ECE440 - Introduction to Random Processes

Midterm Exam

October 28, 2026

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets, smart watches and cell-phones are not allowed.
- Perfect score: 100 (out of 102, extra points are bonus points).
- Duration: 90 minutes.
- This exam has 13 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name: SOLUTIONS

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	18		5.	12	
2.	10		6.	8	
3.	14		7.	20	
4.	20				
			Total	102	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix},$$

and initial distribution $\mathrm{P}(X_0 = 1) = 1/4$ and $\mathrm{P}(X_0 = 2) = 3/4$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 4/9 & 5/9 \\ 5/12 & 7/12 \end{pmatrix} = \begin{pmatrix} 0.44 & 0.56 \\ 0.42 & 0.58 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left(X_5 = 1 \,\middle|\, X_4 = 2, X_3 = 2, X_1 = 1\right) =?$

$\frac{1}{2}$

From the Markov property it follows that

$$\mathrm{P}\left(X_5 = 1 \,\middle|\, X_4 = 2, X_3 = 2, X_1 = 1\right) = \mathrm{P}\left(X_5 = 1 \,\middle|\, X_4 = 2\right) = P_{21} = \frac{1}{2}.$$

(b) (3 points) $\mathrm{P}\left(X_3 = 2, X_2 = 1 \,\middle|\, X_1 = 2\right) =?$

$\frac{1}{3}$

Using the definition of conditional probability and the Markov property, we have

$$\begin{aligned} \mathrm{P}\left(X_3 = 2, X_2 = 1 \,\middle|\, X_1 = 2\right) &= \mathrm{P}\left(X_3 = 2 \,\middle|\, X_2 = 1, X_1 = 2\right) \mathrm{P}\left(X_2 = 1 \,\middle|\, X_1 = 2\right) \\ &= \mathrm{P}\left(X_3 = 2 \,\middle|\, X_2 = 1\right) \mathrm{P}\left(X_2 = 1 \,\middle|\, X_1 = 2\right) \\ &= P_{12} P_{21} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}. \end{aligned}$$

(c) (4 points) $\mathrm{P}(X_2 = 1) =?$

$\frac{61}{144}$

Using the law of total probability (conditioning on X_0) and the two-step transition probabilities given in the problem statement, we find

$$\begin{aligned} \mathrm{P}(X_2 = 1) &= \mathrm{P}\left(X_2 = 1 \,\middle|\, X_0 = 1\right) \mathrm{P}(X_0 = 1) + \mathrm{P}\left(X_2 = 1 \,\middle|\, X_0 = 2\right) \mathrm{P}(X_0 = 2) \\ &= P_{11}^2 \times \frac{1}{4} + P_{21}^2 \times \frac{3}{4} = \frac{4}{9} \times \frac{1}{4} + \frac{5}{12} \times \frac{3}{4} = \frac{1}{9} + \frac{5}{16} = \frac{61}{144}. \end{aligned}$$

(d) (4 points) $\mathrm{P}\left(X_0 = 1 \,\middle|\, X_1 = 1\right) = ?$

$$\boxed{\frac{2}{11}}$$

From Bayes' rule, we have that

$$\mathrm{P}\left(X_0 = 1 \,\middle|\, X_1 = 1\right) = \frac{\mathrm{P}\left(X_1 = 1 \,\middle|\, X_0 = 1\right)\mathrm{P}\left(X_0 = 1\right)}{\mathrm{P}\left(X_1 = 1\right)} = \frac{P_{11}\mathrm{P}\left(X_0 = 1\right)}{P_{11}\mathrm{P}\left(X_0 = 1\right) + P_{21}\mathrm{P}\left(X_0 = 2\right)}.$$

Substituting the given values, the numerator is $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ and the denominator is $\mathrm{P}\left(X_1 = 1\right) = \frac{1}{12} + \frac{1}{2} \times \frac{3}{4} = \frac{11}{24}$. Hence, $\mathrm{P}\left(X_0 = 1 \,\middle|\, X_1 = 1\right) = \frac{1/12}{11/24} = \frac{2}{11}$.

