ECE440 - Introduction to Random Processes

Midterm Exam

October 28, 2019

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 points (out of 102, extra points are bonus points).
- Duration: 90 minutes.
- This exam has 11 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:__________SOLUTIONS__________

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	22		5.	12	
2.	18		6.	14	
3.	8		7.	20	
4.	8				
			Total	102	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 4/5 & 1/5 \\ 1/2 & 1/2 \end{pmatrix}$$

and initial distribution $\mathrm{P}(X_0 = 1) = 1/2$ and $\mathrm{P}(X_0 = 2) = 1/2$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 11/25 & 14/25 \\ 7/20 & 13/20 \end{pmatrix} = \begin{pmatrix} 0.44 & 0.56 \\ 0.35 & 0.65 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left(X_3 = 1 \,\middle|\, X_2 = 2, X_1 = 1\right) = ?$

$\frac{1}{2}$

From the Markov property it follows that

$$\mathrm{P}\left(X_3 = 1 \,\middle|\, X_2 = 2, X_1 = 1\right) = \mathrm{P}\left(X_3 = 1 \,\middle|\, X_2 = 2\right) = P_{21} = \frac{1}{2}.$$

(b) (2 points) $\mathrm{P}\left(X_5 = 2 \,\middle|\, X_3 = 2, X_2 = 1, X_1 = 1, X_0 = 1\right) = ?$

$\frac{13}{20}$

Likewise,

$$\mathrm{P}\left(X_5 = 2 \,\middle|\, X_3 = 2, X_2 = 1, X_1 = 1, X_0 = 1\right) = \mathrm{P}\left(X_5 = 2 \,\middle|\, X_3 = 2\right) = P^2_{22} = \frac{13}{20}.$$

(c) (3 points) $\mathrm{P}(X_2 = 2, X_1 = 1, X_0 = 1) = ?$

$\frac{2}{25}$

Using the definition of conditionaly probability (twice) and the Markov property one finds

$$\begin{aligned} \mathrm{P}(X_2 = 2, X_1 = 1, X_0 = 1) &= \mathrm{P}\left(X_2 = 2 \,\middle|\, X_1 = 1, X_0 = 1\right) \mathrm{P}(X_1 = 1, X_0 = 1) \\ &= \mathrm{P}\left(X_2 = 2 \,\middle|\, X_1 = 1\right) \mathrm{P}\left(X_1 = 1 \,\middle|\, X_0 = 1\right) \mathrm{P}(X_0 = 1) \\ &= P_{12} \times P_{11} \times \frac{1}{2} = \frac{2}{25} \end{aligned}$$

(d) (4 points) $\mathbb{E}\left[X_2 \,\middle|\, X_0 = 2\right] =?$

$$\frac{33}{20}$$

Using the definition of conditional expectation, one obtains

$$\mathbb{E}\left[X_2 \,\middle|\, X_0 = 2\right] = \sum_{i=1}^{2} i \times \mathsf{P}\left(X_2 = i \,\middle|\, X_0 = 2\right) = \sum_{i=1}^{2} i \times P_{2i}^2 = 1 \times \frac{7}{20} + 2 \times \frac{13}{20} = \frac{33}{20}.$$

(e) (8 points) Compute the stationary distribution of $X_{\mathbb{N}}$.

$$\boldsymbol{\pi} = \left[\frac{5}{7}, \frac{2}{7}\right]^T$$

The unique stationary distribution $\boldsymbol{\pi} = [\pi_1, \pi_2]^T$ (the Markov chain is ergodic) satisfies

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 4/5 & 1/2 \\ 1/5 & 1/2 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \quad \pi_1 + \pi_2 = 1.$$

Solving the linear system yields $\boldsymbol{\pi} = [5/7, 2/7]^T$.

