# ECE440 - Introduction to Random Processes

## Midterm Exam

November 1, 2023

**Instructions:**

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 101, extra point is a bonus point).
- Duration: 90 minutes.
- This exam has 11 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name: ____________________________

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<thead>
<tr>
<th>Problem</th>
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<td>3.</td>
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**GOOD LUCK!**
1. Consider three Markov chains with respective state transition diagrams given by

![Markov chain 1](image1)

![Markov chain 2](image2)

![Markov chain 3](image3)

Answer the following questions for each of the Markov chains. Enter your Yes/No responses in the boxes provided. Also provide a brief one-line justification of your answers.

(a) (6 points) Is the Markov chain irreducible?

<table>
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<th>Markov chain 1</th>
<th>Markov chain 2</th>
<th>Markov chain 3</th>
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(b) (6 points) Are all states in the Markov chain aperiodic?

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(c) (6 points) Let $X_n$ be the state of the Markov chain at time $n$, and let $S$ denote the state space. Does

$$\lim_{n \to \infty} P(X_n = j \mid X_0 = i)$$

exist for all $i, j \in S$?

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<th>Markov chain 3</th>
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</table>
(d) (6 points) Does
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{1} \{ X_m = i \}
\]
exist for all \( i \in S \), independently of how the Markov chain is initialized?

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2. (8 points) Let \( X \) and \( Y \) be independent random variables with \( Y \neq 0 \) and \( \mathbb{E}[Y] \neq 0 \). Prove or disprove the following identity:
\[
\mathbb{E} \left[ \frac{X}{Y} \right] = \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}
\]
3. Consider the continuous random variables $X$ and $Y$ with joint probability density function

$$f_{XY}(x, y) = \begin{cases} 
  c(x + y), & 0 < x < 1 \text{ and } 0 < y < 1, \\
  0, & \text{otherwise}.
\end{cases}$$

(a) (2 points) What is the value of $c$? Explain.

(b) (4 points) Find the conditional probability density function $f_{Y|X}(y \mid x)$. 

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(c) (3 points) \( P (Y > 1/2 \mid X = 1/2) = ? \)

(d) (4 points) \( P (Y > 1/2 \mid X < 1/2) = ? \)
4. (a) (4 points) Let $X_N = X_1, X_2, \ldots, X_n, \ldots$ be an i.i.d. sequence of Bernoulli($1/4$) random variables. Calculate
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i^5
\]
and provide justification for the existence of the limit.

(b) (8 points) Suppose that $Y_N = Y_0, Y_1, \ldots, Y_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$ and transition probability matrix
\[
P = \begin{pmatrix}
p & 1 - p \\
1/2 & 1/2
\end{pmatrix}, \quad 0 \leq p < 1.
\]
Determine $p$ so that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{Y_m = 2\}
\]
is equal to the answer you obtained in part (a).
5. Suppose $X$ and $Y$ are random variables with joint probability mass function given by

<table>
<thead>
<tr>
<th></th>
<th>$Y = 1$</th>
<th>$Y = 2$</th>
<th>$Y = 3$</th>
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</thead>
<tbody>
<tr>
<td>$X = 0$</td>
<td>$1/4$</td>
<td>$3/16$</td>
<td>$1/16$</td>
</tr>
<tr>
<td>$X = 1$</td>
<td>$1/8$</td>
<td>0</td>
<td>$3/8$</td>
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</table>

(10 points) $\mathbb{E} \left[ \frac{X}{Y} \mid X^2 + Y^2 \leq 4 \right] =$?
6. Here we study a symmetric random walk on the complete graph with \( N \) vertices. Specifically, we consider undirected graphs without self-loops, where each vertex is connected to all other vertices via edges. For instance, the complete graph on \( N = 5 \) vertices is depicted below.

![Complete graph with 5 vertices](image)

For given positive integer \( N \), suppose that \( X_n \) is the vertex visited by the random walker at time \( n \). Every time period \( n \geq 0 \), the random walker chooses a vertex uniformly at random from the set of all vertices other than \( X_n \), and transitions to the chosen vertex at time \( n+1 \). Accordingly, the process \( X_n = X_0, X_1, \ldots, X_n, \ldots \) is a Markov chain with state space \( S = \{1, \ldots, N\} \).

(a) (5 points) Determine the transition probabilities \( P_{ij} \) for all \( i, j \in S \).
(b) (7 points) Compute the stationary distribution of $X_N$.

(c) (10 points) Suppose that the random walker starts at vertex $i \in S$. Let $T_i$ denote the time until it first returns to $i$. $\mathbb{E}[T_i] = ?$
7. (12 points) Suppose that $X_N = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables with $\mathbb{E}[X_1] = \mu$ and $\text{var}[X_1] = \sigma^2$. Let $N$ be a positive integer-valued random variable independent of $X_N$. Define

$$Y = \frac{1}{N} \sum_{i=1}^{N} X_i.$$

Compute $\text{var}[Y]$. 