(e) (5 points) Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} X_m$$

and provide justification for the existence of the limit.

$$\boxed{\frac{11}{7} \text{ a.s.}}$$

The Markov chain is irreducible (since $P_{12} > 0$ and $P_{21} > 0$) and has a finite state space, hence it is positive recurrent. It is also aperiodic, since $P_{11} > 0$. Therefore, the Markov chain is ergodic. Its stationary distribution $\boldsymbol{\pi} = [\pi_1, \pi_2]^T$ is the unique solution of

$$\pi_1 = \frac{1}{3}\pi_1 + \frac{1}{2}\pi_2, \qquad \pi_1 + \pi_2 = 1.$$

From the first equation $\pi_2 = \frac{4}{3}\pi_1$, and from the normalization constraint we obtain $\pi_1 = \frac{3}{7}$ and $\pi_2 = \frac{4}{7}$. Since $X_m \in \{1, 2\}$, we can write $X_m = 1 \times \mathbb{I}\{X_m = 1\} + 2 \times \mathbb{I}\{X_m = 2\}$. Then, by the Ergodic Theorem

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} X_m &= \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{X_m = 1\} + 2 \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{X_m = 2\} \\ &= \pi_1 + 2\pi_2 = \frac{3}{7} + \frac{8}{7} = \frac{11}{7},\end{aligned}$$

almost surely. Notice that the limit does not depend on the initial distribution.

2. A server processes a batch of 100 jobs, one after the other. The processing times $X_1, X_2, \ldots, X_{100}$ of the jobs are i.i.d. exponential random variables with mean $\mathbb{E}[X_i] = 2$ and $\mathrm{var}[X_i] = 4$. Let

$$S = \sum_{i=1}^{100} X_i$$

denote the total time needed to process the whole batch.

(a) (4 points) Use Chebyshev's inequality to obtain an upper bound for $\mathrm{P}(|S - 200| \geq 40)$.

$$\boxed{\mathrm{P}\left(|S-200| \geq 40\right) \leq \frac{1}{4}}$$

Because the X_i are i.i.d. with $\mathbb{E}\left[X_i\right] = 2$ and $\mathrm{var}\left[X_i\right] = 4$, we have

$$\mathbb{E}\left[S\right] = \sum_{i=1}^{100} \mathbb{E}\left[X_i\right] = 200, \qquad \mathrm{var}\left[S\right] = \sum_{i=1}^{100} \mathrm{var}\left[X_i\right] = 400.$$

Hence, Chebyshev's inequality yields

$$\mathrm{P}\left(|S - \mathbb{E}\left[S\right]| \geq 40\right) \leq \frac{\mathrm{var}\left[S\right]}{40^2} = \frac{400}{1600} = \frac{1}{4}.$$

(b) (6 points) Use the Central Limit Theorem to approximate $\mathrm{P}\left(|S-200| \geq 40\right)$. Write your answer in terms of the cumulative distribution function $\Phi(z)$ of a standard Normal random variable $Z \sim \mathcal{N}(0,1)$, that is

$$\Phi(z) = \mathrm{P}\left(Z \leq z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du.$$

$$\boxed{2\left(1 - \Phi(2)\right) = 2\Phi(-2)}$$

By centering and scaling S (with $\mathbb{E}\left[S\right] = 200$ and $\sqrt{\mathrm{var}\left[S\right]} = 20$ from part (a)) and relying on the Central Limit Theorem, we obtain ($Z \sim \mathcal{N}(0,1)$ below)

$$\begin{aligned}\mathrm{P}\left(|S-200| \geq 40\right) &= \mathrm{P}\left(\left|\frac{S - \mathbb{E}\left[S\right]}{\sqrt{\mathrm{var}\left[S\right]}}\right| \geq \frac{40}{20}\right) \approx \mathrm{P}\left(|Z| \geq 2\right) \\ &= \mathrm{P}\left(Z \leq -2\right) + \mathrm{P}\left(Z \geq 2\right) = \Phi(-2) + 1 - \Phi(2) = 2\left(1 - \Phi(2)\right),\end{aligned}$$

where we used the symmetry of the standard Normal pdf, i.e., $\Phi(-z) = 1 - \Phi(z)$. [Remark: Numerically, $2(1-\Phi(2)) \approx 0.046$, which illustrates how loose the Chebyshev bound in part (a) can be. Of course, the bound in (a) holds for any distribution with the same mean and variance, while the CLT exploits the fact that S is a sum of many i.i.d. random variables.]

