ECE440 - Introduction to Random Processes

Midterm Exam

November 1, 2023

Instructions:

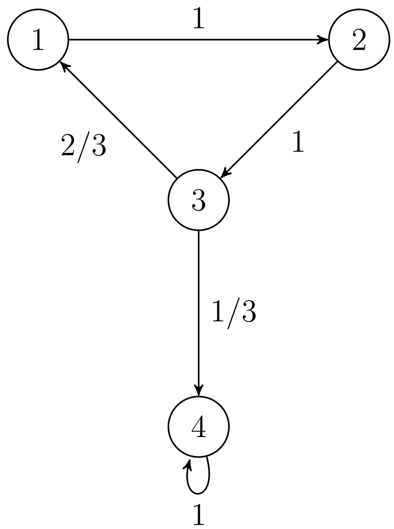
- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 101, extra point is a bonus point).
- Duration: 90 minutes.
- This exam has 11 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:__________SOLUTIONS__________

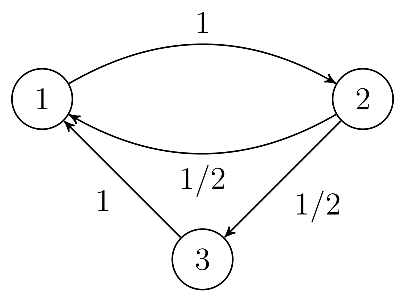
Problem	Max. Points	Score	Problem	Max. Points	Score
1.	24		5.	10	
2.	8		6.	22	
3.	13		7.	12	
4.	12				
			Total	101	

GOOD LUCK!

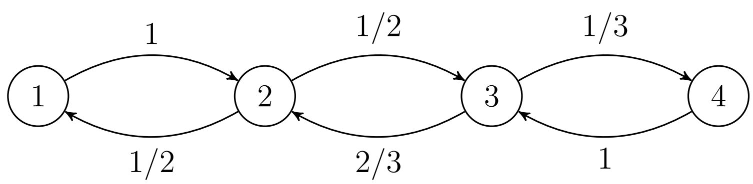
1. Consider three Markov chains with respective state transition diagrams given by



Markov chain 1



Markov chain 2



Markov chain 3

Answer the following questions for each of the Markov chains. Enter your Yes/No responses in the boxes provided. Also provide a brief one-line justification of your answers.

(a) (6 points) Is the Markov chain irreducible?

Markov chain 1	Markov chain 2	Markov chain 3
No	Yes	Yes

Markov chain 1: State 4 is absorbing, hence there are two communication classes $\mathcal{R} = \{4\}$ and $\mathcal{T} = \{1, 2, 3\}$.

Markov chain 2: All states communicate, so there is a single class $\mathcal{R} = \{1, 2, 3\}$.

Markov chain 3: All states communicate, so there is a single class $\mathcal{R} = \{1, 2, 3, 4\}$.

(b) (6 points) Are all states in the Markov chain aperiodic?

Markov chain 1	Markov chain 2	Markov chain 3
No	Yes	No

Markov chain 1: State 4 is aperiodic because $P_{44} = 1$, but all other states have period $d = 3$.

Markov chain 2: Suffices to find the period of state 1, and since $P_{11} = 0$, $P_{11}^2 = 1/2$, $P_{11}^3 = 1/2$, we have $d = \gcd\{2, 3, \ldots\} = 1$.
Markov chain 3: Suffices to find the period of state 1, and since $P_{11}^{2n} > 0$, $P_{11}^{2n+1} = 0$, we have $d = \gcd\{2, 4, \ldots\} = 2$.

(c) (6 points) Let X_n be the state of the Markov chain at time n, and let S denote the state space. Does

$$\lim_{n\to\infty} \mathrm{P}\left(X_n = j \,\middle|\, X_0 = i\right)$$

exist for all $i, j \in S$?

Markov chain 1	Markov chain 2	Markov chain 3
Yes	Yes	No

Markov chain 1: For all $i \in S$ we have $\lim_{n\to\infty} P_{i4}^n = 1$ and $\lim_{n\to\infty} P_{ij}^n = 0$, $j \neq 4$.
Markov chain 2: The Markov chain is ergodic, so $\lim_{n\to\infty} P_{ij}^n$ exist for all $i, j \in S$.
Markov chain 3: All states have period $d = 2$, hence $\lim_{n\to\infty} P_{ij}^n$ does not exist for any $i, j \in S$ (the P_{ij}^n oscillate).

