ECE440 - Introduction to Random Processes

Midterm Exam

November 2, 2022

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 102, extra points are bonus points).
- Duration: 90 minutes.
- This exam has 12 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:__________SOLUTIONS__________

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	20		5.	14	
2.	10		6.	10	
3.	10		7.	22	
4.	16				
			Total	102	

GOOD LUCK!

1. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{pmatrix},$$

and initial distribution $\mathrm{P}(X_0 = 1) = 1/2$ and $\mathrm{P}(X_0 = 2) = 1/2$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 7/12 & 5/12 \\ 5/9 & 4/9 \end{pmatrix} = \begin{pmatrix} 0.58 & 0.42 \\ 0.56 & 0.44 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left(X_3 = 2 \,\middle|\, X_2 = 1, X_1 = 2\right) =?$

$$\boxed{\frac{1}{2}}$$

From the Markov property it follows that

$$\mathrm{P}\left(X_3 = 2 \,\middle|\, X_2 = 1, X_1 = 2\right) = \mathrm{P}\left(X_3 = 2 \,\middle|\, X_2 = 1\right) = P_{12} = \frac{1}{2}.$$

(b) (3 points) $\mathrm{P}\left(X_4 = 1 \,\middle|\, X_2 = 2, X_1 = 1, X_0 = 1\right) =?$

$$\boxed{\frac{5}{9}}$$

Likewise, $\mathrm{P}\left(X_4 = 1 \,\middle|\, X_2 = 2, X_1 = 1, X_0 = 1\right) = \mathrm{P}\left(X_4 = 1 \,\middle|\, X_2 = 2\right) = P^2_{21} = \frac{5}{9}$.

(c) (5 points) $\mathrm{P}(X_2 = 1) =?$

$$\boxed{\frac{41}{72}}$$

From the law of total probability (conditioning on X_0), one has

$$\mathrm{P}(X_2 = 1) = \sum_{i=1}^{2} \mathrm{P}\left(X_2 = 1 \,\middle|\, X_0 = i\right) \mathrm{P}(X_0 = i) = \sum_{i=1}^{2} P^2_{i1} \mathrm{P}(X_0 = i) = \frac{7}{12} \times \frac{1}{2} + \frac{5}{9} \times \frac{1}{2} = \frac{41}{72}.$$

(d) (5 points) $\mathbb{E}\left[X_2 \,\middle|\, X_0 = 1\right] =?$

$$\boxed{\frac{17}{12}}$$

Using the definition of conditional expectation, one obtains

$$\begin{aligned} \mathbb{E}\left[X_2 \,\middle|\, X_0 = 1\right] &= \sum_{i=1}^{2} i \times \mathrm{P}\left(X_2 = i \,\middle|\, X_0 = 1\right) = \sum_{i=1}^{2} i \times P^2_{1i} \\ &= 1 \times \frac{7}{12} + 2 \times \frac{5}{12} = \frac{17}{12}. \end{aligned}$$

(e) (5 points) $\mathbb{E}[X_2] = ?$

$$\boxed{\frac{103}{72}}$$

The unconditional pmf of X_2 is $\mathrm{P}(X_2 = 1) = \frac{41}{72}$ and $\mathrm{P}(X_2 = 2) = \frac{31}{72}$. Hence, the expectation becomes

$$\mathbb{E}[X_2] = 1 \times \frac{41}{72} + 2 \times \frac{31}{72} = \frac{103}{72}.$$

2. (10 points) If a sequence of random variables $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ converges in distribution to a Normal then the delta method allows us to find the limiting distribution of $Y_{\mathbb{N}} = g(X_0), g(X_1), \ldots, g(X_n), \ldots$, where g is any differentiable function with derivative g'.