3. The number N of customers who walk into a store on a given day is a Poisson random variable with parameter λ. Independently of everything else, each customer makes a purchase with probability $p \in (0,1)$. Let $B_i = \mathbb{I}\{\text{Customer } i \text{ makes a purchase}\}$, so that $B_1, B_2, \ldots$ are i.i.d. Bernoulli(p) random variables, independent of N. The number of purchases made on that day is

$$Y = \sum_{i=1}^{N} B_i,$$

with the convention that $Y = 0$ if $N = 0$.

(a) (5 points) What is the conditional distribution of Y given that $N = n$? Explain.

Binomial(n, p)

Since N is independent of the B_i, conditioning on $N = n$ does not change the distribution of $B_1, \ldots, B_n$, and

$$Y \,\big|\, N = n \;\sim\; \sum_{i=1}^{n} B_i.$$

This is the sum of n independent Bernoulli(p) random variables, i.e., the number of successes (purchases) in n independent trials, each with success probability p. Hence, the conditional distribution of Y given $N = n$ is Binomial(n, p), namely

$$\mathrm{P}\left(Y = k \,\middle|\, N = n\right) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

(For $n = 0$ this correctly gives $Y = 0$ with probability one.)

(b) (2 points) $\mathbb{E}\left[Y \,\middle|\, N = n\right] = ?$

np

From part (a), the conditional distribution of Y given $N = n$ is Binomial(n, p), whose mean is np. Equivalently, by linearity of expectation

$$\mathbb{E}\left[Y \,\middle|\, N = n\right] = \mathbb{E}\left[\sum_{i=1}^{n} B_i \,\middle|\, N = n\right] = \sum_{i=1}^{n} \mathbb{E}\left[B_i\right] = np.$$

(c) (3 points) $\mathbb{E}\left[Y\right] = ?$

λp

From part (b), $\mathbb{E}\left[Y \,\middle|\, N\right] = Np$. By the law of iterated expectations,

$$\mathbb{E}\left[Y\right] = \mathbb{E}\left[\mathbb{E}\left[Y \,\middle|\, N\right]\right] = \mathbb{E}\left[Np\right] = p\mathbb{E}\left[N\right] = \lambda p.$$

(d) (4 points) $\text{var}\left[Y\right] =$?

λp

From part (a), $\text{var}\left[Y \,\middle|\, N = n\right] = np(1-p)$ (the variance of a Binomial(n, p) random variable); i.e., $\text{var}\left[Y \,\middle|\, N\right] = Np(1-p)$. Recalling that $\mathbb{E}\left[N\right] = \text{var}\left[N\right] = \lambda$ for a Poisson random variable, the law of total variance yields

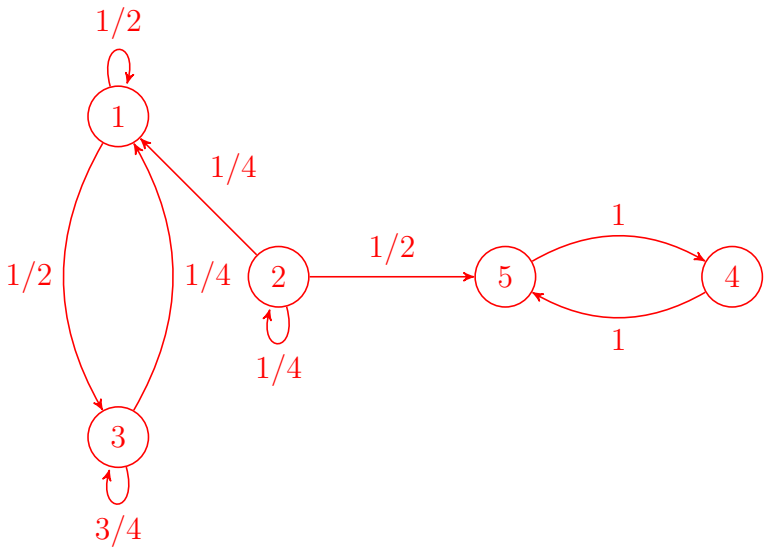
$$\begin{aligned}\text{var}\left[Y\right] &= \mathbb{E}\left[\text{var}\left[Y \,\middle|\, N\right]\right] + \text{var}\left[\mathbb{E}\left[Y \,\middle|\, N\right]\right] \\ &= \mathbb{E}\left[Np(1-p)\right] + \text{var}\left[Np\right] = p(1-p)\lambda + p^2\lambda = \lambda p.\end{aligned}$$

