ECE440 - Introduction to Random Processes

Midterm Exam

October 29, 2025

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 103, extra points are bonus points).
- Duration: 90 minutes.
- This exam has 11 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:_____

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	18		5.	15	
2.	10		6.	8	
3.	12		7.	22	
4.	18				
			Total	103	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix}.$$

To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^3 = \begin{pmatrix} 19/32 & 13/32 \\ 39/64 & 25/64 \end{pmatrix} = \begin{pmatrix} 0.59 & 0.41 \\ 0.61 & 0.39 \end{pmatrix}.$$

(a) (2 points) $P\left(X_4 = 2 \,\middle|\, X_3 = 1, X_2 = 2, X_1 = 1\right) = ?$

(b) (2 points) $P\left(X_5 = 1 \,\middle|\, X_2 = 1, X_0 = 1\right) = ?$

(c) (2 points) $P\left(X_3 = 1 \,\middle|\, X_3 = 2, X_2 = 1, X_1 = 1, X_0 = 2\right) = ?$

(d) (6 points) $\mathbb{E}\left[X_7 \,\middle|\, X_4 = 2\right] =?$

(e) (6 points) Let $N = \min\{n > 0 : X_n = 2\}$. $\mathbb{E}\left[N \,\middle|\, X_0 = 1\right] =?$

2. (10 points) Let $X_1, X_2, \ldots, X_{20}$ be i.i.d. Poisson(10) random variables. For $n = 20$, consider the sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Use the Central Limit Theorem to approximate $P\left(9 \leq \bar{X}_{20} \leq 11\right)$. Write your answer in terms of the cumulative distribution function $\Phi(z)$ of a standard Normal random variable $Z \sim \mathcal{N}(0,1)$, that is

$$\Phi(z) = P\left(Z \leq z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du.$$

3. (12 points) If $T_1, T_2, \ldots, T_n$ are i.i.d. exponential random variables, each with parameter λ, then $T_1 + T_2 + \ldots + T_n$ has the Erlang distribution with parameters n and λ; i.e.,

$$\mathbf{P}\left(T_1 + T_2 + \ldots + T_n \le t\right) = \int_0^t \lambda e^{-\lambda u} \frac{(\lambda u)^{n-1}}{(n-1)!} du.$$

You can use the above result (without proof) in what follows. Also, recall $e^x = \sum_{k=0}^{\infty} x^k/k!$.

Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of exponential random variables, each with parameter μ, so each X_n has cumulative distribution function $F(x) = 1 - e^{-\mu x}$ and probability density function $f(x) = \mu e^{-\mu x}$, $x \ge 0$. Let N be a geometric random variable with parameter p, i.e., its probability mass function is given by $\mathbf{P}\left(N = n\right) = (1-p)^{n-1}p$, $n \ge 1$. Assume that N and $X_{\mathbb{N}}$ are independent, and define

$$Y = \sum_{n=1}^{N} X_n.$$
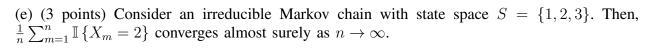
Show that Y has the exponential distribution, and write its parameter in the box.

4. Indicate whether the following claims are True or False **as stated**. Provide a brief justification of your answers.

(a) (3 points) Consider an irreducible Markov chain with state space $S = \{1, 2, 3\}$. Then, state 3 can be transient.

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(b) (3 points) Consider an irreducible Markov chain with state space $S = \{1, 2, 3\}$. Then, state 1 can be absorbing.

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(c) (3 points) Consider an irreducible Markov chain with state space $S = \{1, 2, 3\}$ and $P_{11} = P\left(X_{n+1} = 1 \mid X_n = 1\right) = 1/3$. Then, state 2 can be aperiodic.

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(d) (3 points) Consider an irreducible Markov chain with state space $S = \{1, 2, 3\}$ and transition probability matrix $\mathbf{P}$. Then, $\mathbf{P}^n$ converges as $n \to \infty$.

(e) (3 points) Consider an irreducible Markov chain with state space $S = \{1, 2, 3\}$. Then, $\frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{X_m = 2\}$ converges almost surely as $n \to \infty$.

(f) (3 points) Consider a Markov chain with state space $S = \{1, 2, 3\}$ and two communication classes, $\mathcal{R} = \{1, 2\}$ is recurrent and $\mathcal{T} = \{3\}$ is transient. Then, $P_{13} = \mathrm{P}\left(X_{n+1} = 3 \mid X_n = 1\right) = 0$.

5. Suppose n people go to a restaurant and check their coats. During dinner, the coats get mixed up. After dinner, each person receives a random coat (each person is equally likely to receive any of the n coats). Let $X_i = \mathbb{I}\{$Person i gets their own coat$\}$ be indicator random variables, for $i = 1, \ldots, n$.

(a) (2 points) var$[X_i] =$?

(b) (4 points) For $i \neq j$, calculate $\mathbb{E}[X_i X_j]$.

(c) (3 points) For $i \neq j$, calculate cov$[X_i X_j]$.

(d) (6 points) What is the variance of the total number of people who got their own coats? [Hint: Write said total in terms of the X_i, $i = 1, \ldots, n$]

6. (8 points) As discussed in class, Markov's inequality can be fairly loose in bounding tail probabilities of random variables which exhibit exponential decay. However, the bound cannot be improved without further assumptions, because there exist random variables for which Markov's inequality is tight (meaning, equality holds).

For any $a > 0$ and any $0 < \theta < 1$, construct a nonnegative random variable X such that

$$\mathrm{P}\left(X \geq a\right) = \theta = \frac{\mathbb{E}\left[X\right]}{a}.$$

[Hint: Consider X such that $\mathrm{P}\left(X = a\right) = \theta$.]

7. Suppose customers can arrive to a service station at times $n = 0, 1, 2, \ldots$ In any given period, independent of everything else, there is one arrival with probability p, and there is no arrival with probability $1 - p$. Customers are served one-at-a-time on a first-come-first-served basis. If at the time of an arrival, there are no customers present, then the arriving customer immediately enters service. Otherwise, the arrival joins the back of the queue.

Assume that service times are i.i.d. geometric random variables (each with parameter q) that are independent of the arrival process. So, $P\left(\text{Service time} = \ell\right) = (1 - q)^{\ell-1}q$, for $\ell = 1, 2, \ldots$ Note that a customer who enters service in time n can complete service, at the earliest, in time $n + 1$ (in which case their service time is 1). Upon a service completion, the just-served customer will depart the system with probability α, or will immediately rejoin the back of the queue with probability $1 - \alpha$.

In a time period n, events happen in the following order: (i) arrivals, if any, occur; (ii) service completions followed by departures or rejoins, if any, occur; and (iii) service begins on the next customer if there are customers present in the system.

Let X_n denote the number of customers at the station at the end of time period n. Note that X_n includes both customers waiting as well as any customer being served, and that the random process $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$.

(a) (4 points) Determine the transition probabilities P_{0j} for all $j \geq 0$.

(b) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j = i$.

(c) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j > i$.

(d) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $0 \leq j < i$.