(d) (6 points) Does

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\left\{X_m = i\right\}$$

exist for all $i \in S$, independently of how the Markov chain is initialized?

Markov chain 1	Markov chain 2	Markov chain 3
Yes	Yes	Yes

Markov chain 1: The ergodic limits converge to $\pi_4 = 1$ and $\pi_i = 0$ for $i \in \{1, 2, 3\}$.
Markov chain 2: The Markov chain is ergodic, so the limits exist by the Ergodic Theorem.
Markov chain 3: Even though the single recurrent class has periodic states, the ergodic limits exist.

2. (8 points) Let X and Y be independent random variables with $Y \neq 0$ and $\mathbb{E}[Y] \neq 0$. Prove or disprove the following identity:

$$\mathbb{E}\left[\frac{X}{Y}\right] = \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}.$$

Because X and Y are independent then we have

$$\mathbb{E}\left[\frac{X}{Y}\right] = \mathbb{E}[X] \times \mathbb{E}\left[\frac{1}{Y}\right].$$

But in general $\mathbb{E}\left[\frac{1}{Y}\right] \neq \frac{1}{\mathbb{E}[Y]}$ and hence the identity is not true. As a counter-example, consider Y with pmf $\mathrm{P}(Y = 2) = 1/2$ and $\mathrm{P}(Y = 4) = 1/2$. Hence,

$$\mathbb{E}[Y] = 2 \times \frac{1}{2} + 4 \times \frac{1}{2} = 3 \;\Rightarrow\; \frac{1}{\mathbb{E}[Y]} = \frac{1}{3}$$

$$\mathbb{E}\left[\frac{1}{Y}\right] = \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2} = \frac{3}{8} \neq \frac{1}{\mathbb{E}[Y]}.$$

Together with an X for which $\mathbb{E}[X] \neq 0$, the above Y offers a counter-example for the identity in question.

3. Consider the continuous random variables X and Y with joint probability density function

$$f_{XY}(x,y) = \begin{cases} c(x+y), & 0 < x < 1 \text{ and } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (2 points) What is the value of c? Explain.

1

A valid joint pdf satisfies

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = \int_0^1\int_0^1 c(x+y)dxdy = c\int_0^1\int_0^1 xdxdy + c\int_0^1\int_0^1 ydxdy = c.$$

(b) (4 points) Find the conditional probability density function $f_{Y|X}(y\,|\,x)$.

$f_{Y\|X}(y\,\|\,x) = \begin{cases} \frac{x+y}{x+\frac{1}{2}}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$

From the definition of conditional pdf

$$f_{Y|X}(y\,|\,x) = \frac{f_{XY}(x,y)}{f_X(x)},$$

so we compute the marginal pdf of X. To this end, we marginalize over Y and find

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_0^1 (x+y)dy = xy + \frac{y^2}{2}\Big|_0^1 = x + \frac{1}{2},\ 0 < x < 1.$$

All in all, the desired conditional pdf is given by

$$f_{Y|X}(y\,|\,x) = \begin{cases} \frac{x+y}{x+\frac{1}{2}}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) (3 points) $\mathrm{P}\left(Y > 1/2\,\middle|\,X = 1/2\right) =?$

$\frac{5}{8}$

Using the expression for the conditional pdf we derived in (b), for $X = 1/2$ we obtain

$$\mathrm{P}\left(Y > 1/2\,\middle|\,X = 1/2\right) = \int_{1/2}^1 f_{Y|X}(y\,|\,1/2)dy = \int_{1/2}^1 \left(\frac{1}{2} + y\right)dy = \frac{y}{2} + \frac{y^2}{2}\Big|_{1/2}^1 = \frac{5}{8}.$$

(d) (4 points) $\mathrm{P}\left(Y > 1/2 \,\middle|\, X < 1/2\right) = ?$

$$\frac{2}{3}$$

From the definition of conditional probability, we have

$$\mathrm{P}\left(Y > 1/2 \,\middle|\, X < 1/2\right) = \frac{\mathrm{P}\left(Y > 1/2, X < 1/2\right)}{\mathrm{P}\left(X < 1/2\right)}.$$

