ECE440 - Introduction to Random Processes

Midterm Exam

October 28, 2019

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 points (out of 102, extra points are bonus points).
- Duration: 90 minutes.
- This exam has 11 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:_________________________________

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	22		5.	12	
2.	18		6.	14	
3.	8		7.	20	
4.	8				
			Total	102	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 4/5 & 1/5 \\ 1/2 & 1/2 \end{pmatrix}$$

and initial distribution $\mathrm{P}(X_0 = 1) = 1/2$ and $\mathrm{P}(X_0 = 2) = 1/2$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 11/25 & 14/25 \\ 7/20 & 13/20 \end{pmatrix} = \begin{pmatrix} 0.44 & 0.56 \\ 0.35 & 0.65 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left(X_3 = 1 \,\middle|\, X_2 = 2, X_1 = 1\right) =?$

(b) (2 points) $\mathrm{P}\left(X_5 = 2 \,\middle|\, X_3 = 2, X_2 = 1, X_1 = 1, X_0 = 1\right) =?$

(c) (3 points) $\mathrm{P}(X_2 = 2, X_1 = 1, X_0 = 1) =?$

(d) (4 points) $\mathbb{E}\left[X_2 \,\middle|\, X_0 = 2\right] = ?$

(e) (8 points) Compute the stationary distribution of $X_{\mathbb{N}}$.

(f) (3 points) In the long run, what fraction of time will you find $X_{\mathbb{N}}$ in state 1?

2. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables, where $\mathrm{P}(X_1 = 1) = 1/4$, $\mathrm{P}(X_1 = 2) = 1/4$, $\mathrm{P}(X_1 = 3) = 1/3$, and $\mathrm{P}(X_1 = 4) = 1/6$. Define

$$T = \min\{n \geq 1 : X_n \notin \{1, 2\}\} \quad \text{and} \quad Y = \sum_{i=1}^{T} X_i$$

(a) (6 points) Compute $\mathbb{E}\left[X_i \,\middle|\, T = t\right]$, for $i = 1, \ldots, t-1$.

(b) (6 points) Compute $\mathbb{E}\left[X_t \,\middle|\, T = t\right]$.

(c) (6 points) Compute $\mathbb{E}\left[Y \,\middle|\, T = t\right]$.

3. (8 points) Consider a continuous random variable X with probability density function $f_X(x)$. Let $A \subset \mathbb{R}$ be a subset of the real line and define the indicator random variable $Y = \mathbb{I}\{X \in A\}$.

Find an expression for $F_Y(y) = \mathrm{P}(Y \leq y)$, the cumulative distribution function of Y. (Hint: first find the probability mass function of Y)

4. (8 points) Consider a continuous random variable X that is uniformly distributed in the interval $[0, 1]$. Suppose that $Y_{\mathbb{N}} = Y_1, Y_2, \ldots, Y_n, \ldots$ is an i.i.d. sequence of random variables and let A be a set such that $\mathrm{P}\left(Y_1 \in A \,\middle|\, X = x\right) = x^2$. Calculate

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\left\{Y_i \notin A\right\}$$

and provide justification for the existence of the limit.

5. Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 \\ * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & 0 \end{pmatrix}$$

where the $*$ denote possibly different, but strictly positive numbers.

(a) (6 points) Draw the corresponding state transition diagram. If you can infer some of the $*$ values, indicate them in your diagram.

(b) (3 points) Is the Markov chain ergodic? Explain.

(c) (3 points) Is the period of state 2 equal to the period of state 5? Explain.

6. Consider a continuous random variable X that is uniformly distributed in the interval $[0, 1]$. Let a and b be deterministic constants such that $0 < a < b < 1$. Define the random variables

$$Y = \begin{cases} 1, & 0 < X < b, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad Z = \begin{cases} 1, & a < X < 1, \\ 0, & \text{otherwise} \end{cases} .$$

(a) (6 points) Compute $p_{Y,Z}(y, z) = \mathrm{P}(Y = y, Z = z)$, the joint probability mass function of Y and Z.

(b) (4 points) Are Y and Z independent? Justify your answer.

(c) (4 points) $\mathbb{E}\left[Y \,\middle|\, Z = 0\right] = ?$

7. Suppose customers can arrive to a service system at times $n = 0, 1, 2, \ldots$ In any given period, independent of everything else, there is one arrival with probability p, and there is no arrival with probability $1-p$. If, upon arrival, a customer finds k other customers present in the system ($k = 0, 1, 2, \ldots$), then that arriving customer will enter the system with probability $\alpha(k)$ and will depart without entering the system with probability $1 - \alpha(k)$.

Customers that enter the system are served one-at-a-time on a first-come-first-served basis. If at the time of entrance there are no customers present, then the entering customer immediately begins service. Otherwise, the entering customer joins the back of the queue.

Assume that service times are i.i.d. geometric random variables (each with parameter q) that are independent of the arrival/entrance process. So, $\mathrm{P}(\text{Service time} = \ell) = (1-q)^{\ell-1}q$, for $\ell = 1, 2, \ldots$ Note that a customer who enters service in time n can complete service, at the earliest, in time $n+1$ (in which case her service time is 1). Upon a service completion, the just-served customer departs the system.

In a time period n, events happen in the following order: (i) arrivals, if any, occur; (ii) any arrival decides whether or not to enter the system; (iii) service completions, if any, occur; (iv) service begins on a new customer if there are customers present in the system.

Let X_n denote the number of customers in the system at the end of time period n, i.e., after the time-n arrivals and services. Note that X_n includes both customers waiting as well as any customer being served, and that the random process $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$.

(a) (5 points) Determine the transition probabilities P_{0j} for all $j \geq 0$.

(b) (5 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j = i$.

(c) (5 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j > i$.

(d) (5 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $0 \leq j < i$.