ECE440 - Introduction to Random Processes

Midterm Exam

November 1, 2021

Instructions:

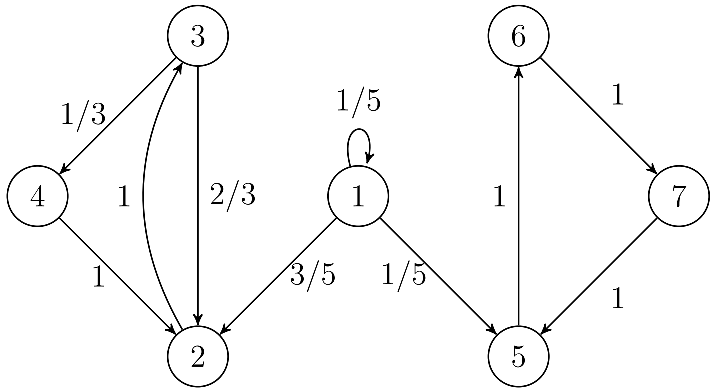
- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 points.
- Duration: 90 minutes.
- This exam has 13 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name: SOLUTIONS

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	24		5.	12	
2.	10		6.	10	
3.	12		7.	8	
4.	12		8.	12	
			Total	100	

GOOD LUCK!

1. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2, 3, 4, 5, 6, 7\}$, state transition diagram



and initial distribution $\mathrm{P}(X_0 = 1) = 1$ and $\mathrm{P}(X_0 = i) = 0$ for $2 \leq i \leq 7$.

(a) (2 points) $\mathrm{P}(X_5 = 4, X_4 = 3, X_3 = 2, X_2 = 3, X_1 = 2, X_0 = 1) = ?$

$\frac{2}{15}$

Repeated application of the definition of conditional probability and the Markov property yield

$$\begin{aligned}\mathrm{P}(X_5 = 4, X_4 = 3, X_3 = 2, X_2 = 3, X_1 = 2, X_0 = 1) &= \mathrm{P}(X_0 = 1) P_{12} P_{23} P_{32} P_{23} P_{34} \\ &= 1 \times \frac{3}{5} \times 1 \times \frac{2}{3} \times 1 \times \frac{1}{3} = \frac{2}{15}.\end{aligned}$$

(b) (6 points) Specify the communication classes and determine whether they are transient or recurrent.

There are three communication classes. State 1 only communicates with itself and since there is positive probability of leaving to never come back, it comprises a transient class $\mathcal{T} = \{1\}$. The other two classes are recurrent, namely $\mathcal{R}_1 = \{2, 3, 4\}$ and $\mathcal{R}_2 = \{5, 6, 7\}$.

(c) (6 points) What is the period of state 6?

3

The behavior in class $\mathcal{R}_2$ is deterministic, with visits repeating cyclicly $\cdots 5 \to 6 \to 7 \to 5 \to 6 \cdots$. States are revisited every 3 steps, hence the period of state 6 is 3.

(d) (4 points) $\lim_{n\to\infty} P_{11}^n = ?$

0

State 1 is transient so visits to this state eventually stop almost surely. Thus $\lim_{n\to\infty} P_{11}^n = 0$.

(e) (6 points) Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=2}^{n} \mathbb{I}\left\{X_i = 7 \,\middle|\, X_1 = 5\right\}$$

and provide justification for the existence of the limit.

$\frac{1}{3}$

The event $X_1 = 5$ indicates the Markov chain is absorbed by the recurrent class $\mathcal{R}_2$. Even though this class is not ergodic (recall its states have period 3), the ergodic limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=2}^{n} \mathbb{I}\left\{X_i = 7 \,\middle|\, X_1 = 5\right\} = \frac{1}{3} \quad \text{a. s.},$$

because the process spends exactly a third of the time in each state.