We compute each of the probabilities by integrating the appropriate pdfs, namely

$$\mathrm{P}\left(Y > 1/2, X < 1/2\right) = \int_{1/2}^{1} \int_{0}^{1/2} f_{XY}(x, y) dx dy = \int_{1/2}^{1} \int_{0}^{1/2} (x + y) dx dy = \frac{1}{4},$$
$$\mathrm{P}\left(X < 1/2\right) = \int_{0}^{1/2} f_X(x) dx = \int_{0}^{1/2} \left(x + \frac{1}{2}\right) dx = \frac{3}{8}.$$

Putting the pieces together, we arrive at the result $\mathrm{P}\left(Y > 1/2 \,\middle|\, X < 1/2\right) = \frac{2}{3}$.

4. (a) (4 points) Let $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ be an i.i.d. sequence of Bernoulli$(1/4)$ random variables. Calculate

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i^5$$

and provide justification for the existence of the limit.

$$\frac{1}{4}$$

Because $X_{\mathbb{N}}$ is i.i.d., then $Z_{\mathbb{N}} = X_1^5, X_2^5, \ldots, X_n^5, \ldots$ is also i.i.d. By the strong law of large numbers the limit exists and is equal to

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i^5 = \mathbb{E}\left[X_1^5\right], \quad \text{w.p. } 1.$$

But notice that since $X_i \sim \text{Bernoulli}(1/4)$, then $X_i^5 \sim \text{Bernoulli}(1/4)$ as well. Thus, $\mathbb{E}\left[X_i^5\right] = \frac{1}{4}$.

(b) (8 points) Suppose that $Y_{\mathbb{N}} = Y_0, Y_1, \ldots, Y_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} p & 1-p \\ 1/2 & 1/2 \end{pmatrix}, \quad 0 \leq p < 1.$$

Determine p so that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\left\{Y_m = 2\right\}$$

is equal to the answer you obtained in part (a).

$$\frac{5}{6}$$

The long-run fraction of time spent in state 2 is π_2 almost surely, where $\boldsymbol{\pi} = [\pi_1, \pi_2]^\top$ is the unique (the Markov chain is ergodic for $0 \le p < 1$) stationary distribution which satisfies

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} p & 1/2 \\ 1-p & 1/2 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \quad \pi_1 + \pi_2 = 1.$$

Solving the linear system yields $\pi_2 = \frac{2(1-p)}{3-2p}$. Imposing $\pi_2 = 1/4$, we find $p = 5/6$.

5. Suppose X and Y are random variables with joint probability mass function given by

	$Y=1$	$Y=2$	$Y=3$
$X=0$	$1/4$	$3/16$	$1/16$
$X=1$	$1/8$	0	$3/8$

(10 points) $\mathbb{E}\left[\frac{X}{Y} \,\middle|\, X^2+Y^2 \le 4\right] = ?$

$$\frac{2}{9}$$

To evaluate $\mathbb{E}\left[\frac{X}{Y} \,\middle|\, X^2+Y^2 \le 4\right]$ the relevant joint conditional pmf $\mathsf{P}\left(X=1, Y=y \,\middle|\, X^2+Y^2 \le 4\right)$ is

$$\mathsf{P}\left(X=1, Y=1 \,\middle|\, X^2+Y^2 \le 4\right) = \frac{\mathsf{P}\left(\{X=1, Y=1\}, \{X^2+Y^2 \le 4\}\right)}{\mathsf{P}\left(X^2+Y^2 \le 4\right)}$$

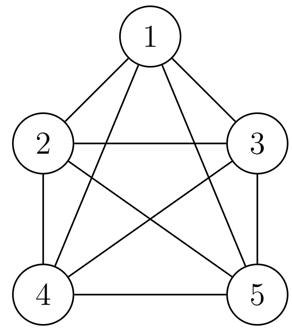
$$= \frac{\mathsf{P}(X=1, Y=1)}{\mathsf{P}(X=0, Y=1) + \mathsf{P}(X=0, Y=2) + \mathsf{P}(X=1, Y=1)} = \frac{2}{9},$$

$\mathsf{P}\left(X=1, Y=2 \,\middle|\, X^2+Y^2 \le 4\right) = 0,$
$\mathsf{P}\left(X=1, Y=3 \,\middle|\, X^2+Y^2 \le 4\right) = 0.$