[Remark: The fact that $\mathbb{E}\left[Y\right] = \text{var}\left[Y\right] = \lambda p$ is no coincidence. One can show that Y has the Poisson(λp) distribution, a result known as *thinning* of a Poisson random variable.]

4. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) (3 points) Draw the corresponding state transition diagram.



(b) (4 points) Specify the communication classes and determine whether they are transient or recurrent.

$\mathcal{R}_1 = \{1, 3\}$, recurrent

$\mathcal{T} = \{2\}$, transient

$\mathcal{R}_2 = \{4, 5\}$, recurrent

States 1 and 3 communicate since $P_{13} > 0$ and $P_{31} > 0$, and likewise states 4 and 5 communicate. No transitions leave the sets $\{1, 3\}$ or $\{4, 5\}$, so these are closed classes; because they are finite and closed, they are recurrent. State 2 is accessible from no other state, so it forms a class of its own. Starting from state 2, with probability $1 - P_{22} = 3/4$ the chain leaves to state 1 or 5 and never returns, so $\{2\}$ is transient.

(c) (3 points) What is the period of state 5? Does $\lim_{n\to\infty} P_{55}^n$ exist? Explain.

$d = 2$, the limit does not exist

Once in $\{4, 5\}$ the chain deterministically alternates $5 \to 4 \to 5 \to \cdots$, so $P^n_{55} = 1$ if n is even and $P^n_{55} = 0$ if n is odd. Thus $\gcd\{n : P^n_{55} > 0\} = 2$, and the sequence P^n_{55} oscillates between 0 and 1, so its limit does not exist.

(d) (3 points) $\lim_{n\to\infty} P^n_{13} =$?

$\frac{2}{3}$

Starting from state 1, the chain remains in the closed class $\{1, 3\}$ forever. Restricted to this class, the chain is irreducible, finite (hence positive recurrent) and aperiodic (since $P_{11} > 0$), i.e., ergodic. Hence $\lim_{n\to\infty} P^n_{13} = \pi_3$, where $[\pi_1, \pi_3]^T$ solves

$$\pi_1 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_3, \qquad \pi_1 + \pi_3 = 1.$$

From the first equation $\pi_3 = 2\pi_1$, so $\pi_1 = 1/3$ and $\pi_3 = 2/3$.

(e) (4 points) Suppose that $X_0 = 2$. What is the probability that the Markov chain eventually enters the set of states $\{4, 5\}$? [Hint: Apply the law of total probability by conditioning on X_1, the first transition.]

$\frac{2}{3}$

Let $h = \mathrm{P}\left(\text{chain eventually enters } \{4, 5\} \,\middle|\, X_0 = 2\right)$. Conditioning on the first transition (first-step analysis), and noting that from state 1 the chain never enters $\{4, 5\}$ while from state 5 it already has, we obtain

$$h = P_{21} \times 0 + P_{22} \times h + P_{25} \times 1 = \frac{1}{4}h + \frac{1}{2} \quad \Rightarrow \quad h = \frac{2}{3}.$$

Alternatively, the chain stays in state 2 for a geometric number of steps, and when it leaves, it goes to state 5 with probability $P_{25}/(P_{21} + P_{25}) = (1/2)/(3/4) = 2/3$.

