ECE440 - Introduction to Random Processes

Midterm Exam

October 29, 2025

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 103, extra points are bonus points).
- Duration: 90 minutes.
- This exam has 11 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:__________SOLUTIONS__________

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	18		5.	15	
2.	10		6.	8	
3.	12		7.	22	
4.	18				
			Total	103	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix}.$$

To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^3 = \begin{pmatrix} 19/32 & 13/32 \\ 39/64 & 25/64 \end{pmatrix} = \begin{pmatrix} 0.59 & 0.41 \\ 0.61 & 0.39 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left(X_4 = 2 \,\middle|\, X_3 = 1, X_2 = 2, X_1 = 1\right) =?$

$$\frac{1}{2}$$

From the Markov property it follows that

$$\mathrm{P}\left(X_4 = 2 \,\middle|\, X_3 = 1, X_2 = 2, X_1 = 1\right) = \mathrm{P}\left(X_4 = 2 \,\middle|\, X_3 = 1\right) = P_{12} = \frac{1}{2}.$$

(b) (2 points) $\mathrm{P}\left(X_5 = 1 \,\middle|\, X_2 = 1, X_0 = 1\right) =?$

$$\frac{19}{32}$$

Likewise, $\mathrm{P}\left(X_5 = 1 \,\middle|\, X_2 = 1, X_0 = 1\right) = \mathrm{P}\left(X_5 = 1 \,\middle|\, X_2 = 1\right) = P^3_{11} = \frac{19}{32}$.

(c) (2 points) $\mathrm{P}\left(X_3 = 1 \,\middle|\, X_3 = 2, X_2 = 1, X_1 = 1, X_0 = 2\right) =?$

$$0$$

Given $X_3 = 2$, then $X_3 = 1$ is impossible so $\mathrm{P}\left(X_3 = 1 \,\middle|\, X_3 = 2, X_2 = 1, X_1 = 1, X_0 = 2\right) = 0$.

(d) (6 points) $\mathbb{E}\left[X_7 \,\middle|\, X_4 = 2\right] =?$

$$\frac{89}{64}$$

The conditional pmf of X_7 given $X_4 = 2$ is

$$\mathrm{P}\left(X_7 = 1 \,\middle|\, X_4 = 2\right) = P^3_{21} = \frac{39}{64},$$
$$\mathrm{P}\left(X_7 = 2 \,\middle|\, X_4 = 2\right) = P^3_{22} = \frac{25}{64}.$$

Hence, the conditional expectation is $\mathbb{E}\left[X_7 \,\middle|\, X_4 = 2\right] = 1 \times \frac{39}{64} + 2 \times \frac{25}{64} = \frac{89}{64}$.

(e) (6 points) Let $N = \min\{n > 0 : X_n = 2\}$. $\mathbb{E}\left[N \,\middle|\, X_0 = 1\right] =?$

2

The random variable N indicates the first (strictly positive) time instant the Markov chain visits state 2. Notice that conditioned on $X_0 = 1$, then N is a Geometric random variable with parameter $p := P_{12} = 1/2$. Accordingly,

$$\mathbb{E}\left[N \,\middle|\, X_0 = 1\right] = \frac{1}{P_{12}} = 2.$$

2. (10 points) Let $X_1, X_2, \ldots, X_{20}$ be i.i.d. Poisson(10) random variables. For $n = 20$, consider the sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Use the Central Limit Theorem to approximate $\mathrm{P}\left(9 \leq \bar{X}_{20} \leq 11\right)$. Write your answer in terms of the cumulative distribution function $\Phi(z)$ of a standard Normal random variable $Z \sim \mathcal{N}(0,1)$, that is

$$\Phi(z) = \mathrm{P}\left(Z \leq z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du.$$

$\Phi(\sqrt{2}) - \Phi(-\sqrt{2})$

Let X_i, $i = 1, \ldots, 20$, be i.i.d. Poisson(10) random variables with $\mathbb{E}\left[X_i\right] = 10$ and $\mathrm{var}\left[X_i\right] = 10$. Let $\bar{X}_{20} = \frac{1}{20} \sum_{i=1}^{20} X_i$ be the sample mean, for which we have

$$\mathbb{E}\left[\bar{X}_{20}\right] = \mathbb{E}\left[\frac{1}{20} \sum_{i=1}^{20} X_i\right] = \frac{1}{20} \sum_{i=1}^{20} \mathbb{E}\left[X_i\right] = 10,$$
$$\mathrm{var}\left[\bar{X}_{20}\right] = \mathrm{var}\left[\frac{1}{20} \sum_{i=1}^{20} X_i\right] = \frac{1}{20^2} \sum_{i=1}^{20} \mathrm{var}\left[X_i\right] = \frac{10}{20} = \frac{1}{2}.$$

