# ECE440 - Introduction to Random Processes

## Midterm Exam

October 28, 2026

**Instructions:**

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets, smart watches and cell-phones are not allowed.
- Perfect score: 100 (out of 102, extra points are bonus points).
- Duration: 90 minutes.
- This exam has 13 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:_____

| Problem | Max. Points | Score | Problem | Max. Points | Score |
|---|---|---|---|---|---|
| 1. | 18 | | 5. | 12 | |
| 2. | 10 | | 6. | 8 | |
| 3. | 14 | | 7. | 20 | |
| 4. | 20 | | | | |
| | | | Total | 102 | |

**GOOD LUCK!**

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix},$$

and initial distribution $\mathrm{P}(X_0 = 1) = 1/4$ and $\mathrm{P}(X_0 = 2) = 3/4$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 4/9 & 5/9 \\ 5/12 & 7/12 \end{pmatrix} = \begin{pmatrix} 0.44 & 0.56 \\ 0.42 & 0.58 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left(X_5 = 1 \,\middle|\, X_4 = 2, X_3 = 2, X_1 = 1\right) =?$

(b) (3 points) $\mathrm{P}\left(X_3 = 2, X_2 = 1 \,\middle|\, X_1 = 2\right) =?$

(c) (4 points) $\mathrm{P}(X_2 = 1) =?$

(d) (4 points) $\mathrm{P}\left(X_0 = 1 \,\middle|\, X_1 = 1\right) = ?$

(e) (5 points) Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} X_m$$

and provide justification for the existence of the limit.

2. A server processes a batch of $100$ jobs, one after the other. The processing times $X_1, X_2, \ldots, X_{100}$ of the jobs are i.i.d. exponential random variables with mean $\mathbb{E}[X_i] = 2$ and $\text{var}[X_i] = 4$. Let

$$S = \sum_{i=1}^{100} X_i$$

denote the total time needed to process the whole batch.

(a) (4 points) Use Chebyshev's inequality to obtain an upper bound for $\text{P}(|S - 200| \geq 40)$.

(b) (6 points) Use the Central Limit Theorem to approximate $\text{P}(|S - 200| \geq 40)$. Write your answer in terms of the cumulative distribution function $\Phi(z)$ of a standard Normal random variable $Z \sim \mathcal{N}(0, 1)$, that is

$$\Phi(z) = \text{P}(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du.$$

3. The number $N$ of customers who walk into a store on a given day is a Poisson random variable with parameter $\lambda$. Independently of everything else, each customer makes a purchase with probability $p \in (0,1)$. Let $B_i = \mathbb{I}\{\text{Customer } i \text{ makes a purchase}\}$, so that $B_1, B_2, \ldots$ are i.i.d. Bernoulli$(p)$ random variables, independent of $N$. The number of purchases made on that day is

$$Y = \sum_{i=1}^{N} B_i,$$

with the convention that $Y = 0$ if $N = 0$.

(a) (5 points) What is the conditional distribution of $Y$ given that $N = n$? Explain.

(b) (2 points) $\mathbb{E}\left[Y \,\middle|\, N = n\right] = ?$

(c) (3 points) $\mathbb{E}[Y] = ?$

(d) (4 points) $\text{var}[Y] = ?$

4. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) (3 points) Draw the corresponding state transition diagram.

(b) (4 points) Specify the communication classes and determine whether they are transient or recurrent.

(c) (3 points) What is the period of state $5$? Does $\lim_{n\to\infty} P_{55}^n$ exist? Explain.

(d) (3 points) $\lim_{n\to\infty} P_{13}^n =?$

(e) (4 points) Suppose that $X_0 = 2$. What is the probability that the Markov chain eventually enters the set of states $\{4, 5\}$? [Hint: Apply the law of total probability by conditioning on $X_1$, the first transition.]

(f) (3 points) $\lim_{n\to\infty} P_{23}^n = ?$

5. Suppose $X$ and $Y$ are discrete random variables with joint probability mass function $p_{XY}(x, y) = \mathrm{P}(X = x, Y = y)$ given by

| | $X = 0$ | $X = 1$ | $X = 2$ |
|---|---|---|---|
| $Y = 1$ | $1/8$ | $1/8$ | $0$ |
| $Y = 2$ | $1/8$ | $1/4$ | $3/8$ |

(a) (3 points) Find the marginal probability mass function $p_Y(y) = \mathrm{P}(Y = y)$.

(b) (3 points) Find the conditional probability mass function $p_{X|Y}(x \,|\, y) = \mathrm{P}\left(X = x \,\middle|\, Y = y\right)$, for $y = 1$ and $y = 2$.

(c) (3 points) Determine $\mathbb{E}\left[X \,\middle|\, Y = y\right]$, for $y = 1$ and $y = 2$.

(d) (3 points) Use the law of iterated expectations to compute $\mathbb{E}\left[X\right]$.

6. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of Bernoulli$(1/2)$ random variables (e.g., the outcomes of independent tosses of a fair coin), and consider the sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Indicate whether the following claims are True or False **as stated**. Provide a brief justification of your answers.

(a) (3 points) $\bar{X}_n$ converges almost surely to $1/2$ as $n \to \infty$.

(b) (5 points) $\sqrt{n}\left(\bar{X}_n - 1/2\right)$ converges in distribution to a standard Normal random variable $Z \sim \mathcal{N}(0, 1)$ as $n \to \infty$.

7. A total of $3$ balls are distributed between two urns, labeled A and B. At each time period $n = 0, 1, 2, \ldots$, one of the $3$ balls is selected uniformly at random (each ball is selected with probability $1/3$, independently of the past), and it is moved from the urn it is currently in to the other urn. Let $X_n$ denote the number of balls in urn A at the end of time period $n$. The random process $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, 3\}$.

(a) (5 points) Determine the transition probability matrix $\mathbf{P}$.

(b) (3 points) Is the Markov chain irreducible? What is the period of state $0$? Explain.

(c) (7 points) Compute the stationary distribution of $X_{\mathbb{N}}$.

(d) (2 points) Is the Markov chain ergodic? Explain.

(e) (3 points) In the long run, what fraction of time is urn A empty? That is, calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{X_m = 0\}$$

and provide justification for the existence of the limit.