(f) (3 points) In the long run, what fraction of time will you find $X_{\mathbb{N}}$ in state 1?

$$\frac{5}{7}$$

By the ergodic theorem, the long-run fraction of time spent in state 1 is $\pi_1 = 5/7$

2. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables, where $\mathsf{P}(X_1 = 1) = 1/4$, $\mathsf{P}(X_1 = 2) = 1/4$, $\mathsf{P}(X_1 = 3) = 1/3$, and $\mathsf{P}(X_1 = 4) = 1/6$. Define

$$T = \min\{n \geq 1 : X_n \notin \{1, 2\}\} \quad \text{and} \quad Y = \sum_{i=1}^{T} X_i$$

(a) (6 points) Compute $\mathbb{E}\left[X_i \,\middle|\, T = t\right]$, for $i = 1, \ldots, t-1$.

$$\frac{3}{2}$$

From the definition of T, then for $i = 1, \ldots, t-1$ one has

$$\begin{aligned}\mathbb{E}\left[X_i \,\middle|\, T = t\right] &= \mathbb{E}\left[X_i \,\middle|\, X_1 \in \{1,2\}, \ldots, X_i \in \{1,2\}, \ldots, X_{t-1} \in \{1,2\}, X_t \in \{3,4\}\right] \\ &= \mathbb{E}\left[X_i \,\middle|\, X_i \in \{1,2\}\right]\end{aligned}$$

where the last equality follows from the independence of the sequence $X_{\mathbb{N}}$. The relevant conditional pmf $\mathrm{P}\left[X_i = x \,\middle|\, X_i \in \{1,2\}\right]$ is

$$\begin{aligned}
\mathrm{P}\left[X_i = 1 \,\middle|\, X_i \in \{1,2\}\right] &= \frac{\mathrm{P}\left[X_i = 1, X_i \in \{1,2\}\right]}{\mathrm{P}\left[X_i \in \{1,2\}\right]} = \frac{\mathrm{P}\left[X_i = 1\right]}{\mathrm{P}\left[X_i = 1\right] + \mathrm{P}\left[X_i = 2\right]} = \frac{1}{2} \\
\mathrm{P}\left[X_i = 2 \,\middle|\, X_i \in \{1,2\}\right] &= \frac{\mathrm{P}\left[X_i = 2, X_i \in \{1,2\}\right]}{\mathrm{P}\left[X_i \in \{1,2\}\right]} = \frac{\mathrm{P}\left[X_i = 2\right]}{\mathrm{P}\left[X_i = 1\right] + \mathrm{P}\left[X_i = 2\right]} = \frac{1}{2} \\
\mathrm{P}\left[X_i = 3 \,\middle|\, X_i \in \{1,2\}\right] &= 0 \\
\mathrm{P}\left[X_i = 4 \,\middle|\, X_i \in \{1,2\}\right] &= 0.
\end{aligned}$$

Hence, the conditional expectation is $\mathbb{E}\left[X_i \,\middle|\, T = t\right] = 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = \frac{3}{2}$, for $i = 1, \ldots, t-1$.

(b) (6 points) Compute $\mathbb{E}\left[X_t \,\middle|\, T = t\right]$.

$$\frac{10}{3}$$

Reasoning as in the previous part, from the definition of T and using the independence of the sequence $X_{\mathbb{N}}$

$$\begin{aligned}
\mathbb{E}\left[X_t \,\middle|\, T = t\right] &= \mathbb{E}\left[X_t \,\middle|\, X_1 \in \{1,2\}, \ldots, X_i \in \{1,2\}, \ldots, X_{t-1} \in \{1,2\}, X_t \in \{3,4\}\right] \\
&= \mathbb{E}\left[X_t \,\middle|\, X_t = X_t \in \{3,4\}\right].
\end{aligned}$$

The relevant conditional pmf $\mathrm{P}\left[X_t = x \,\middle|\, X_t \in \{3,4\}\right]$ is

$$\begin{aligned}
\mathrm{P}\left[X_t = 1 \,\middle|\, X_t \in \{3,4\}\right] &= 0 \\
\mathrm{P}\left[X_t = 2 \,\middle|\, X_t \in \{3,4\}\right] &= 0 \\
\mathrm{P}\left[X_t = 3 \,\middle|\, X_t \in \{3,4\}\right] &= \frac{\mathrm{P}\left[X_t = 3, X_t \in \{3,4\}\right]}{\mathrm{P}\left[X_t \in \{3,4\}\right]} = \frac{\mathrm{P}\left[X_t = 3\right]}{\mathrm{P}\left[X_t = 3\right] + \mathrm{P}\left[X_t = 4\right]} = \frac{2}{3} \\
\mathrm{P}\left[X_t = 4 \,\middle|\, X_t \in \{3,4\}\right] &= \frac{\mathrm{P}\left[X_t = 4, X_t \in \{3,4\}\right]}{\mathrm{P}\left[X_t \in \{3,4\}\right]} = \frac{\mathrm{P}\left[X_t = 4\right]}{\mathrm{P}\left[X_t = 3\right] + \mathrm{P}\left[X_t = 4\right]} = \frac{1}{3}.
\end{aligned}$$

Hence, the conditional expectation is $\mathbb{E}\left[X_i \,\middle|\, T = t\right] = 3 \times \frac{2}{3} + 4 \times \frac{1}{3} = \frac{10}{3}$.