We do not need to compute the values of $\mathsf{P}\left(X=0, Y=y \,\middle|\, X^2+Y^2 \le 4\right)$ because if $X=0$, then $\frac{X}{Y} = 0$ and those terms will not contribute to the expectation. Hence,

$$\mathbb{E}\left[\frac{X}{Y} \,\middle|\, X^2+Y^2 \le 4\right] = \sum_{x=0}^{1}\sum_{y=1}^{3}\left(\frac{x}{y}\right) \times \mathsf{P}\left(X=x, Y=y \,\middle|\, X^2+Y^2 \le 4\right) = 1 \times \frac{2}{9} = \frac{2}{9}.$$

6. Here we study a symmetric random walk on the complete graph with N_v vertices. Specifically, we consider undirected graphs without self-loops, where each vertex is connected to all other vertices via edges. For instance, the complete graph on $N_v = 5$ vertices is depicted below.



For given positive integer N_v, suppose that X_n is the vertex visited by the random walker at time n. Every time period $n \geq 0$, the random walker chooses a vertex uniformly at random from the set of all vertices *other than* X_n, and transitions to the chosen vertex at time $n+1$. Accordingly, the process $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, \ldots, N_v\}$.

(a) (5 points) Determine the transition probabilities P_{ij} for all $i, j \in S$.

$$P_{ij} = \begin{cases} \frac{1}{N_v - 1}, & i \neq j, \\ 0, & i = j. \end{cases}$$

Suppose that $X_n = i$, $i \in S$. Since the random walker chooses the vertex j its going to visit at time $n+1$ uniformly at random from $S \setminus \{i\}$, then it immediately follows that

$$P_{ij} = \mathsf{P}\left(X_{n+1} = j \,\middle|\, X_n = i\right) = \frac{1}{N_v - 1}$$

for all $i \in S$ all $j \in S \setminus \{i\}$. Moreover, we have $P_{ii} = 0$ for all $i \in S$.

(b) (7 points) Compute the stationary distribution of $X_{\mathbb{N}}$.

$$\pi_i = \frac{1}{N_v}, \; i \in S$$

There is no need for any calculations here. Since at each time step $n \geq 0$, all vertices other than X_n are equally likely to be visited at time $n+1$, then the long-run fraction of time spent in every vertex will be the same. In other words, the stationary distribution will be uniform over $S = \{1, \ldots, N_v\}$.

(c) (10 points) Suppose that the random walker starts at vertex $i \in S$. Let T_i denote the time until it first returns to i. $\mathbb{E}\left[T_i\right] = ?$

$$N_v$$

Notice first that $T_i > 1$, because if we are initially at i then necessarily we are going to be at $j \neq i$ in the next time step. In each of the subsequent steps, there is a probability $P_{ji} = \frac{1}{N_v - 1}$ of returning to i. Given these considerations, we can write $T_i = 1 + N$, where $N \sim$ Geometric$(\frac{1}{N_v - 1})$. Recalling that $\mathbb{E}\left[N\right] = N_v - 1$, then we find $\mathbb{E}\left[T_i\right] = 1 + \mathbb{E}\left[N\right] = N_v$.

7. (12 points) Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables with $\mathbb{E}[X_1] = \mu$ and $\text{var}[X_1] = \sigma^2$. Let N be a positive integer-valued random variable independent of $X_{\mathbb{N}}$. Define

$$Y = \frac{1}{N} \sum_{i=1}^{N} X_i.$$

Compute $\text{var}[Y]$.

$$\boxed{\sigma^2 \times \mathbb{E}\left[\frac{1}{N}\right]}$$

To compute $\text{var}[Y]$, we condition on N. Because the X_n are i.i.d. and independent of N, we find that $\mathbb{E}\left[Y \mid N\right] = \mu$ and $\text{var}\left[Y \mid N\right] = \frac{\sigma^2}{N}$. Using the conditional variance formula

$$\begin{aligned}
\text{var}[Y] &= \mathbb{E}\left[\text{var}\left[Y \mid N\right]\right] + \text{var}\left[\mathbb{E}\left[Y \mid N\right]\right] \\
&= \mathbb{E}\left[\frac{\sigma^2}{N}\right] + \text{var}[\mu] \\
&= \sigma^2 \times \mathbb{E}\left[\frac{1}{N}\right].
\end{aligned}$$

To arrive at the last equality we used that the variance of a point mass random variable is zero.