We are asked to approximate $\mathrm{P}\left(|\bar{X}_{20} - \mathbb{E}\left[\bar{X}_{20}\right]| \leq 1\right) = \mathrm{P}\left(9 \leq \bar{X}_{20} \leq 11\right)$. By centering and scaling $\bar{X}_{20}$ and relying on the Central Limit Theorem, we obtain ($Z \sim \mathcal{N}(0,1)$ below)

$$\begin{aligned}
\mathrm{P}\left(9 \leq \bar{X}_{20} \leq 11\right) &= \mathrm{P}\left(\frac{9 - \mathbb{E}\left[\bar{X}_{20}\right]}{\sqrt{\mathrm{var}\left[\bar{X}_{20}\right]}} \leq \frac{\bar{X}_{20} - \mathbb{E}\left[\bar{X}_{20}\right]}{\sqrt{\mathrm{var}\left[\bar{X}_{20}\right]}} \leq \frac{11 - \mathbb{E}\left[\bar{X}_{20}\right]}{\sqrt{\mathrm{var}\left[\bar{X}_{20}\right]}}\right) \\
&= \mathrm{P}\left(-\frac{1}{\sqrt{1/2}} \leq \frac{\bar{X}_{20} - \mathbb{E}\left[\bar{X}_{20}\right]}{\sqrt{\mathrm{var}\left[\bar{X}_{20}\right]}} \leq \frac{1}{\sqrt{1/2}}\right) \\
&\approx \mathrm{P}\left(-\sqrt{2} \leq Z \leq \sqrt{2}\right) = \Phi(\sqrt{2}) - \Phi(-\sqrt{2}).
\end{aligned}$$

3. (12 points) If $T_1, T_2, \ldots, T_n$ are i.i.d. exponential random variables, each with parameter λ, then $T_1 + T_2 + \ldots + T_n$ has the Erlang distribution with parameters n and λ; i.e.,

$$\mathrm{P}\left(T_1 + T_2 + \ldots + T_n \le t\right) = \int_0^t \lambda e^{-\lambda u} \frac{(\lambda u)^{n-1}}{(n-1)!} du.$$

You can use the above result (without proof) in what follows. Also, recall $e^x = \sum_{k=0}^{\infty} x^k/k!$.

Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of exponential random variables, each with parameter μ, so each X_n has cumulative distribution function $F(x) = 1 - e^{-\mu x}$ and probability density function $f(x) = \mu e^{-\mu x}$, $x \ge 0$. Let N be a geometric random variable with parameter p, i.e., its probability mass function is given by $\mathrm{P}\left(N = n\right) = (1-p)^{n-1} p$, $n \ge 1$. Assume that N and $X_{\mathbb{N}}$ are independent, and define

$$Y = \sum_{n=1}^{N} X_n.$$

Show that Y has the exponential distribution, and write its parameter in the box.

μp

We derive the cumulative distribution function of the compound random variable Y by conditioning on $N \sim \text{Geometric}(p)$. Accordingly,

$$\begin{aligned}
\mathrm{P}\left(Y \le y\right) &= \sum_{n=1}^{\infty} \mathrm{P}\left(Y \le y \,\middle|\, N = n\right) \mathrm{P}\left(N = n\right) \\
&= \sum_{n=1}^{\infty} \mathrm{P}\left(\sum_{i=1}^{N} X_i \le y \,\middle|\, N = n\right) \mathrm{P}\left(N = n\right) \\
&= \sum_{n=1}^{\infty} \mathrm{P}\left(\sum_{i=1}^{n} X_i \le y \,\middle|\, N = n\right) \mathrm{P}\left(N = n\right) \\
&= \sum_{n=1}^{\infty} \mathrm{P}\left(\sum_{i=1}^{n} X_i \le y\right) \mathrm{P}\left(N = n\right) \\
&= \sum_{n=1}^{\infty} \int_0^y \mu e^{-\mu x} \frac{(\mu x)^{n-1}}{(n-1)!} dx \times (1-p)^{n-1} p \\
&= \int_0^y \mu p e^{-\mu x} \sum_{n=1}^{\infty} \frac{[\mu x(1-p)]^{n-1}}{(n-1)!} dx \\
&= \int_0^y \mu p e^{-\mu x} e^{\mu x(1-p)} dx \\
&= \int_0^y \mu p e^{-\mu p x} dx.
\end{aligned}$$

Hence, Y has the exponential distribution with parameter μp.

4. Indicate whether the following claims are True or False **as stated**. Provide a brief justification of your answers.

(a) (3 points) Consider an irreducible Markov chain with state space $S = \{1, 2, 3\}$. Then, state 3 can be transient.

False

Because the Markov chain is irreducible and $|S| < \infty$, then all states must be recurrent.

