ECE440 - Introduction to Random Processes

Midterm Exam

October 9, 2014

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100.
- Duration: 75 minutes.
- This exam has 10 numbered pages, check now that all pages are present.
- Show all your work, and write your final answers in the boxes when provided.

Name:__________________________________

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	22		5.	10	
2.	10		6.	14	
3.	10		7.	18	
4.	16				
			Total	100	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix}$$

and initial distribution $\mathrm{P}[X_0 = 1] = 1$ and $\mathrm{P}[X_0 = 2] = 0$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 4/9 & 5/9 \\ 5/12 & 7/12 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left[X_3 = 1 \,\middle|\, X_2 = 2, X_0 = 1\right] =?$

(b) (6 points) $\mathrm{P}\left[X_4 = 2 \,\middle|\, X_2 = 1, X_1 = 1, X_0 = 1\right] =?$

(c) (8 points) $\mathbb{E}[X_2] = ?$

(d) (6 points) $\mathrm{P}[X_0 = 2, X_2 = 1] = ?$

2. (10 points) Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables and that A and B are sets such that $\mathrm{P}[X_1 \in A] = p$, $\mathrm{P}[X_1 \in B] = q$, and $\mathrm{P}[X_1 \in A \cap B] = r$. Compute

$$\mathbb{E}\left[\sum_{j=1}^{n}\sum_{i=1}^{n} \mathbb{I}\{X_j \in A\}\mathbb{I}\{X_i \in B\}\right]$$

in terms of $n, p, q,$ and r. Show your work.

3. (10 points) Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space S, and transition probability matrix $\mathbf{P}$ with entries P_{ij}, $i, j \in S$. For $n = 0, 1, 2 \ldots$ define $Y_n = X_{2n}$. Is $Y_{\mathbb{N}} = Y_0, Y_1, \ldots, Y_n, \ldots$ a Markov chain? If so, provide an expression for the transition probabilities of $Y_{\mathbb{N}}$. Justify your answer.

4. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables, where $\mathrm{P}[X_1 = 1] = \mathrm{P}[X_1 = 2] = \mathrm{P}[X_1 = 3] = 1/3$. Define

$$T = \min\{n \geq 1 : X_n \notin \{2, 3\}\}.$$

(a) (6 points) Compute $\mathbb{E}\left[X_i \,\middle|\, T = t\right]$, for $i = 1, \ldots, t-1$.

(b) (5 points) Compute $\mathbb{E}\left[X_t \,\middle|\, T = t\right]$.

(c) (5 points) Compute $\mathbb{E}\left[X_{t+1} \,\middle|\, T = t\right]$.

5. (10 points) Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables, where $\mathrm{P}[X_n = 1] = 1/3$, $\mathrm{P}[X_n = 2] = 1/6$, $\mathrm{P}[X_n = 3] = 1/2$. Calculate

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\left\{X_{2i-1} = X_{2i}\right\}$$

and provide justification for the existence of the limit.

6. (14 points) Suppose that days are either rainy (r) or sunny (s). If on any particular day it is rainy, the the next day will be rainy with probability $2/3$ and sunny with probability $1/3$. Similarly, if on any particular day it is sunny, then the next day will be rainy with probability $1/4$ and sunny with probability $3/4$. What is the long-run fraction of days that will be rainy?

7. Consider a branching process and suppose that X_n is the number of individuals in generation n. Suppose the k-th individual in generation n creates $Q_{k,n+1}$ individuals in generation $n+1$, and that the $Q_{k,n}$ are i.i.d. across individuals and generations, and independent of X_0. Under the preceding assumptions, $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$ for which

$$X_{n+1} = \sum_{k=1}^{X_n} Q_{k,n+1} \quad \text{if } X_n > 0,$$

and $X_{n+1} = 0$ if $X_n = 0$. Let

$$p_1(x) = \mathrm{P}\left[Q_{1,1} = x\right] \;\; \text{and}$$
$$p_k(x) = \sum_{y=0}^{x} p_1(y) p_{k-1}(x-y), \;\; k \geq 2.$$

You may want to recall that for independent, non-negative, integer-valued random variables U and V, the pmf $p_W(x)$ of $W = U + V$ is given by the discrete convolution of the pmfs $p_U(x)$ and $p_V(x)$ of U and V, that is

$$p_W(x) = \sum_{y=0}^{x} p_U(y) p_V(x-y).$$

(a) (10 points) Determine the transition probability matrix of $X_{\mathbb{N}}$ in terms of $p_k(\cdot)$, $k \geq 1$.

(b) (4 points) From the information given, can you determine whether any of the states is recurrent? Justify your answer.

(c) (4 points) From the information given, can you determine whether the Markov chain is irreducible? Justify your answer.