(c) (6 points) Compute $\mathbb{E}\left[Y \,\middle|\, T = t\right]$.

$$(t-1)\frac{3}{2} + \frac{10}{3}$$

From the definition of Y and noting that T is not independent of the $X_1, X_2, \ldots, X_T$, one finds

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{T} X_i \,\middle|\, T = t\right] &= \mathbb{E}\left[\sum_{i=1}^{t} X_i \,\middle|\, T = t\right] \\
&= \sum_{i=1}^{t-1} \mathbb{E}\left[X_i \,\middle|\, T = t\right] + +\mathbb{E}\left[X_t \,\middle|\, T = t\right] \\
&= (t-1)\mathbb{E}\left[X_i \,\middle|\, T = t\right] + \mathbb{E}\left[X_t \,\middle|\, T = t\right] = (t-1)\frac{3}{2} + \frac{10}{3}
\end{aligned}$$

3. (8 points) Consider a continuous random variable X with probability density function $f_X(x)$. Let $A \subset \mathbb{R}$ be a subset of the real line and define the indicator random variable $Y = \mathbb{I}\{X \in A\}$.

Find an expression for $F_Y(y) = \mathrm{P}(Y \leq y)$, the cumulative distribution function of Y. (Hint: first find the probability mass function of Y)

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ 1 - \int_A f_X(x)dx, & 0 \leq y < 1 \\ 1, & \text{otherwise} \end{cases}$$

Being an indicator random variable, then Y has Bernoulli distribution with parameter $p = \mathrm{P}(X \in A) = \int_A f_X(x)dx$. Hence, the cumulative distribution function of Y is given by

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ 1 - \int_A f_X(x)dx, & 0 \leq y < 1 \\ 1, & \text{otherwise} \end{cases} .$$

4. (8 points) Consider a continuous random variable X that is uniformly distributed in the interval $[0, 1]$. Suppose that $Y_{\mathbb{N}} = Y_1, Y_2, \ldots, Y_n, \ldots$ is an i.i.d. sequence of random variables and let A be a set such that $\mathrm{P}\left(Y_1 \in A \,\middle|\, X = x\right) = x^2$. Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{Y_i \notin A\}$$

and provide justification for the existence of the limit.

$$\frac{2}{3}$$

Because $Y_{\mathbb{N}}$ is i.i.d., then $Z_{\mathbb{N}} = \mathbb{I}\{Y_1 \notin A\}, \mathbb{I}\{Y_2 \notin A\}, \ldots, \mathbb{I}\{Y_n \notin A\}, \ldots$ is also i.i.d. By the strong law of large numbers the limit exists and is equal to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{Y_i \notin A\} = \mathbb{E}\left[\mathbb{I}\{Y_1 \notin A\}\right] = \mathrm{P}(Y_1 \notin A), \quad \text{w.p. } 1.$$

To compute the probability, notice first that $\mathrm{P}(Y_1 \notin A) = 1 - \mathrm{P}(Y_1 \in A)$. Moreover, upon conditioning on $X = x$

$$\mathrm{P}(Y_1 \in A) = \int_{-\infty}^{\infty} \mathrm{P}\left(Y_1 \in A \,\middle|\, X = x\right) f_X(x)dx = \int_0^1 x^2 dx = \frac{1}{3},$$

where the second equality follows because $\mathrm{P}\left(Y_1 \in A \,\middle|\, X = x\right) = x^2$ and X has density $f_X(x) = 1$ in $[0, 1]$, and $f_X(x) = 0$ elsewhere. Putting all the pieces together, the limit is $2/3$ almost surely.