(b) (3 points) Consider an irreducible Markov chain with state space $S = \{1, 2, 3\}$. Then, state 1 can be absorbing.

False

If state 1 were absorbing, then the number of classes would be strictly greater than one (an absorbing state always forms a recurrent class of its own, and the remaining states will be in at least one other class). This cannot be since the Markov chain is assumed to be irreducible.

(c) (3 points) Consider an irreducible Markov chain with state space $S = \{1, 2, 3\}$ and $P_{11} = \mathrm{P}\left(X_{n+1} = 1 \,\middle|\, X_n = 1\right) = 1/3$. Then, state 2 can be aperiodic.

True

Because $P_{11} > 0$, then state 1 is aperiodic. Since the Markov chain is irreducible, all states are aperiodic.

(d) (3 points) Consider an irreducible Markov chain with state space $S = \{1, 2, 3\}$ and transition probability matrix $\mathbf{P}$. Then, $\mathbf{P}^n$ converges as $n \to \infty$.

False

Convergence of limiting probabiltiies requires ergodicity. $\mathbf{P}^n$ may oscillate if the states are periodic.

(e) (3 points) Consider an irreducible Markov chain with state space $S = \{1, 2, 3\}$. Then, $\frac{1}{n}\sum_{m=1}^{n} \mathbb{I}\{X_m = 2\}$ converges almost surely as $n \to \infty$.

True

Even if the recurrent states were periodic, the ergodic average will converge almost surely.

(f) (3 points) Consider a Markov chain with state space $S = \{1, 2, 3\}$ and two communication classes, $\mathcal{R} = \{1, 2\}$ is recurrent and $\mathcal{T} = \{3\}$ is transient. Then, $P_{13} = \mathrm{P}\left(X_{n+1} = 3 \,\middle|\, X_n = 1\right) = 0$.

True

Since state 3 is transient, then necessarily $P_{31} > 0$ or $P_{32} > 0$. If $P_{13} > 0$, this would imply states 1 and 3 communicate, which cannot be since they are in different classes.

5. Suppose n people go to a restaurant and check their coats. During dinner, the coats get mixed up. After dinner, each person receives a random coat (each person is equally likely to receive any of the n coats). Let $X_i = \mathbb{I}\{\text{Person } i \text{ gets their own coat}\}$ be indicator random variables, for $i = 1, \ldots, n$.

(a) (2 points) var $[X_i]$ =?

$$\frac{n-1}{n^2}$$

The indicator random variables $X_i = \mathbb{I}\{\text{Person } i \text{ gets their own coat}\}$, $i = 1, \ldots, n$, have the Bernoulli$(1/n)$ distribution. Hence, var $[X_i] = \frac{1}{n}\left(1 - \frac{1}{n}\right)$.

(b) (4 points) For $i \neq j$, calculate $\mathbb{E}[X_i X_j]$.

$$\frac{1}{n(n-1)}$$

Noting that X_i and X_j are identically distributed but not independent, we have that

$$\begin{aligned}
\mathbb{E}[X_i X_j] &= \mathbb{E}\left[\mathbb{I}\{\text{Person } i \text{ gets their own coat}\}\mathbb{I}\{\text{Person } j \text{ gets their own coat}\}\right] \\
&= \mathbb{E}\left[\mathbb{I}\{\text{Person } i \text{ gets their own coat} \cap \text{Person } j \text{ gets their own coat}\}\right] \\
&= \mathrm{P}(X_i = 1, X_j = 1) \\
&= \mathrm{P}\left(X_i = 1 \,\middle|\, X_j = 1\right)\mathrm{P}(X_j = 1) = \frac{1}{n-1} \times \frac{1}{n}.
\end{aligned}$$

(c) (3 points) For $i \neq j$, calculate cov$[X_i X_j]$.

$$\frac{1}{n^2(n-1)}$$

From the covariance definition, we find

$$\text{cov}[X_i X_j] = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\,\mathbb{E}[X_j] = \frac{1}{n-1} \times \frac{1}{n} - \frac{1}{n} \times \frac{1}{n}.$$

(d) (6 points) What is the variance of the total number of people who got their own coats? [Hint: Write said total in terms of the X_i, $i = 1, \ldots, n$]

1

Let $X = \sum_{i=1}^{n} X_i$ be the total number of people who got their own coats. The variance is

$$\begin{aligned}
\text{var}[X] &= \text{var}\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n} \text{var}[X_i] + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \text{cov}[X_i X_j] \\
&= n \times \frac{n-1}{n^2} + n(n-1) \times \frac{1}{n^2(n-1)} = 1.
\end{aligned}$$

6. (8 points) As discussed in class, Markov's inequality can be fairly loose in bounding tail probabilities of random variables which exhibit exponential decay. However, the bound cannot be improved without further assumptions, because there exist random variables for which Markov's inequality is tight (meaning, equality holds).

