ECE440 - Introduction to Random Processes

Midterm Exam

November 1, 2023

Instructions:

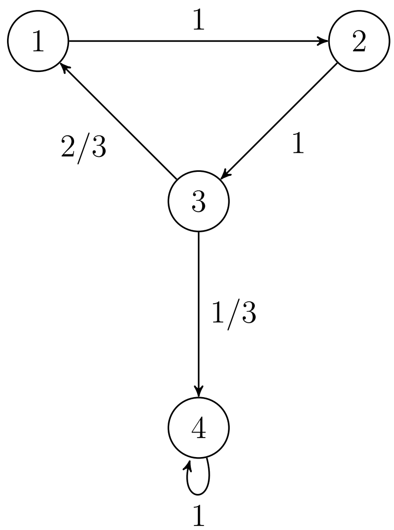
- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 101, extra point is a bonus point).
- Duration: 90 minutes.
- This exam has 11 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:__________________________________

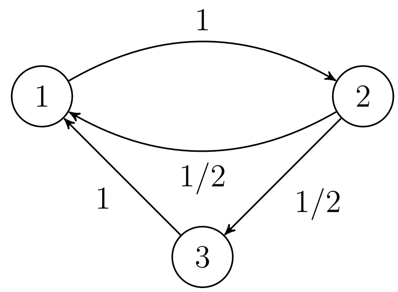
Problem	Max. Points	Score	Problem	Max. Points	Score
1.	24		5.	10	
2.	8		6.	22	
3.	13		7.	12	
4.	12				
			Total	101	

GOOD LUCK!

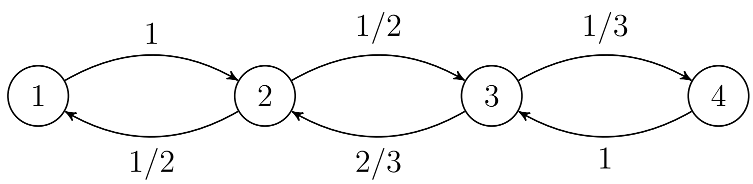
1. Consider three Markov chains with respective state transition diagrams given by



Markov chain 1



Markov chain 2



Markov chain 3

Answer the following questions for each of the Markov chains. Enter your Yes/No responses in the boxes provided. Also provide a brief one-line justification of your answers.

(a) (6 points) Is the Markov chain irreducible?

Markov chain 1	Markov chain 2	Markov chain 3

(b) (6 points) Are all states in the Markov chain aperiodic?

Markov chain 1	Markov chain 2	Markov chain 3

(c) (6 points) Let X_n be the state of the Markov chain at time n, and let S denote the state space. Does

$$\lim_{n\to\infty} \mathrm{P}\left(X_n = j \,\middle|\, X_0 = i\right)$$

exist for all $i, j \in S$?

Markov chain 1	Markov chain 2	Markov chain 3

(d) (6 points) Does

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{X_m = i\}$$

exist for all $i \in S$, independently of how the Markov chain is initialized?

Markov chain 1	Markov chain 2	Markov chain 3

2. (8 points) Let X and Y be independent random variables with $Y \neq 0$ and $\mathbb{E}[Y] \neq 0$. Prove or disprove the following identity:

$$\mathbb{E}\left[\frac{X}{Y}\right] = \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}.$$

3. Consider the continuous random variables X and Y with joint probability density function

$$f_{XY}(x,y) = \begin{cases} c(x+y), & 0 < x < 1 \text{ and } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (2 points) What is the value of c? Explain.

(b) (4 points) Find the conditional probability density function $f_{Y|X}(y \,|\, x)$.

(c) (3 points) $\mathrm{P}\left(Y > 1/2 \,\middle|\, X = 1/2\right) =?$

(d) (4 points) $\mathrm{P}\left(Y > 1/2 \,\middle|\, X < 1/2\right) =?$

4. (a) (4 points) Let $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ be an i.i.d. sequence of Bernoulli$(1/4)$ random variables. Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i^5$$

and provide justification for the existence of the limit.

(b) (8 points) Suppose that $Y_{\mathbb{N}} = Y_0, Y_1, \ldots, Y_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} p & 1-p \\ 1/2 & 1/2 \end{pmatrix}, \quad 0 \le p < 1.$$

Determine p so that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{Y_m = 2\}$$

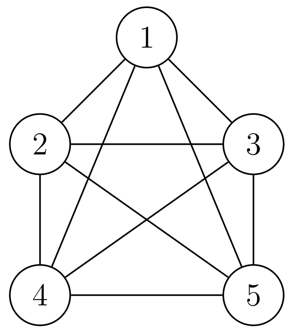
is equal to the answer you obtained in part (a).

5. Suppose X and Y are random variables with joint probability mass function given by

	$Y = 1$	$Y = 2$	$Y = 3$
$X = 0$	$1/4$	$3/16$	$1/16$
$X = 1$	$1/8$	0	$3/8$

(10 points) $\mathbb{E}\left[\frac{X}{Y} \,\middle|\, X^2 + Y^2 \leq 4\right] =$?

6. Here we study a symmetric random walk on the complete graph with N_v vertices. Specifically, we consider undirected graphs without self-loops, where each vertex is connected to all other vertices via edges. For instance, the complete graph on $N_v = 5$ vertices is depicted below.



For given positive integer N_v, suppose that X_n is the vertex visited by the random walker at time n. Every time period $n \geq 0$, the random walker chooses a vertex uniformly at random from the set of all vertices *other than* X_n, and transitions to the chosen vertex at time $n+1$. Accordingly, the process $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, \ldots, N_v\}$.

(a) (5 points) Determine the transition probabilities P_{ij} for all $i, j \in S$.

(b) (7 points) Compute the stationary distribution of $X_{\mathbb{N}}$.

(c) (10 points) Suppose that the random walker starts at vertex $i \in S$. Let T_i denote the time until it first returns to i. $\mathbb{E}[T_i] = ?$

7. (12 points) Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables with $\mathbb{E}[X_1] = \mu$ and $\operatorname{var}[X_1] = \sigma^2$. Let N be a positive integer-valued random variable independent of $X_{\mathbb{N}}$. Define

$$Y = \frac{1}{N} \sum_{i=1}^{N} X_i.$$

Compute $\operatorname{var}[Y]$.