duration: 90 minutes.
- This exam has 12 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name: SOLUTIONS

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	28		5.	16	
2.	14		6.	10	
3.	8		7.	16	
4.	8				
			Total	100	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/4 & 3/4 \\ 2/3 & 1/3 \end{pmatrix}$$

and initial distribution $\mathrm{P}(X_0 = 1) = 1/3$ and $\mathrm{P}(X_0 = 2) = 2/3$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 9/16 & 7/16 \\ 7/18 & 11/18 \end{pmatrix} = \begin{pmatrix} 0.56 & 0.44 \\ 0.39 & 0.61 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left(X_4 = 2 \,\middle|\, X_2 = 2, X_0 = 1\right) = ?$

$$\boxed{\frac{11}{18}}$$

From the Markov property it follows that

$$\mathrm{P}\left(X_4 = 2 \,\middle|\, X_2 = 2, X_0 = 1\right) = \mathrm{P}\left(X_4 = 2 \,\middle|\, X_2 = 2\right) = P^2_{22} = \frac{11}{18}.$$

(b) (3 points) $\mathrm{P}(X_1 = 1, X_0 = 1) = ?$

$$\boxed{\frac{1}{12}}$$

From the definition of conditional probability one finds

$$\mathrm{P}(X_1 = 1, X_0 = 1) = \mathrm{P}\left(X_1 = 1 \,\middle|\, X_0 = 1\right) \mathrm{P}(X_0 = 1) = P_{11} \times \frac{1}{3} = \frac{1}{12}$$

(c) (4 points) $\mathrm{P}\left(X_0 = 1 \,\middle|\, X_1 = 1\right) = ?$

$$\boxed{\frac{3}{19}}$$

Once more, the definition of conditional probability yields

$$\mathrm{P}\left(X_0 = 1 \,\middle|\, X_1 = 1\right) = \frac{\mathrm{P}(X_0 = 1, X_1 = 1)}{\mathrm{P}(X_1 = 1)}.$$

We already computed $\mathrm{P}(X_0 = 1, X_1 = 1) = \frac{1}{12}$, and using the law of total probability one has

$$\mathrm{P}(X_1 = 1) = \sum_{i=1}^{2} \mathrm{P}\left(X_1 = 1 \,\middle|\, X_0 = i\right) \mathrm{P}(X_0 = i) = \sum_{i=1}^{1} P_{i1} \mathrm{P}(X_0 = i) = \frac{1}{4} \times \frac{1}{3} + \frac{2}{3} \times \frac{2}{3} = \frac{19}{36}.$$

So the conditional probability is $\mathrm{P}\left(X_0 = 1 \,\middle|\, X_1 = 1\right) = \frac{3}{19}$.

(d) (4 points) $\mathbb{E}[X_1] =?$

$\frac{53}{36}$

The unconditional pmf of X_1 is $\mathrm{P}(X_1 = 1) = \frac{19}{36}$ and $\mathrm{P}(X_1 = 2) = \frac{17}{36}$. So it follows the expectation is $\mathbb{E}[X_1] = 1 \times \frac{19}{36} + 2 \times \frac{17}{36} = \frac{53}{36}$.

(e) (3 points) $\mathbb{E}\left[X_2 \,\middle|\, X_1 = 2\right] =?$

$\frac{4}{3}$

Using the definition of conditional expectation, one obtains

$$\mathbb{E}\left[X_2 \,\middle|\, X_1 = 2\right] = \sum_{i=1}^{2} i \times \mathrm{P}\left(X_2 = i \,\middle|\, X_1 = 2\right) = \sum_{i=1}^{2} i \times P_{2i} = 1 \times \frac{2}{3} + 2 \times \frac{1}{3} = \frac{4}{3}.$$

(f) (8 points) Compute the stationary distribution of $X_{\mathbb{N}}$.

$\boldsymbol{\pi} = \left[\frac{8}{17}, \frac{9}{17}\right]^T$

The unique stationary distribution $\boldsymbol{\pi} = [\pi_1, \pi_2]^T$ (the Markov chain is ergodic) satisfies

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 1/4 & 2/3 \\ 3/4 & 1/3 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \qquad \pi_1 + \pi_2 = 1.$$

Solving the linear system yields $\boldsymbol{\pi} = [8/17, 9/17]^T$.