For any $a > 0$ and any $0 < \theta < 1$, construct a nonnegative random variable X such that

$$\text{P}(X \geq a) = \theta = \frac{\mathbb{E}[X]}{a}.$$

[Hint: Consider X such that $\text{P}(X = a) = \theta$.]

Consider the discrete nonnegative random variable

$$X = \begin{cases} a & \text{w.p. } \theta, \\ 0 & \text{w.p. } 1 - \theta. \end{cases}$$

Then, $\mathbb{E}[X] = a\theta$ and so $\text{P}(X \geq a) = \theta = \frac{\mathbb{E}[X]}{a}$ as desired.

7. Suppose customers can arrive to a service station at times $n = 0, 1, 2, \ldots$ In any given period, independent of everything else, there is one arrival with probability p, and there is no arrival with probability $1 - p$. Customers are served one-at-a-time on a first-come-first-served basis. If at the time of an arrival, there are no customers present, then the arriving customer immediately enters service. Otherwise, the arrival joins the back of the queue.

Assume that service times are i.i.d. geometric random variables (each with parameter q) that are independent of the arrival process. So, $\text{P}(\text{Service time} = \ell) = (1 - q)^{\ell-1} q$, for $\ell = 1, 2, \ldots$ Note that a customer who enters service in time n can complete service, at the earliest, in time $n+1$ (in which case their service time is 1). Upon a service completion, the just-served customer will depart the system with probability α, or will immediately rejoin the back of the queue with probability $1-\alpha$.

In a time period n, events happen in the following order: (i) arrivals, if any, occur; (ii) service completions followed by departures or rejoins, if any, occur; and (iii) service begins on the next customer if there are customers present in the system.

Let X_n denote the number of customers at the station at the end of time period n. Note that X_n includes both customers waiting as well as any customer being served, and that the random process $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$.

(a) (4 points) Determine the transition probabilities P_{0j} for all $j \geq 0$.

$$P_{00} = 1-p, \; P_{01} = p, \; P_{0j} = 0, \; j > 1$$

If the present state is $X_n = 0$ (empty station), then there are only two possible transitions:

1) If an arrival does not occur at time instant $n+1$ then $X_{n+1} = 0$. Note that because $X_n = 0$, then it is also impossible that some user exits the system or rejoins the queue after being served because the system was empty. Then we conclude that $P_{00} = 1-p$.
2) If an arrival occurs at time instant $n+1$ then $X_{n+1} = 1$. Then we conclude that $P_{01} = p$.

All other transition probabilities from state 0 are null, namely $P_{0j} = 0, \; j > 1$.

(b) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j = i$.

$$P_{ii} = pq\alpha + (1-p)(1-q\alpha), \; i > 0$$

Suppose the present state is $X_n = i, \; i > 0$. If an arrival does not occur at time instant $n+1$ and the user being served does not complete service, then $X_{n+1} = i$. This will happen with probability $(1-p)(1-q)$. Furthermore, if an arrival does not occur at time instant $n+1$ and the user being served completes service but rejoins the back of the queue, then also $X_{n+1} = i$. This will happen with probability $(1-p)q(1-\alpha)$. Finally, if an arrival occurs at time instant $n+1$ and the user currently being served exits the system after completing service, then $X_{n+1} = i$ as well. This will happen with probability $pq\alpha$. The conclusion is that $P_{ii} = pq\alpha + (1-p)(1-q\alpha)$.

(c) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $j > i$.

$$P_{i,i+1} = p(1-q\alpha), \; P_{ij} = 0, \; i > 0 \text{ and } j > i+1$$

Suppose the present state is $X_n = i$, $i > 0$. If an arrival occurs at time instant $n+1$ and the user being served does not complete service, then $X_{n+1} = i+1$. This will happen with probability $p(1-q)$. Furthermore, if an arrival occurs at time instant $n+1$ and the user being served completes service but rejoins the back of the queue, then also $X_{n+1} = i+1$. This will happen with probability $pq(1-\alpha)$. The conclusion is that $P_{i,i+1} = p(1-q\alpha)$.

All other transition probabilities from state $i > 0$ to $j > i+1$ are null.

(d) (6 points) Determine the transition probabilities P_{ij} for all $i > 0$ and $0 \leq j < i$.

$P_{i,i-1} = (1-p)q\alpha$, $P_{ij} = 0$, $i > 0$ and $0 \leq j < i-1$

Suppose the present state is $X_n = i$, $i > 0$. If an arrival does not occur at time instant $n+1$ and the user exits the system after being served, then $X_{n+1} = i-1$. This will happen with probability $(1-p)q\alpha$. The conclusion is that $P_{i,i-1} = (1-p)q\alpha$.

All other transition probabilities from state $i > 0$ to $0 \leq j < i-1$ are null.