ECE440 - Introduction to Random Processes

Midterm Exam

November 2, 2022

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 102, extra points are bonus points).
- Duration: 90 minutes.
- This exam has 12 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:___________________________________

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	20		5.	14	
2.	10		6.	10	
3.	10		7.	22	
4.	16				
			Total	102	

GOOD LUCK!

1. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{pmatrix},$$

and initial distribution $\mathrm{P}(X_0 = 1) = 1/2$ and $\mathrm{P}(X_0 = 2) = 1/2$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 7/12 & 5/12 \\ 5/9 & 4/9 \end{pmatrix} = \begin{pmatrix} 0.58 & 0.42 \\ 0.56 & 0.44 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\big(X_3 = 2 \,\big|\, X_2 = 1, X_1 = 2\big) =?$

(b) (3 points) $\mathrm{P}\big(X_4 = 1 \,\big|\, X_2 = 2, X_1 = 1, X_0 = 1\big) =?$

(c) (5 points) $\mathrm{P}\left(X_2 = 1\right) =?$

(d) (5 points) $\mathbb{E}\left[X_2 \,\middle|\, X_0 = 1\right] =?$

(e) (5 points) $\mathbb{E}\left[X_2\right] =?$

2. (10 points) If a sequence of random variables $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ converges in distribution to a Normal then the delta method allows us to find the limiting distribution of $Y_{\mathbb{N}} = g(X_0), g(X_1), \ldots, g(X_n), \ldots$, where g is any differentiable function with derivative g'.

Theorem (The delta method). Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is such that

$$\frac{\sqrt{n}(X_n - \mu)}{\sigma} \quad \text{converges in distribution to a standard Normal as } n \to \infty,$$

and that g is a differentiable function such that $g'(\mu) \neq 0$. Then

$$\frac{\sqrt{n}(g(X_n) - g(\mu))}{|g'(\mu)|\sigma} \quad \text{converges in distribution to a standard Normal as } n \to \infty.$$

In other words, for sufficiently large n

$$X_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \quad \text{implies that } g(X_n) \sim \mathcal{N}\left(g(\mu), (g'(\mu))^2 \frac{\sigma^2}{n}\right).$$

Suppose that $U_{\mathbb{N}} = U_1, U_2, \ldots, U_n, \ldots$ is an i.i.d. sequence of random variables, which are uniformly distributed in the interval $[0, 1]$. Let

$$Z_n = \left(\frac{1}{n}\sum_{i=1}^{n} U_i\right)^2, \quad n = 1, 2, \ldots.$$

Find the distribution of Z_n for sufficiently large n.

3. Consider a probability space $(S, \mathcal{F}, \mathrm{P}(\cdot))$. Suppose that A and B are events in $\mathcal{F}$.

(a) (5 points) Derive a simple expression for $\mathrm{P}(B \cap A^c)$ in terms of $\mathrm{P}(B)$ and $\mathrm{P}(A \cap B)$. Show your work.

(b) (5 points) Now suppose that A and B are independent. Prove that A^c and B are independent.

4. Consider n independent trials, each of which results in one of the outcomes $1, \ldots, r$, with respective probabilities $p_1, \ldots, p_r$, $\sum_{i=1}^{r} p_i = 1$. If we let N_i denote the number of trials that result in outcome i, then the random vector $\mathbf{N} = [N_1, \ldots, N_r]^\top$ is said to have a multinomial distribution.

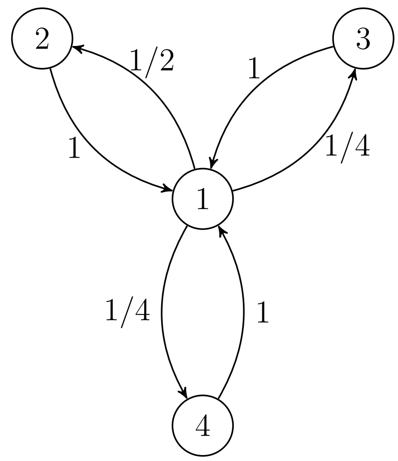
(a) (2 points) $\mathbb{E}[N_j] =?$

(b) (3 points) $\mathbb{E}\left[N_j^2\right] =?$

(c) (5 points) Explain why the conditional distribution of N_i, given that $N_j = k$ for $j \neq i$, is Binomial$(n-k, \frac{p_i}{1-p_j})$.

(d) (6 points) Let $i \neq j$. $\mathbb{E}[N_i N_j] =?$ (Hint: condition on N_j)

5. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3, 4\}$, state transition diagram



and initial distribution $\mathrm{P}(X_0 = 1) = 1$ and $\mathrm{P}(X_0 = i) = 0$ for $2 \leq i \leq 4$.

(a) (3 points) What is the period of state 2?

(b) (3 points) $\sum_{i=1}^{9} \mathbb{I}\{X_i = 1\} =?$

(c) (8 points) Calculate

$$\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \mathbb{I}\left\{X_i = 3\right\}$$

and provide justification for the existence of the limit.

6. Consider two independent random variables U and T. Suppose that U is uniformly distributed in the interval $[0, 2]$ and that T has the exponential distribution with mean $1/2$.

(a) (6 points) $\mathbb{E}\left[Ue^T\right] =?$

(b) (4 points) Note that $\text{var}\left[U\right] = 1/3$ and $\text{var}\left[T\right] = 1/4$. $\text{var}\left[2U - 2T\right] =?$

7. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 3/4 & 1/4 & 0 \\ 0 & p & 2/5 \end{pmatrix}.$$

(a) (2 points) What is the value of p? Explain.

(b) (6 points) Draw the corresponding state transition diagram.

(c) (4 points) Specifiy the communication classes and determine whether they are transient or recurrent.

(d) (10 points) Determine the limit probabilities $\lim_{n\to\infty} P_{ij}^n$ for each $i, j \in S$.