ECE440 - Introduction to Random Processes

Midterm Exam

October 23, 2020

Instructions:

- This is an **individual** take-home exam, **collaborations are not allowed**.
- Write clearly and show all your work.
- Your solutions should be submitted via Gradescope as a single pdf file.
- The estimated amount of time required to complete this exam is 2.5 hours.
- **The submission deadline is 10 pm ET, Friday October 23, 2020.**
- Late submissions will not be accepted.
- Perfect score: 100 points.
- This exam has 12 numbered pages.

Name:________________________________

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	18		6.	10	
2.	10		7.	10	
3.	10		8.	8	
4.	8		9.	16	
5.	10				
			Total	100	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

and initial distribution $\mathrm{P}(X_0 = 1) = 1$ and $\mathrm{P}(X_0 = 2) = 0$. Unless otherwise stated, suppose that $0 < a < 1$ and $0 < b < 1$.

(a) (1 points) $\mathrm{P}\left(X_5 = 2 \,\middle|\, X_4 = 1, X_3 = 2, X_1 = 1\right) =?$

(b) (2 points) $\mathrm{P}\left(X_3 = 2, X_2 = 2 \,\middle|\, X_1 = 1\right) =?$

(c) (3 points) $\mathbb{E}\left[X_1\right] =?$

(d) (8 points) Prove that

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}$$

and provide justification for the existence of the limit.

(e) (2 points) From now on, suppose that $a = b = 1$. $\mathrm{P}\left(X_{26} = 1 \,\middle|\, X_1 = 2\right) =?$

(f) (2 points) Still with $a = b = 1$, calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \mathbb{I}\left\{X_i = 2\right\}$$

and provide justification for the existence of the limit.

2. Consider a probability space $(S, \mathcal{F}, \mathrm{P}(\cdot))$.

(a) (4 points) Let $E \in \mathcal{F}$ be an event. Show that if E is independent of itself the $\mathrm{P}(E)$ is either 0 or 1.

(b) (6 points) Suppose that $A, B \in \mathcal{F}$ are independent events. Show that A^c and B^c are independent events.

3. (a) (5 points) Let X and Y be random variables. Show that if $\mathbb{E}\left[X \,\middle|\, Y = y\right] = c$ for some deterministic constant c, then X and Y are uncorrelated.

(b) (5 points) Let U and V be random variables. Suppose that $\mathbb{E}\left[V \,\middle|\, U\right] = U$. Show that $\text{cov}[U, V] = \text{var}[U]$.

4. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables, which are uniformly distributed in the interval $[0, 1]$.

(a) (4 points) For fixed $n > 1$, consider the random variable

$$Y_n = \sum_{i=1}^{n} \mathbb{I}\{X_i > 0.3\}.$$

Write down an expression for $\mathrm{P}(Y_n = y)$, the probability mass function of Y_n. Make sure you also specify the range of values of y for which $\mathrm{P}(Y_n = y) = 0$.

(b) (4 points) Suppose that a, b are deterministic constants such that $a < b$. Using the Central Limit Theorem, write down an approximate expression for $\mathrm{P}(a \leq Y_{1000} \leq b)$.

5. (10 points) Consider a model for the evolution of a population and suppose that X_n is the number of individuals in generation n. Suppose the k-th individual in generation n creates $Q_{k,n+1}$ individuals in generation $n+1$, and that the $Q_{k,n}$ are i.i.d. across individuals and generations, and independent of X_0. Let $\mu = \mathbb{E}\left[Q_{k,n}\right]$ and $\sigma^2 = \text{var}[Q_{k,n}]$. Under the preceding assumptions, $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$ for which

$$X_{n+1} = Q_{1,n+1} + \ldots + Q_{X_n,n+1} \quad \text{if } X_n > 0,$$

and $X_{n+1} = 0$ if $X_n = 0$. Let $M_n = \mathbb{E}\left[X_n\right]$ and $V_n = \text{var}[X_n]$.

Derive an expression for V_{n+1} in terms of V_n, M_n, μ and σ^2.

6. (10 points) Consider a continuous random variable X with probability density function

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = 0$ if $X < 1/2$ and $Y = 2X$ otherwise. Compute $F_Y(y) = \mathrm{P}\,(Y \leq y)$, the cumulative distribution function of Y.

7. (10 points) Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables such that

$$\mathrm{P}(X_1 = 1) = 4/10, \quad \mathrm{P}(X_1 = 2) = 1/10,$$
$$\mathrm{P}(X_1 = 3) = 3/10, \quad \mathrm{P}(X_1 = 4) = 2/10.$$

Define

$$T = \min\{n \geq 1 : X_n \in \{1, 2\}\}.$$

$\mathbb{E}\left[X_1^2 + X_2^2 + X_5^2 \,\middle|\, T = 5\right] = ?$

8. (8 points) Consider flipping a coin for which the probability of heads is $p = 1/2$. Let $X_i \in \{0, 1\}$ denote the outcome of a single toss, and let $X_i = 1$ if said outcome is heads. The fraction of heads after n independent tosses is

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

According to the Weak Law of Large Numbers, $\bar{X}_n$ converges to p in probability as $n \to \infty$.

How large should n be so that $\mathrm{P}\left(0.4 \leq \bar{X}_n \leq 0.6\right) \geq 0.7$? [Hint: Use Chebyshev's inequality]

9. A fair die is tossed many times in succession. The tosses are independent of each other. Initialize $X_0 = 6$ and for each $n \geq 1$, let X_n denote the minimum among the first n tosses.

(a) (12 points) Show that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain, specify its state space and determine the transition probability matrix.

(b) (4 points) Specifiy the communication classes and determine whether they are transient or recurrent.