(f) (3 points) $\lim_{n\to\infty} P^n_{23} =$?

$\frac{2}{9}$

Starting from state 2, with probability $1 - 2/3 = 1/3$ (from part (e)) the chain is eventually absorbed in the class $\{1, 3\}$, after which it behaves as the ergodic chain in part (d). With the complementary probability $2/3$ it is absorbed in $\{4, 5\}$ and never visits state 3. Hence

$$\lim_{n\to\infty} P^n_{23} = \frac{1}{3} \times \pi_3 + \frac{2}{3} \times 0 = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}.$$

More formally, the chain can only enter $\{1,3\}$ from state 2 through state 1. Conditioning on the time k at which the chain leaves state 2 for state 1, we have $P_{23}^n = \sum_{k=1}^n P_{22}^{k-1} P_{21} P_{13}^{n-k} = \sum_{k=1}^n (1/4)^k P_{13}^{n-k}$, which converges to $\pi_3 \sum_{k=1}^\infty (1/4)^k = \frac{2}{3} \times \frac{1}{3}$ as $n \to \infty$.

5. Suppose X and Y are discrete random variables with joint probability mass function $p_{XY}(x,y) = \mathrm{P}(X = x, Y = y)$ given by

	$X=0$	$X=1$	$X=2$
$Y=1$	$1/8$	$1/8$	0
$Y=2$	$1/8$	$1/4$	$3/8$

(a) (3 points) Find the marginal probability mass function $p_Y(y) = \mathrm{P}(Y = y)$.

$$p_Y(1) = \frac{1}{4}, \qquad p_Y(2) = \frac{3}{4}$$

To obtain the marginal pmf $p_Y(y)$ we sum $p_{XY}(x,y)$ over all values of x for each y, i.e., we add up the entries in each row of the table:

$$p_Y(1) = \sum_{x=0}^{2} p_{XY}(x,1) = \frac{1}{8} + \frac{1}{8} + 0 = \frac{1}{4},$$
$$p_Y(2) = \sum_{x=0}^{2} p_{XY}(x,2) = \frac{1}{8} + \frac{1}{4} + \frac{3}{8} = \frac{3}{4}.$$

As a sanity check, $p_Y(1) + p_Y(2) = 1$.

(b) (3 points) Find the conditional probability mass function $p_{X|Y}(x \,|\, y) = \mathrm{P}\big(X = x \,\big|\, Y = y\big)$, for $y = 1$ and $y = 2$.

$$p_{X|Y}(0 \,|\, 1) = \frac{1}{2}, \;\; p_{X|Y}(1 \,|\, 1) = \frac{1}{2}, \;\; p_{X|Y}(2 \,|\, 1) = 0$$
$$p_{X|Y}(0 \,|\, 2) = \frac{1}{6}, \;\; p_{X|Y}(1 \,|\, 2) = \frac{1}{3}, \;\; p_{X|Y}(2 \,|\, 2) = \frac{1}{2}$$

From the definition of conditional pmf and part (a),

$$p_{X|Y}(x \,|\, y) = \frac{p_{XY}(x,y)}{p_Y(y)}.$$

For $y = 1$ we divide the first row of the table by $p_Y(1) = 1/4$, obtaining $1/2$, $1/2$ and 0 for $x = 0, 1, 2$, respectively. For $y = 2$ we divide the second row by $p_Y(2) = 3/4$, obtaining $\frac{1/8}{3/4} = \frac{1}{6}$, $\frac{1/4}{3/4} = \frac{1}{3}$ and $\frac{3/8}{3/4} = \frac{1}{2}$. As a sanity check, each conditional pmf sums up to one.