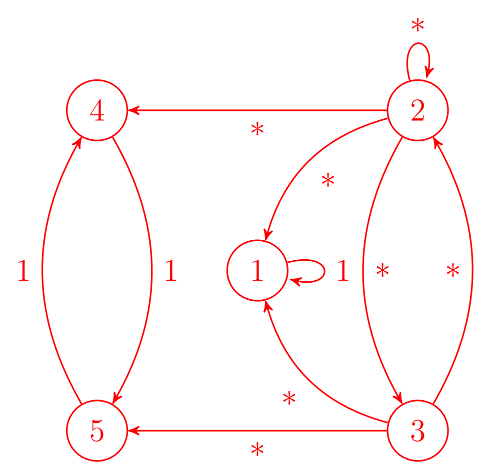
5. Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 \\ * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & 0 \end{pmatrix}$$

where the $*$ denote possibly different, but strictly positive numbers.

(a) (6 points) Draw the corresponding state transition diagram. If you can infer some of the $*$ values, indicate them in your diagram.

The state transition diagram is



(b) (3 points) Is the Markov chain ergodic? Explain.

No

The Markov chain is not ergodic because it is not irreducible. It has three communication classes, namely $\mathcal{R}_1 = \{1\}$, $\mathcal{R}_2 = \{4, 5\}$ and $\mathcal{T} = \{2, 3\}$.

(c) (3 points) Is the period of state 2 equal to the period of state 5? Explain.

No, state 5 has period 2 and state 2 is aperiodic

From the state transition diagram, it is apparent that $P_{55}^{2n+1} = 0$ and $P_{55}^{2n} = 1$ so $\gcd\{2, 4, \ldots\} = 2$. This implies state 5 has period 2. On the other hand $P_{22} = * > 0$, so state 2 is aperiodic and thus states 2 and 5 have different periods.

6. Consider a continuous random variable X that is uniformly distributed in the interval $[0, 1]$. Let a and b be deterministic constants such that $0 < a < b < 1$. Define the random variables

$$Y = \begin{cases} 1, & 0 < X < b, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad Z = \begin{cases} 1, & a < X < 1, \\ 0, & \text{otherwise} \end{cases} .$$

(a) (6 points) Compute $p_{Y,Z}(y, z) = \mathrm{P}(Y = y, Z = z)$, the joint probability mass function of Y and Z.

$p_{Y,Z}(0, 0) = 0$, $p_{Y,Z}(0, 1) = 1-b$, $p_{Y,Z}(1, 0) = a$ and $p_{Y,Z}(1, 1) = b-a$

Since X is uniformly distributed in $[0, 1]$ and $0 < a < b < 1$, the joint probability mass function values are

$$\begin{aligned}
\mathrm{P}(Y = 0, Z = 0) &= \mathrm{P}(b < X < 1, 0 < X < a) = \mathrm{P}(\emptyset) = 0, \\
\mathrm{P}(Y = 0, Z = 1) &= \mathrm{P}(b < X < 1, a < X < 1) = \mathrm{P}(b < X < 1) = 1 - b, \\
\mathrm{P}(Y = 1, Z = 0) &= \mathrm{P}(0 < X < b, 0 < X < a) = \mathrm{P}(0 < X < a) = a, \\
\mathrm{P}(Y = 1, Z = 1) &= \mathrm{P}(0 < X < b, a < X < 1) = \mathrm{P}(a < X < b) = b - a.
\end{aligned}$$

(b) (4 points) Are Y and Z independent? Justify your answer.

No

The random variables Y and Z are dependent because

$$0 = \mathrm{P}(Y = 0, Z = 0) \neq \mathrm{P}(Y = 0)\,\mathrm{P}(Z = 0) = (1 - b)a > 0.$$

(c) (4 points) $\mathbb{E}\left[Y \,\middle|\, Z = 0\right] =?$

1

The conditional pmf of Y given $Z = 0$ is

$$\begin{aligned}
\mathrm{P}\left(Y = 0 \,\middle|\, Z = 0\right) &= \frac{\mathrm{P}(Y = 0, Z = 0)}{\mathrm{P}(Z = 0)} = 0, \\
\mathrm{P}\left(Y = 1 \,\middle|\, Z = 0\right) &= \frac{\mathrm{P}(Y = 1, Z = 0)}{\mathrm{P}(Z = 0)} = 1.
\end{aligned}$$

Hence, the conditional expectation is $\mathbb{E}\left[Y \,\middle|\, Z = 0\right] = 0 \times 0 + 1 \times 1 = 1$.