Theorem (The delta method). Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is such that

$$\frac{\sqrt{n}(X_n - \mu)}{\sigma} \quad \text{converges in distribution to a standard Normal as } n \to \infty,$$

and that g is a differentiable function such that $g'(\mu) \neq 0$. Then

$$\frac{\sqrt{n}(g(X_n) - g(\mu))}{|g'(\mu)|\sigma} \quad \text{converges in distribution to a standard Normal as } n \to \infty.$$

In other words, for sufficiently large n

$$X_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \quad \text{implies that } g(X_n) \sim \mathcal{N}\left(g(\mu), (g'(\mu))^2 \frac{\sigma^2}{n}\right).$$

Suppose that $U_{\mathbb{N}} = U_1, U_2, \ldots, U_n, \ldots$ is an i.i.d. sequence of random variables, which are uniformly distributed in the interval $[0, 1]$. Let

$$Z_n = \left(\frac{1}{n}\sum_{i=1}^{n} U_i\right)^2, \quad n = 1, 2, \ldots.$$

Find the distribution of Z_n for sufficiently large n.

$$\boxed{Z_n \sim \mathcal{N}\left(\frac{1}{4}, \frac{1}{12n}\right)}$$

Notice first that because $U_n \sim \text{Uniform}[0, 1]$, then $\mathbb{E}[U_n] = \frac{1}{2}$ and $\operatorname{var}[U_n] = \frac{1}{12}$. By the Central Limit Theorem, we can assert that the limiting distribution of the sample mean is given by

$$\frac{1}{n}\sum_{i=1}^{n} U_i \sim \mathcal{N}\left(\frac{1}{2}, \frac{1}{12n}\right).$$

Now, Z_n is a quadratic transformation of said sample mean (i.e., $g(x) = x^2$ and $g'(x) = 2x$) so the delta method yields the desired distribution

$$Z_n \sim \mathcal{N}\left(\frac{1}{4}, \frac{1}{12n}\right).$$

3. Consider a probability space $(S, \mathcal{F}, \mathrm{P}(\cdot))$. Suppose that A and B are events in $\mathcal{F}$.

(a) (5 points) Derive a simple expression for $\mathrm{P}(B \cap A^c)$ in terms of $\mathrm{P}(B)$ and $\mathrm{P}(A \cap B)$. Show your work.

$$\boxed{\mathrm{P}(B \cap A^c) = \mathrm{P}(B) - \mathrm{P}(A \cap B)}$$

Because $B = \{B \cap A\} \cup \{B \cap A^c\}$ and $\{B \cap A\} \cap \{B \cap A^c\} = \emptyset$, then

$$\mathrm{P}(B) = \mathrm{P}(B \cap A) + \mathrm{P}(B \cap A^c) \Rightarrow \mathrm{P}(B \cap A^c) = \mathrm{P}(B) - \mathrm{P}(A \cap B).$$

(b) (5 points) Now suppose that A and B are independent. Prove that A^c and B are independent.

The independence of A and B implies $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$, hence

$$\begin{aligned} \mathrm{P}(B \cap A^c) =& \mathrm{P}(B) - \mathrm{P}(A \cap B) \\ =& \mathrm{P}(B) - \mathrm{P}(A)\,\mathrm{P}(B) \\ =& \mathrm{P}(B)\,(1 - \mathrm{P}(A)) \\ =& \mathrm{P}(B)\,\mathrm{P}(A^c). \end{aligned}$$

The conclusion is that A^c and B are independent as well.

4. Consider n independent trials, each of which results in one of the outcomes $1, \ldots, r$, with respective probabilities $p_1, \ldots, p_r$, $\sum_{i=1}^{r} p_i = 1$. If we let N_i denote the number of trials that result in outcome i, then the random vector $\mathbf{N} = [N_1, \ldots, N_r]^\top$ is said to have a multinomial distribution.

(a) (2 points) $\mathbb{E}[N_j] =$?

$$\boxed{np_j}$$

Recognizing that $N_j \sim \text{Binomial}(n, p_j)$, we find $\mathbb{E}[N_j] = np_j$.

(b) (3 points) $\mathbb{E}\left[N_j^2\right] =$?

$$\boxed{np_j(1 - p_j) + n^2 p_j^2}$$

Recall that $\mathrm{var}[N_j] = np_j(1 - p_j)$. The second moment is thus given by

$$\mathbb{E}\left[N_j^2\right] = \mathrm{var}[N_j] + (\mathbb{E}[N_j])^2 = np_j(1 - p_j) + (np_j)^2.$$

(c) (5 points) Explain why the conditional distribution of N_i, given that $N_j = k$ for $j \neq i$, is Binomial$(n-k, \frac{p_i}{1-p_j})$.