(g) (4 points) Calculate

$$\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} (X_n)^2$$

and provide justification for the existence of the limit.

$\frac{44}{17}$

Since the Markov chain is ergodic, the ergodic limit is

$$\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} (X_n)^2 = \sum_{i=1}^{2} i^2 \times \pi_i = 1 \times \frac{8}{17} + 4 \times \frac{9}{17} = \frac{44}{17}, \quad \text{a. s.}$$

2. Consider a random variable X that is uniformly distributed in the interval $[0, 1]$, something we denote as $X \sim \text{Uniform}[0, 1]$. Let Y be a random variable which, conditioned on $X = x$, is uniformly distributed over the interval $[x, 1]$, that is $Y \mid X = x \sim \text{Uniform}[x, 1]$.

(a) (4 points) What is $f_{Y|X}(y|x)$, the conditional probability density function of Y given $X = x$?

$$f_{Y|X}(y|x) = \frac{1}{1-x}, \; x < y < 1$$

Because $Y \mid X = x \sim \text{Uniform}[x, 1]$, then it follows that

$$f_{Y|X}(y|x) = \frac{1}{1-x}, \quad x < y < 1.$$

Of course, one also has $f_{Y|X}(y|x) = 0$ outside of the interval $[x, 1]$.

(b) (5 points) $\mathbb{E}\left[Y \mid X = x\right] =?$

$$\frac{1+x}{2}$$

From the definition of conditional mean for continuous random variables, we obtain

$$\mathbb{E}\left[Y \mid X = x\right] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy = \frac{1}{1-x} \int_x^1 y dy = \frac{1+x}{2}.$$

(c) (5 points) $\mathbb{E}\left[Y\right] =?$

$$\frac{3}{4}$$

Using the law of iterated expectations along with $\mathbb{E}_Y\left[Y \mid X\right] = \frac{1+X}{2}$, we find

$$\mathbb{E}\left[Y\right] = \mathbb{E}_X\left[\mathbb{E}_Y\left[Y \mid X\right]\right] = \mathbb{E}_X\left[\frac{1+X}{2}\right] = \frac{1 + \mathbb{E}\left[X\right]}{2}.$$

Because $X \sim \text{Uniform}[0, 1]$, then $\mathbb{E}\left[X\right] = \frac{1}{2}$. Putting the pieces together, we obtain $\mathbb{E}\left[Y\right] = \frac{3}{4}$.

3. (8 points) Consider a computer program having $n = 100$ pages of code. Let X_i be the number of bugs on the ith page of code. Suppose that the X_i, $i = 1, \ldots, n$, are i.i.d. random variables having Poisson distribution with mean 1. Let $Y = \sum_{i=1}^{n} X_i$ be the total number of bugs.

Use the Central Limit Theorem to approximate $\text{P}\left(Y < 90\right)$. Write your result in terms of the complementary cumulative distribution function Φ of a standard Normal random variable, that is for $Z \sim \mathcal{N}(0, 1)$ then

$$\Phi(z) = \text{P}\left(Z \geq z\right) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-u^2/2} du.$$

$\Phi(1)$

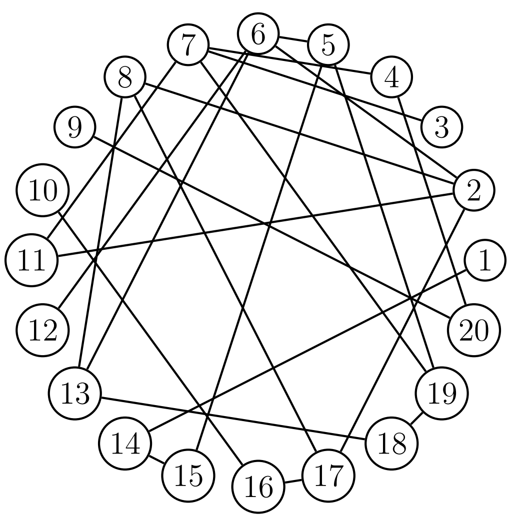
Because $X_i \sim \text{Poisson}(1)$, then $\mathbb{E}[X_i] = 1$ and $\text{var}[X_i] = 1$, for $i = 1, \ldots, 100$. Moreover, it follows that $\mathbb{E}[Y] = 100$ and $\text{var}[Y] = 100$ (to compute the variance we relied on the independence of the X_i). By virtue of the Central Limit Theorem, we can approximate the distribution of $\frac{Y - \mathbb{E}[Y]}{\sqrt{\text{var}[Y]}}$ with that of a standard Normal, namely