(c) (3 points) Determine $\mathbb{E}\big[X \,\big|\, Y = y\big]$, for $y = 1$ and $y = 2$.

$$\boxed{\mathbb{E}\left[X \,\middle|\, Y=1\right] = \frac{1}{2},\ \ \mathbb{E}\left[X \,\middle|\, Y=2\right] = \frac{4}{3}}$$

Using the conditional pmfs from part (b),

$$\mathbb{E}\left[X \,\middle|\, Y=1\right] = \sum_{x=0}^{2} x\, p_{X|Y}(x \,|\, 1) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} + 2 \times 0 = \frac{1}{2},$$
$$\mathbb{E}\left[X \,\middle|\, Y=2\right] = \sum_{x=0}^{2} x\, p_{X|Y}(x \,|\, 2) = 0 \times \frac{1}{6} + 1 \times \frac{1}{3} + 2 \times \frac{1}{2} = \frac{4}{3}.$$

(d) (3 points) Use the law of iterated expectations to compute $\mathbb{E}[X]$.

$$\boxed{\frac{9}{8}}$$

The conditional expectation $\mathbb{E}\left[X \,\middle|\, Y\right]$ is a random variable taking the value $1/2$ with probability $p_Y(1) = 1/4$, and the value $4/3$ with probability $p_Y(2) = 3/4$. Hence, by the law of iterated expectations

$$\begin{aligned}\mathbb{E}[X] = \mathbb{E}\left[\mathbb{E}\left[X \,\middle|\, Y\right]\right] &= \mathbb{E}\left[X \,\middle|\, Y=1\right] p_Y(1) + \mathbb{E}\left[X \,\middle|\, Y=2\right] p_Y(2) \\ &= \frac{1}{2} \times \frac{1}{4} + \frac{4}{3} \times \frac{3}{4} = \frac{1}{8} + 1 = \frac{9}{8}.\end{aligned}$$

As a check, summing the columns of the table gives the marginal pmf $p_X(0) = \frac{1}{4}$, $p_X(1) = \frac{3}{8}$, $p_X(2) = \frac{3}{8}$, and so $\mathbb{E}[X] = \frac{3}{8} + 2 \times \frac{3}{8} = \frac{9}{8}$.

6. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of Bernoulli$(1/2)$ random variables (e.g., the outcomes of independent tosses of a fair coin), and consider the sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Indicate whether the following claims are True or False **as stated**. Provide a brief justification of your answers.

(a) (3 points) $\bar{X}_n$ converges almost surely to $1/2$ as $n \to \infty$.

$$\boxed{\text{True}}$$

The X_i are i.i.d. with mean $\mathbb{E}[X_1] = 1/2$, so the Strong Law of Large Numbers asserts that $\mathrm{P}\left(\lim_{n\to\infty} \bar{X}_n = 1/2\right) = 1$.

(b) (5 points) $\sqrt{n}\left(\bar{X}_n - 1/2\right)$ converges in distribution to a standard Normal random variable $Z \sim \mathcal{N}(0,1)$ as $n \to \infty$.

False

The variance of a Bernoulli$(1/2)$ random variable is $\sigma^2 = \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{4}$, so $\sigma = 1/2$. The Central Limit Theorem asserts that

$$\frac{\sum_{i=1}^{n} X_i - n/2}{\sqrt{n\sigma^2}} = \frac{\sqrt{n}\left(\bar{X}_n - 1/2\right)}{\sigma} = 2\sqrt{n}\left(\bar{X}_n - 1/2\right)$$

converges in distribution to $Z \sim \mathcal{N}(0,1)$. Hence $\sqrt{n}\left(\bar{X}_n - 1/2\right)$ converges in distribution to $Z/2 \sim \mathcal{N}(0, 1/4)$, not to a standard Normal. The normalization must account for the variance of the X_i.

7. A total of 3 balls are distributed between two urns, labeled A and B. At each time period $n = 0, 1, 2, \ldots$, one of the 3 balls is selected uniformly at random (each ball is selected with probability $1/3$, independently of the past), and it is moved from the urn it is currently in to the other urn. Let X_n denote the number of balls in urn A at the end of time period n. The random process $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, 3\}$.

(a) (5 points) Determine the transition probability matrix $\mathbf{P}$.

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Rows and columns are indexed by the states $0, 1, 2, 3$, in that order. Suppose the present state is $X_n = i$, i.e., there are i balls in urn A and $3 - i$ balls in urn B. Exactly one ball changes urns in every time period, so the number of balls in urn A either decreases or increases by one.