2. (10 points) Consider i.i.d. continuous random variables $X_1, \ldots, X_{10}$ with probability density function

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the approximate probability that $Y = \sum_{i=1}^{10} X_i$ will exceed 7. Write your result in terms of the complementary cumulative distribution function Φ of a standard Normal random variable, that is for $Z \sim \mathcal{N}(0,1)$ then

$$\Phi(z) = \mathrm{P}\left(Z \geq z\right) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-u^2/2} du.$$

$\Phi\left(\frac{1}{\sqrt{5}}\right)$

We have $\mathbb{E}[Y] = 10\mathbb{E}[X_1]$ and $\text{var}[Y] = 10\text{var}[X_1]$. Some simple calculations yield

$$\mathbb{E}[X_1] = \int_0^1 2x^2 dx = \frac{2}{3} \quad \Rightarrow \mathbb{E}[Y] = \frac{20}{3}$$

and

$$\text{var}[X_1] = \mathbb{E}\left[X_1^2\right] - \mathbb{E}[X_1]^2 = \int_0^1 2x^3 dx - \frac{4}{9} = \frac{1}{18} \quad \Rightarrow \text{var}[Y] = \frac{10}{18}.$$

By virtue of the Central Limit Theorem, we can approximate the distribution of Y with that of a Normal, namely

$$Y \sim \mathcal{N}\left(\frac{20}{3}, \frac{10}{18}\right).$$

With $Z \sim \mathcal{N}(0,1)$, the desired probability can thus be approximated as

$$\mathrm{P}\left(Y > 7\right) \approx \mathrm{P}\left(Z > \frac{7 - \mathbb{E}[Y]}{\sqrt{\text{var}[Y]}}\right) = \Phi\left(\frac{1}{\sqrt{5}}\right).$$

3. Draw a county at random from the United States. Then draw n people at random from that county. Let $0 \leq X \leq n$ be the number of those people who are infected with COVID-19. If Q denotes the proportion of people in the county with the virus, then Q is also a random variable since it varies from county to county. Given $Q = q$, we have that $X \sim \text{Binomial}(n, q)$. Also, suppose that the random variable Q is uniformly distributed in the interval $[0, 1]$. The distribution of X is thus constructed in two steps, leading to a so-termed hierarchical model that we write

$$Q \sim \text{Uniform}[0, 1]$$
$$X \,\big|\, Q = q \sim \text{Binomial}(n, q).$$

(a) (2 points) $\mathbb{E}\left[X \,\middle|\, Q = q\right] =?$

$$nq$$

Because $X \,\big|\, Q = q \sim \text{Binomial}(n, q)$ it follows that $\mathbb{E}\left[X \,\middle|\, Q = q\right] = nq$.

(b) (4 points) $\mathbb{E}\left[X\right] =?$

$$\frac{n}{2}$$

Notice that $\mathbb{E}\left[Q\right] = \frac{1}{2}$ since $Q \sim \text{Uniform}[0, 1]$. Using the law of iterated expectations we find

$$\mathbb{E}\left[X\right] = \mathbb{E}_Q\left[\mathbb{E}_X\left[X \,\middle|\, Q\right]\right] = \mathbb{E}_Q\left[nQ\right] = n\mathbb{E}\left[Q\right] = \frac{n}{2}.$$

(c) (6 points) $\text{var}\left[X\right] =?$

$$\frac{n}{6} + \frac{n^2}{12}$$

To compute $\text{var}\left[X\right]$ we condition on Q. Because $X \,\big|\, Q = q \sim \text{Binomial}(n, q)$ it follows that $\mathbb{E}\left[X \,\middle|\, Q\right] = nQ$ and $\text{var}\left[X \,\middle|\, Q\right] = nQ(1 - Q)$. Using the conditional variance formula

$$\begin{aligned}
\text{var}\left[X\right] &= \mathbb{E}_Q\left[\text{var}_X\left[X \,\middle|\, Q\right]\right] + \text{var}_Q\left[\mathbb{E}_X\left[X \,\middle|\, Q\right]\right] \\
&= \mathbb{E}\left[nQ(1 - Q)\right] + \text{var}\left[nQ\right] \\
&= n\int_0^1 q(1 - q)dq + n^2\text{var}\left[Q\right] = \frac{n}{6} + \frac{n^2}{12}.
\end{aligned}$$

In arriving at the final result we used that $\text{var}\left[Q\right] = \frac{1}{12}$ since $Q \sim \text{Uniform}[0, 1]$.

4. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables, which are uniformly distributed in the interval $[0, 1]$.

(a) (8 points) Define the random variable

$$Y = \min\{X_1, X_2\}.$$

Write down an expression for $f_Y(y)$, the probability density function of Y. [Hint: it might be easier to first compute $\mathrm{P}(Y > y)$.]

$$f_Y(y) = \begin{cases} 2(1-y), & 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We derive first the complementary cumulative distribution function $\mathrm{P}(Y > y)$. To start, note that because $X_1, X_2 \sim$ Uniform$[0, 1]$ then $0 \leq Y \leq 1$. Accordingly, $\mathrm{P}(Y > y) = 1$ for $y \leq 0$ and $\mathrm{P}(Y > y) = 0$ for $y \geq 1$. For $0 < y < 1$, we have

$$\begin{aligned} \mathrm{P}(Y > y) &= \mathrm{P}(\min\{X_1, X_2\} > y) \\ &= \mathrm{P}(X_1 > y, X_2 > y) \\ &= \mathrm{P}(X_1 > y) \times \mathrm{P}(X_2 > y) = (1-y)^2. \end{aligned}$$

We used that: (i) events $\{\min\{X_1, X_2\} > y\}$ and $\{X_1 > y, X_2 > y\}$ are equivalent; and (ii) X_1 and X_2 are independent. Moving on, since the cumulative distribution function $F_Y(y) = \mathrm{P}(Y \leq y) = 1 - \mathrm{P}(Y > y)$, we have

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ 1 - (1-y)^2, & 0 \leq y < 1, \\ 1, & y \geq 1. \end{cases}$$

Recalling that the density $f_Y(y) = dF_Y(y)/dy$, we arrive at the desired result

$$f_Y(y) = \begin{cases} 2(1-y), & 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) (4 points) Let $Y_{\mathbb{N}} = Y_1, Y_2, \ldots, Y_n, \ldots$ be the sequence of random variables given by

$$Y_n = \min\{X_1, \ldots, X_n\}, \quad n \geq 1.$$

Show that Y_n converges in probability to 0 as $n \to \infty$.

To show that Y_n converges in probability to 0, notice that

$$\begin{aligned} \mathrm{P}(|Y_n - 0| > \epsilon) &= \mathrm{P}(\min\{X_1, \ldots, X_n\} > \epsilon) \\ &= \mathrm{P}(X_1 > \epsilon) \times \mathrm{P}(X_2 > \epsilon) \times \ldots \times \mathrm{P}(X_n > \epsilon) = (1-\epsilon)^n, \end{aligned}$$

which goes to 0 as $n \to \infty$.

5. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix},$$

where $0 < a < 1$ and $0 < b < 1$. We define the recurrence time of state $i \in S$ as

$$T_i = \min\{n > 0 : X_n = i\} \text{ given that } X_0 = i.$$

Accordingly, T_i is a discrete random variable taking values on the integers $\{1, 2, 3, \ldots\}$.
(a) (6 points) Compute $p_{T_1}(n) = \mathrm{P}\left(T_1 = n \,\middle|\, X_0 = 1\right)$, the probability mass function of T_1.

$$p_{T_1}(n) = \begin{cases} 1-a, & n = 1, \\ ab(1-b)^{n-2}, & n \geq 2. \end{cases}$$