$$\frac{Y - 100}{10} \sim \mathcal{N}(0, 1).$$

The desired probability can thus be approximated as

$$\begin{aligned} \text{P}(Y < 90) &= \text{P}\left(\frac{Y - 100}{10} < \frac{90 - 100}{10}\right) \\ &\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} e^{-u^2/2} du \\ &= \frac{1}{\sqrt{2\pi}} \int_{1}^{\infty} e^{-u^2/2} du \\ &= \Phi(1). \end{aligned}$$

4. (8 points) The Erdős-Rényi model specifies the simplest mechanism to generate a random graph on N vertices. It yields undirected graphs (edges do not have directionality) without self loops (an edge connecting a vertex to itself is not allowed). For fixed parameters N and $0 \leq p \leq 1$, the Erdős-Rényi model specifies that each of the possible $\binom{N}{2}$ edges is included in the graph with probability p, independently from every other edge. For $p = 0$, the graph has no edges. For $p = 1$, one obtains a complete graph where every pair of vertices is connected by an edge. A sample realization of an Erdős-Rényi graph with $N = 20$ and $p = 0.15$ is shown below.



In graph theory, the degree of a vertex is the number of incident edges to that vertex. In the sample graph above, the degree of vertex 13 is 3 while the degree of vertex 4 is 2. For the

Erdős-Rényi model, the degrees D_v of vertices $v = 1, \ldots, N$ are identically distributed random variables. Name the distribution of the random variable D_v and specify its parameters.

$$D_v \sim \text{Binomial}(N-1, p)$$

For a given vertex v, consider the edge indicator random variables

$$B_u = \mathbb{I}\{\text{edge } (v,u) \text{ is included in the graph}\}$$

for all other $N-1$ vertices $u \neq v$. The Erdős-Rényi model specifies that the B_u are i.i.d., with distribution $B_u \sim \text{Bernoulli}(p)$. Now, notice that the degree of vertex v can be expressed as

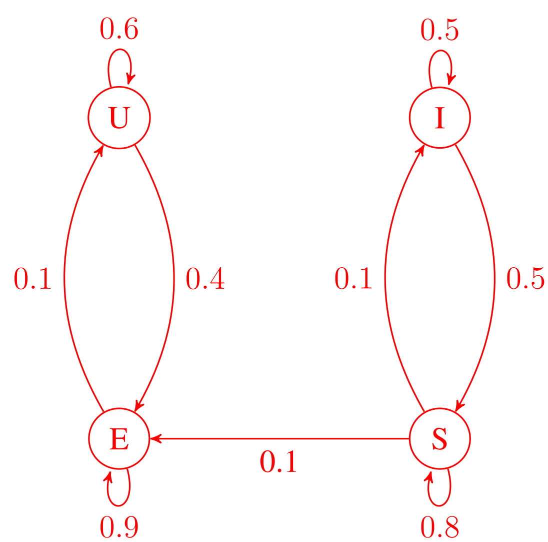
$$D_v = \sum_{u \neq v} B_u,$$

namely a sum of $N-1$ i.i.d. Bernoulli random variables with parameter p. The conclusion is that $D_v \sim \text{Binomial}(N-1, p)$

5. As part of her thesis work, a graduate student from the Warner School is interested in modeling the employment dynamics of young people using a Markov chain. After carrying out a field survey and processing the data, she was able to estimate the following transition probabilities.

	Student	Intern	Employed	Unemployed
Student	0.8	0.1	0.1	0
Intern	0.5	0.5	0	0
Employed	0	0	0.9	0.1
Unemployed	0	0	0.4	0.6

(a) (6 points) Draw the corresponding state transition diagram.