7. Suppose customers can arrive to a service system at times $n = 0, 1, 2, \ldots$ In any given period, independent of everything else, there is one arrival with probability p, and there is no arrival with probability $1-p$. If, upon arrival, a customer finds k other customers present in the system ($k = 0, 1, 2, \ldots$), then that arriving customer will enter the system with probability $\alpha(k)$ and will depart without entering the system with probability $1 - \alpha(k)$.

Customers that enter the system are served one-at-a-time on a first-come-first-served basis. If at the time of entrance there are no customers present, then the entering customer immediately begins service. Otherwise, the entering customer joins the back of the queue.

Assume that service times are i.i.d. geometric random variables (each with parameter q) that are independent of the arrival/entrance process. So, $\mathrm{P}(\text{Service time} = \ell) = (1-q)^{\ell-1}q$, for $\ell = 1, 2, \ldots$ Note that a customer who enters service in time n can complete service, at the earliest, in time $n+1$ (in which case her service time is 1). Upon a service completion, the just-served customer departs the system.

In a time period n, events happen in the following order: (i) arrivals, if any, occur; (ii) any arrival decides whether or not to enter the system; (iii) service completions, if any, occur; (iv) service begins on a new customer if there are customers present in the system.

Let X_n denote the number of customers in the system at the end of time period n, i.e., after the time-n arrivals and services. Note that X_n includes both customers waiting as well as any customer being served, and that the random process $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$.

(a) (5 points) Determine the transition probabilities P_{0j} for all $j \geq 0$.

$$P_{00} = 1 - p\alpha(0),\ P_{01} = p\alpha(0),\ P_{0j} = 0,\ j > 1$$

If the present state is $X_n = 0$ (empty system), then there are only two possible transitions:

1) If an arrival does not occur at time instant $n+1$ or if there is an arrival that decides to depart without entering the system then $X_{n+1} = 0$. Then we conclude that $P_{00} = 1 - p + p(1 - \alpha(0)) = 1 - p\alpha(0)$.
2) If an arrival occurs at time instant $n+1$ and decides to enter they system then $X_{n+1} = 1$. Then we conclude that $P_{01} = p\alpha(0)$.

All other transition probabilities from state 0 are null, namely $P_{0j} = 0,\ j > 1$.

(b) (5 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j = i$.

$$P_{ii} = [1 - p\alpha(i)](1-q) + pq\alpha(i),\ i > 0$$

Suppose the present state is $X_n = i, i > 0$. If an arrival does not occur at time instant $n+1$ and the user being served does not complete service, then $X_{n+1} = i$. This will happen with probability $(1-p)(1-q)$. Furthermore, if an arrival occurs at time instant $n+1$ that departs without entering the system and the user being served does not complete service, then also $X_{n+1} = i$. This will happen with probability $p[1 - \alpha(i)](1-q)$. Finally, if an arrival occurs at time instant

$n+1$ and decides to enter they system while the user currently being served exits the system after completing service, then $X_{n+1} = i$ as well. This will happen with probability $p\alpha(i)q$. The conclusion is that $P_{ii} = (1-p)(1-q) + p[1-\alpha(i)](1-q) + p\alpha(i)q = [1-p\alpha(i)](1-q) + pq\alpha(i)$.

(c) (5 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j > i$.

$P_{i,i+1} = p\alpha(i)(1-q),\ P_{ij} = 0,\ i > 0$ and $j > i+1$

Suppose the present state is $X_n = i$, $i > 0$. If an arrival occurs at time instant $n+1$ that decides to enter the system and the user being served does not complete service, then $X_{n+1} = i+1$. The conclusion is that $P_{i,i+1} = p\alpha(i)(1-q)$.

All other transition probabilities from state $i > 0$ to $j > i+1$ are null.

(d) (5 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $0 \le j < i$.

$P_{i,i-1} = [1-p\alpha(i)]q, P_{ij} = 0, i > 0$ and $0 \le j < i-1$

Suppose the present state is $X_n = i$, $i > 0$. If an arrival does not occur at time instant $n+1$ and the user being served completes service, then $X_{n+1} = i-1$. This will happen with probability $(1-p)q$. Furthermore, if an arrival occurs at time instant $n+1$ but does not enter the system and the user being served completes service, then also $X_{n+1} = i+1$. This will happen with probability $p[1-\alpha(i)]q$. The conclusion is that $P_{i,i-1} = [1-p\alpha(i)]q$.

All other transition probabilities from state $i > 0$ to $0 \le j < i-1$ are null.