We have that

$$\begin{aligned}
&\mathrm{P}\left(T_1 = 1 \,\middle|\, X_0 = 1\right) = \mathrm{P}\left(X_1 = 1 \,\middle|\, X_0 = 1\right) = 1 - a, \\
&\mathrm{P}\left(T_1 = 2 \,\middle|\, X_0 = 1\right) = \mathrm{P}\left(X_2 = 1, X_1 = 2 \,\middle|\, X_0 = 1\right) = a \times b, \\
&\mathrm{P}\left(T_1 = 3 \,\middle|\, X_0 = 1\right) = \mathrm{P}\left(X_3 = 1, X_2 = 2, X_1 = 2 \,\middle|\, X_0 = 1\right) = a \times b \times (1-b), \\
&\qquad\qquad\qquad\qquad \vdots \\
&\mathrm{P}\left(T_1 = n \,\middle|\, X_0 = 1\right) = \mathrm{P}\left(X_n = 1, X_{n-1} = 2, \ldots, X_1 = 2 \,\middle|\, X_0 = 1\right) = a \times b \times (1-b)^{n-2}.
\end{aligned}$$

All in all, the conclusion is that the probability mass function of T_1 is given by

$$p_{T_1}(n) = \begin{cases} 1-a, & n = 1, \\ ab(1-b)^{n-2}, & n \geq 2. \end{cases}$$

(b) (6 points) $\mathbb{E}\left[T_1 \,\middle|\, X_0 = 1\right] =?$ [Reminder: for your calculations, it may be useful to recall the sum of the geometric series $\sum_{r=1}^{\infty} \alpha^{r-1} = 1/(1-\alpha)$, for $0 < \alpha < 1$.]

$$\frac{a+b}{b}$$

From the definition of expectation

$$\begin{aligned}
\mathbb{E}\left[T_1 \,\middle|\, X_0 = 1\right] &= \sum_{n=1}^{\infty} n p_{T_1}(n) \\
&= 1 - a + \sum_{n=2}^{\infty} nab(1-b)^{n-2} \\
&= 1 - a + \sum_{k=1}^{\infty} (k+1)ab(1-b)^{k-1} \\
&= 1 - a + ab\sum_{k=1}^{\infty} (1-b)^{k-1} + a\sum_{k=1}^{\infty} kb(1-b)^{k-1} \\
&= 1 - a + \frac{ab}{1-(1-b)} + \frac{a}{b} = \frac{a+b}{b}.
\end{aligned}$$

In addition to the change of variables $n = k+1$ and the sum of the geometric series, we used that $\sum_{k=1}^{\infty} kb(1-b)^{k-1} = \frac{1}{b}$ is the expectation of a Geometric(b) random variable.

6. Suppose that we want to evaluate the integral

$$I = \int_a^b f(x)dx$$

for some integrable function f. Unlike polynomial, rational or trigonometric functions, if f is complicated then there may be no known closed form expression for I. In these cases, numerical integration methods are appropriate to approximate the value of I.

Here we will explore the simplest version of Monte Carlo integration. Start by writing

$$I = \int_a^b f(x)dx = \int_a^b w(x)g(x)dx,$$

where $w(x) = f(x)(b - a)$ and $g(x) = 1/(b - a)$.

(a) (4 points) Show that $I = \mathbb{E}[w(X)]$, where X is a random variable. Specify the distribution of X.

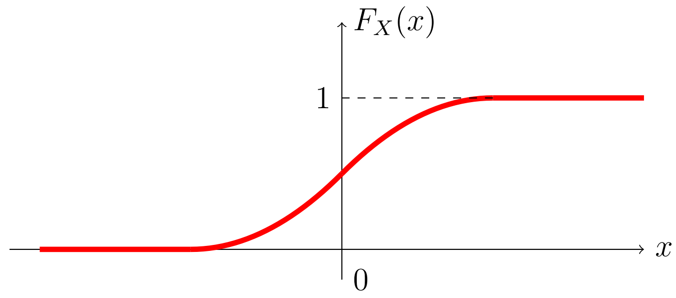
Recognizing $g(x) = 1/(b - a)$ as the probability density function of a random variable that is uniformly distributed in $[a, b]$, then it follows that $I = \mathbb{E}[w(X)]$, where $X \sim \text{Uniform}[a, b]$.