(b) (3 points) Is the Markov chain ergodic? Explain.

No

The Markov chain has a recurrent communication class $\mathcal{R} = \{\text{Employed,Unemployed}\}$ and a transient class $\mathcal{T} = \{\text{Student,Intern}\}$. This implies $X_{\mathbb{N}}$ is neither irreducible, nor ergordic.

(c) (7 points) In the long run, what fraction of time will an individual be unemployed?

$\frac{1}{5}$

In the long run, visits to the transient states $\mathcal{T} = \{\text{Student,Intern}\}$ stop almost surely. Hence $X_{\mathbb{N}}$ behaves as an ergodic Markov chain $\bar{X}_{\mathbb{N}}$ on the reduced state space $\mathcal{R} = \{\text{Employed,Unemployed}\}$, with transition probabilities

$$\bar{\mathbf{P}} = \begin{pmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{pmatrix}.$$

The unique stationary distribution $\boldsymbol{\pi} = [\pi_{\mathrm{E}}, \pi_{\mathrm{U}}]^T$ of this ergodic Markov chain satisfies

$$\begin{pmatrix} \pi_{\mathrm{E}} \\ \pi_{\mathrm{U}} \end{pmatrix} = \begin{pmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} \pi_{\mathrm{E}} \\ \pi_{\mathrm{U}} \end{pmatrix}, \quad \pi_{\mathrm{E}} + \pi_{\mathrm{U}} = 1.$$

Solving the linear system yields $\boldsymbol{\pi} = [4/5, 1/5]^T$, which implies and individual will be unemployed $\frac{1}{5}$ of the time.

6. Consider two random variables X and Y. Let c be a deterministic constant.

(a) (3 points) Derive a simple expression for $\mathrm{cov}[X, cY]$ in terms of c and $\mathrm{cov}[X, Y]$.

$c \times \mathrm{cov}[X, Y]$

Recall the identity $\mathrm{cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. Then it follows from linearity of the expectation operator that

$$\mathrm{cov}[X, cY] = \mathbb{E}[cXY] - \mathbb{E}[X]\mathbb{E}[cY] = c\mathbb{E}[XY] - c\mathbb{E}[X]\mathbb{E}[Y] = c \times \mathrm{cov}[X, Y].$$

(b) (3 points) Derive a simple expression for $\mathrm{cov}[X, X+Y]$ in terms of $\mathrm{var}[X]$ and $\mathrm{cov}[X, Y]$.

$\mathrm{var}[X] + \mathrm{cov}[X, Y]$

Similarly,

$$\begin{aligned} \mathrm{cov}[X, X+Y] &= \mathbb{E}[X(X+Y)] - \mathbb{E}[X]\mathbb{E}[X+Y] \\ &= \mathbb{E}\left[X^2\right] + \mathbb{E}[XY] - (\mathbb{E}[X])^2 - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathrm{var}[X] + \mathrm{cov}[X, Y]. \end{aligned}$$

(c) (4 points) If X and Y have a covariance of $\text{cov}[X,Y]$, we can transform them to a new pair of random variables whose covariance is zero. To do so, we consider the linear transformation

$$W = X$$
$$Z = X + aY,$$

where a is a deterministic constant. Find the value of a so that W and Z are uncorrelated.

$$a = -\frac{\text{var}[X]}{\text{cov}[X,Y]}$$

Leveraging the identities just derived, we find

$$\begin{aligned}\text{cov}[W,Z] &= \text{cov}[X, X + aY] \\ &= \text{var}[X] + \text{cov}[X, aY] \\ &= \text{var}[X] + a \times \text{cov}[X,Y].\end{aligned}$$

By definition, W and Z are uncorrelated if $\text{cov}[W,Z] = 0$. This requires choosing $a = -\frac{\text{var}[X]}{\text{cov}[X,Y]}$.

7. During each day, a non-negative integer number of customers arrives to a store to purchase a particular product. Each customer purchases a unit of the product when the product is in stock. Customers who do not find the product in stock depart without making a purchase. The store may order new units of the product at the end of the day (after that day's demand has materialized), and any such orders arrive to the store before the beginning of the next day.