Conditioned on $N_j = k$ for $j \neq i$, outcome i could have only resulted from the remaining $n-k$ trials. These remaining trials are independent, and the probability of outcome i given than the outcome cannot be j (i.e., our "success" probability) is simply

$$\mathrm{P}\left(i \,\middle|\, \text{not } j\right) = \frac{\mathrm{P}\left(\{i\} \cap \{\text{not } j\}\right)}{\mathrm{P}\left(\text{not } j\right)} = \frac{\mathrm{P}\left(i\right)}{\mathrm{P}\left(\text{not } j\right)} = \frac{p_i}{1-p_j}.$$

The conclusion is that $N_i \,\middle|\, N_j = k \sim$ Binomial$(n-k, \frac{p_i}{1-p_j})$ as desired.

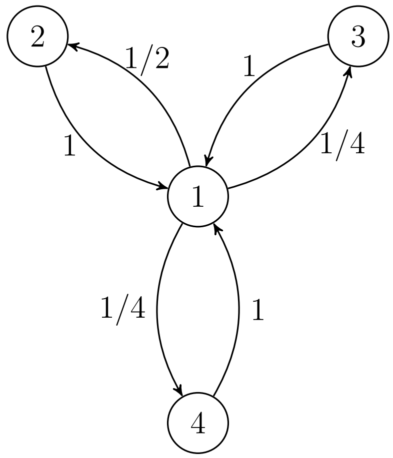
(d) (6 points) Let $i \neq j$. $\mathbb{E}\left[N_i N_j\right] =?$ (Hint: condition on N_j)

$$\boxed{p_i p_j n(n-1)}$$

We compute the expectation by conditioning on N_j, thus obtaining

$$\begin{aligned}
\mathbb{E}\left[N_i N_j\right] &= \sum_{k=0}^{n} \mathbb{E}\left[N_i N_j \,\middle|\, N_j = k\right] \mathrm{P}\left(N_j = k\right) \\
&= \sum_{k=0}^{n} \mathbb{E}\left[N_i k \,\middle|\, N_j = k\right] \mathrm{P}\left(N_j = k\right) \\
&= \sum_{k=0}^{n} k\mathbb{E}\left[N_i \,\middle|\, N_j = k\right] \mathrm{P}\left(N_j = k\right) \\
&= \sum_{k=0}^{n} k(n-k)\frac{p_i}{1-p_j} \mathrm{P}\left(N_j = k\right) \\
&= \frac{p_i}{1-p_j}\left(\sum_{k=0}^{n} nk\mathrm{P}\left(N_j = k\right) - \sum_{k=0}^{n} k^2\mathrm{P}\left(N_j = k\right)\right) \\
&= \frac{p_i}{1-p_j}\left(n\mathbb{E}\left[N_j\right] - \mathbb{E}\left[N_j^2\right]\right) \\
&= \frac{p_i}{1-p_j}\left(n^2 p_j - np_j(1-p_j) - n^2 p_j^2\right) \\
&= p_i p_j n(n-1).
\end{aligned}$$

5. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3, 4\}$, state transition diagram



and initial distribution $\mathrm{P}(X_0 = 1) = 1$ and $\mathrm{P}(X_0 = i) = 0$ for $2 \le i \le 4$.

(a) (3 points) What is the period of state 2?

2

Notice that all states communicate, hence they have the same period. From the state transition diagram, it is apparent that the Markov chain (deterministically) re-enters state 1 every two time steps. This implies that state 1 is periodic with period 2, and so are all other states.

(b) (3 points) $\sum_{i=1}^{9} \mathbb{I}\{X_i = 1\} =?$

4

Because $X_0 = 1$ almost surely, the sample paths over the interval $n = 1, \ldots, 9$ will look like

$X_1 \neq 1 \to X_2 = 1 \to X_3 \neq 1 \to X_4 = 1 \to X_5 \neq 1 \to X_6 = 1 \to X_7 \neq 1 \to X_8 = 1 \to X_9 \neq 1.$

All in all, the number of visits to state 1 during the interval $n = 1, \ldots, 9$ is 4.

(c) (8 points) Calculate

$$\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \mathbb{I}\{X_i = 3\}$$

and provide justification for the existence of the limit.