1) If the selected ball is in urn A, which happens with probability $i/3$, then it is moved to urn B and $X_{n+1} = i - 1$. Thus $P_{i,i-1} = i/3$, for $i = 1, 2, 3$.
2) If the selected ball is in urn B, which happens with probability $(3-i)/3$, then it is moved to urn A and $X_{n+1} = i + 1$. Thus $P_{i,i+1} = (3-i)/3$, for $i = 0, 1, 2$.

All other transition probabilities are null. In particular, $P_{01} = 1$ and $P_{32} = 1$. As a check, all rows of $\mathbf{P}$ sum up to one.

(b) (3 points) Is the Markov chain irreducible? What is the period of state 0? Explain.

Irreducible, $d = 2$

From part (a), $0 \to 1 \to 2 \to 3 \to 2 \to 1 \to 0$ is a path with positive probability, so all states communicate and the chain is irreducible. Since $X_{n+1} - X_n \in \{-1, +1\}$ for all n, the parity of X_n alternates at every step, and the chain can only return to state 0 after an even number of steps. Because $P_{00}^2 = P_{01}P_{10} = 1/3 > 0$, we conclude $d = \gcd\{n : P_{00}^n > 0\} = 2$. (By irreducibility, all states have period 2.)

(c) (7 points) Compute the stationary distribution of $X_{\mathbb{N}}$.

$$\boldsymbol{\pi} = [\pi_0, \pi_1, \pi_2, \pi_3]^T = \left[\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right]^T$$

The chain is finite and irreducible, so it has a unique stationary distribution solving $\pi_j = \sum_{i=0}^{3} \pi_i P_{ij}$ for all $j \in S$, together with $\sum_{j=0}^{3} \pi_j = 1$. Using the transition probabilities from part (a), the balance equations are

$$\begin{aligned}
\pi_0 &= \frac{1}{3}\pi_1, \\
\pi_1 &= \pi_0 + \frac{2}{3}\pi_2, \\
\pi_2 &= \frac{2}{3}\pi_1 + \pi_3, \\
\pi_3 &= \frac{1}{3}\pi_2.
\end{aligned}$$

The first and last equations give $\pi_1 = 3\pi_0$ and $\pi_2 = 3\pi_3$. Substituting in the second equation yields $3\pi_0 = \pi_0 + 2\pi_3$, i.e., $\pi_3 = \pi_0$ (the third equation is then redundant). Hence $\boldsymbol{\pi} = \pi_0[1, 3, 3, 1]^T$, and the normalization constraint $8\pi_0 = 1$ gives $\pi_0 = \pi_3 = 1/8$ and $\pi_1 = \pi_2 = 3/8$. [Remark: This is the Binomial$(3, 1/2)$ distribution, i.e., in the long run each ball is equally likely to be in either urn, independently of the other balls. More generally, for N balls the stationary distribution is Binomial$(N, 1/2)$. This is the celebrated Ehrenfest model of diffusion.]

(d) (2 points) Is the Markov chain ergodic? Explain.

No

The chain is irreducible and positive recurrent (finite and irreducible), but it is periodic with period $d = 2$ from part (b). Ergodicity requires aperiodicity. Consequently, $P_{00}^n = 0$ for all odd n while $P_{00}^n > 0$ for all even n, so the limiting probabilities $\lim_{n\to\infty} P_{ij}^n$ do not exist.

(e) (3 points) In the long run, what fraction of time is urn A empty? That is, calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{X_m = 0\}$$

and provide justification for the existence of the limit.

$\frac{1}{8}$ a.s.

Even though the chain is periodic (and hence not ergodic), it is irreducible and positive recurrent, so the Ergodic Theorem still applies to the long-run fraction of time spent in each state. Therefore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{X_m = 0\} = \pi_0 = \frac{1}{8}, \quad \text{almost surely,}$$

regardless of the initial state. Note the contrast with part (d): time averages converge even though the limiting probabilities P_{i0}^n oscillate.