Each day orders are made as follows. If, at the end of the day, there are 5 or fewer units of the product in stock, then an order is placed so that there will be exactly 10 units of inventory present at the start of the next day. If there are more than 5 units of inventory present, no order is placed. Suppose that the daily demand $D_{\mathbb{N}} = D_1, D_2, \ldots, D_n, \ldots$ is an i.i.d. sequence of non-negative integer-valued random variables, each with probability mass function $p(\cdot)$; i.e.,

$$\text{P}[D_1 = i] = p(i), \quad i = 0, 1, 2, \ldots$$

Suppose that at the beginning of the first day of operation $(n = 0)$, the stock level is an arbitrary fixed non-negative integer z.

Let $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ be the Markov chain that represents the amount of the product in stock at the beginning of each day.

(a) (4 points) Determine the transition probabilities P_{ij} for all $i \leq 5$ and $j \geq 0$.

$$P_{ij} = \mathbb{I}\{j = 10\}$$

In the current state $X_n = i \leq 5$, the stock is already below the replenishment level. During day n, the inventory can only decrease (if $D_n > 0$) or stay the same (if $D_n = 0$). Hence, the next transition is surely going to be to state $X_{n+1} = 10$. In summary, for all $i \leq 5$ then $P_{i10} = 1$ while $P_{ij} = 0$ for all $j \neq 10$. These probabilities can be compactly expressed as $P_{ij} = \mathbb{I}\{j = 10\}$.

(b) (4 points) Determine the transition probabilities P_{ij} for all $i \geq j > 5$ and $j \neq 10$.

$$P_{ij} = p(i-j)$$

In the current state $X_n = i > 5$, the stock is above the replenishment level. A transition to state $X_{n+1} = j > 5$ with $j \neq 10$, can only occur if there is no enough day n demand to trigger a product order. That is, when $X_n - D_n = j > 5$. So in this setting,

$$\begin{aligned}\mathrm{P}\left(X_{n+1} = j \,\middle|\, X_n = i\right) &= \mathrm{P}\left(X_n - D_n = j \,\middle|\, X_n = i\right) = \mathrm{P}\left(i - D_n = j \,\middle|\, X_n = i\right) \\ &= \mathrm{P}\left(i - D_n = j\right) = \mathrm{P}\left(D_n = i - j\right) = p(i-j).\end{aligned}$$

(c) (4 points) Determine the transition probabilities P_{i10} for all $5 < i < 9$.

$$P_{i10} = \sum_{k=i-5}^{\infty} p(k)$$

In the current state $X_n = i > 5$, the stock is above the replenishment level. But because $i < 9$, the only possibly way to transition to $X_{n+1} = 10$ is when the day n demand is so large that a product order has to be placed. That is, when $X_n - D_n \leq 5$. Accordingly,

$$\begin{aligned}\mathrm{P}\left(X_{n+1} = 10 \,\middle|\, X_n = i\right) &= \mathrm{P}\left(X_n - D_n \leq 5 \,\middle|\, X_n = i\right) = \mathrm{P}\left(i - D_n \leq 5 \,\middle|\, X_n = i\right) \\ &= \mathrm{P}\left(i - D_n \leq 5\right) = \mathrm{P}\left(D_n \geq i - 5\right) = \sum_{k=i-5}^{\infty} p(k).\end{aligned}$$

(d) (4 points) Determine the transition probabilities P_{i10} for all $i \geq 10$.

$$P_{i10} = p(i-10) + \sum_{k=i-5}^{\infty} p(k)$$

In the current state $X_n = i \geq 10$, the stock is above the replenishment level. But because $i \geq 10$, there are two potential ways to transition to $X_{n+1} = 10$: i) if there is no enough day n demand to trigger a product order (when $X_n - D_n = 10$); or ii) if the day n demand is so large that a product order has to be placed (when $X_n - D_n \leq 5$). Putting the pieces together, we obtain

$$\mathrm{P}\left(X_{n+1} = 10 \,\middle|\, X_n = i\right) = p(i-10) + \sum_{k=i-5}^{\infty} p(k).$$