$\frac{1}{2}$

Even though all states are periodic, because the single communication class is positive recurrent then the ergodic theorem holds. Hence, the limit of interest is

$$\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \mathbb{I}\{X_i = 3\} = 4 \times \pi_3 \quad \text{a.s.}$$

Now, the long-run fraction of time spent in each state can be determined by inspection. The Markov chain (deterministically) spends half of the time in state 1, hence $\pi_1 = 1/2$. From the probabilities out of state 1, we find that $\pi_2 = 1/2 \times P_{12} = 1/4$, and $\pi_3 = 1/2 \times P_{13} = 1/8 = \pi_4$. Substituting the value of π_3 we obtain

$$\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \mathbb{I}\{X_i = 3\} = \frac{1}{2} \quad \text{a.s.}$$

6. Consider two independent random variables U and T. Suppose that U is uniformly distributed in the interval $[0, 2]$ and that T has the exponential distribution with mean $1/2$.

(a) (6 points) $\mathbb{E}\left[Ue^T\right] =$?

$$\boxed{2}$$

Because U and T are independent, then $\mathbb{E}\left[Ue^T\right] = \mathbb{E}[U]\,\mathbb{E}\left[e^T\right]$. The mean of $U \sim \text{Uniform}[0, 2]$ is $\mathbb{E}[U] = 1$. Now, since $T \sim \text{Geometric}(2)$, then

$$\mathbb{E}\left[e^T\right] = \int_0^\infty e^t 2e^{-2t} dt = 2\int_0^\infty e^{-t} dt = 2$$

Putting all pieces together, we arrive at $\mathbb{E}\left[Ue^T\right] = 2$.

(b) (4 points) Note that $\text{var}[U] = 1/3$ and $\text{var}[T] = 1/4$. $\text{var}[2U - 2T] =$?

$$\boxed{\frac{7}{3}}$$

Once more, because U and T are independent, then

$$\text{var}[2U - 2T] = 4\text{var}[U] + 4\text{var}[T] = 4 \times \frac{1}{3} + 4 \times \frac{1}{4} = \frac{7}{3}.$$

7. Suppose that $X_\mathbb{N} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 3/4 & 1/4 & 0 \\ 0 & p & 2/5 \end{pmatrix}.$$

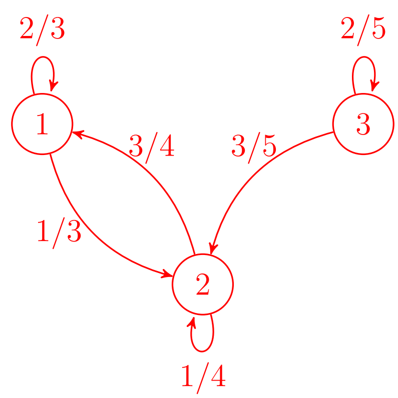
(a) (2 points) What is the value of p? Explain.

$$\boxed{\frac{3}{5}}$$

Rows of transition probability matrices correspond to conditional pmfs over states, hence by the axioms of probability all entries in each row should sum up to 1. Accordingly, $p = 3/5$.

(b) (6 points) Draw the corresponding state transition diagram.

The state transition diagram is



(c) (4 points) Specify the communication classes and determine whether they are transient or recurrent.

This Markov chain has two communication classes. One of them is recurrent $\mathcal{R} = \{1, 2\}$ and the other is transient $\mathcal{T} = \{3\}$.

(d) (10 points) Determine the limit probabilities $\lim_{n\to\infty} P_{ij}^n$ for each $i, j \in S$.

Because state 3 is transient, visits to that state stop almost surely and hence

$$\lim_{n\to\infty} P_{i3}^n = 0, \quad \text{for all } i \in S.$$

The other two states belong to the ergodic class $\mathcal{R}$ and the limit probabilities satisfy

$$\lim_{n\to\infty} P_{i1}^n = \pi_1 \text{ and } \lim_{n\to\infty} P_{i2}^n = \pi_2, \quad \text{for all } i \in S,$$

where the stationary distribution $\boldsymbol{\pi} = [\pi_1, \pi_2]^\top$ satisfies

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 2/3 & 3/4 \\ 1/3 & 1/4 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \quad \pi_1 + \pi_2 = 1.$$

Solving the linear system yields $\boldsymbol{\pi} = \left[\frac{9}{13}, \frac{4}{13}\right]^\top$.