(b) (6 points) Suppose that you can generate N i.i.d. samples from the distribution of X. Describe a method to estimate the value of I, and state any result you use to justify your approximation.

Supose we generate i.i.d. samples $X_1, \ldots, X_N \sim \text{Uniform}[a, b]$. The law of large numbers justifies the Monte Carlo approach of estimating the integral $I = \mathbb{E}[w(X)]$ via the sample mean, namely

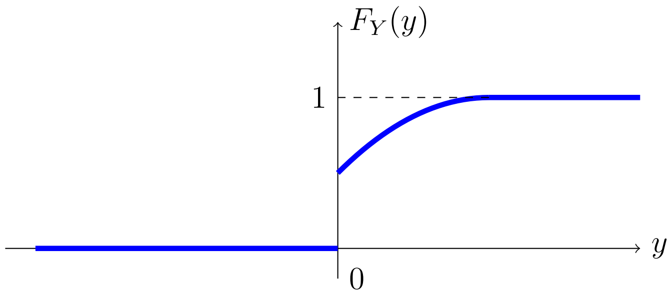
$$\hat{I} = \frac{1}{N}\sum_{i=1}^{N} w(X_i).$$

7. (8 points) Consider a random variable X with cumulative distribution function $F_X(x) = \text{P}(X \leq x)$ given in the following figure.



Sketch $F_Y(y) = \text{P}(Y \leq y)$, the cumulative distribution function of $Y = \max\{0, X\}$.

Because $Y = \max\{0, X\} \geq 0$, then it follows immediately that $\text{P}(Y \leq y) = 0$ for $y < 0$. Now, for $y \geq 0$ then $\text{P}(Y \leq y) = \text{P}(X \leq y) = F_X(y)$. The resulting cumulative distribution function is depicted in the following figure.



8. (12 points) Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $0 < p < 1$ and $q = 1 - p$.

Determine the stationary distribution of $X_{\mathbb{N}}$. [Reminder: for your calculations, it might useful to recall the partial geometric sum $\sum_{r=0}^{k} \alpha^r = \frac{1-\alpha^{k+1}}{1-\alpha}$, for $\alpha \neq 1$.]

The Markov Chain is ergodic and has a unique stationary distribution $\boldsymbol{\pi} = [\pi_1, \pi_2, \pi_3, \pi_4, \pi_5]^\top$. Writing down the balance equations for states $2, \ldots, 5$ we obtain (recall $p + q = 1$)

$$\begin{aligned} \pi_2 &= p\pi_1 \\ \pi_3 &= p\pi_2 \;\Rightarrow \pi_3 = p^2\pi_1 \\ \pi_4 &= p\pi_3 \;\Rightarrow \pi_4 = p^3\pi_1 \\ \pi_5 &= p\pi_4 \;\Rightarrow \pi_5 = p^4\pi_1. \end{aligned}$$

Finally, since $\sum_{i=1}^{5} \pi_i = 1$ we can readily solve for π_1 to obtain

$$\pi_1 \left(1 + p + p^2 + p^3 + p^4\right) = 1 \;\Rightarrow \pi_1 = \frac{1}{\sum_{i=0}^{4} p^i} = \frac{q}{1 - p^5}.$$

Putting all the pieces together, the stationary distribution is

$$\boldsymbol{\pi} = \left[\frac{q}{1-p^5}, \frac{pq}{1-p^5}, \frac{p^2 q}{1-p^5}, \frac{p^3 q}{1-p^5}, \frac{p^4 q}{1-p^5}\right]^\top.$$