ECE440 - Introduction to Random Processes

Midterm Exam

October 9, 2014

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100.
- Duration: 75 minutes.
- This exam has 10 numbered pages, check now that all pages are present.
- Show all your work, and write your final answers in the boxes when provided.

Name: SOLUTIONS

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	22		5.	10	
2.	10		6.	14	
3.	10		7.	18	
4.	16				
			Total	100	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix}$$

and initial distribution $\mathrm{P}[X_0 = 1] = 1$ and $\mathrm{P}[X_0 = 2] = 0$. To spare you of pointless calculations, if needed you may use that

$$\mathbf{P}^2 = \begin{pmatrix} 4/9 & 5/9 \\ 5/12 & 7/12 \end{pmatrix}.$$

(a) (2 points) $\mathrm{P}\left[X_3 = 1 \,\middle|\, X_2 = 2, X_0 = 1\right] =?$

$\frac{1}{2}$

From the Markov property it follows that

$$\mathrm{P}\left[X_3 = 1 \,\middle|\, X_2 = 2, X_0 = 1\right] = \mathrm{P}\left[X_3 = 1 \,\middle|\, X_2 = 2\right] = P_{21} = \frac{1}{2}.$$

(b) (6 points) $\mathrm{P}\left[X_4 = 2 \,\middle|\, X_2 = 1, X_1 = 1, X_0 = 1\right] =?$

$\frac{5}{9}$

Likewise,

$$\mathrm{P}\left[X_4 = 2 \,\middle|\, X_2 = 1, X_1 = 1, X_0 = 1\right] = \mathrm{P}\left[X_4 = 2 \,\middle|\, X_2 = 1\right] = P^2_{12} = \frac{5}{9}.$$

(c) (8 points) $\mathbb{E}[X_2] =?$

$\frac{14}{9}$

The unconditional pmf of X_2 is (note that $\mathrm{P}[X_0 = 1] = 1$ and $\mathrm{P}[X_0 = 2] = 0$)

$$\mathrm{P}[X_2 = 1] = \mathrm{P}\left[X_2 = 1 \,\middle|\, X_0 = 1\right] \mathrm{P}[X_0 = 1] + \mathrm{P}\left[X_2 = 1 \,\middle|\, X_0 = 2\right] \mathrm{P}[X_0 = 2] = P^2_{11} = \frac{4}{9}$$

$$\mathrm{P}[X_2 = 2] = \mathrm{P}\left[X_2 = 2 \,\middle|\, X_0 = 1\right] \mathrm{P}[X_0 = 1] + \mathrm{P}\left[X_2 = 2 \,\middle|\, X_0 = 2\right] \mathrm{P}[X_0 = 2] = P^2_{12} = \frac{5}{9}.$$

Hence, the expectation is $\mathbb{E}[X_2] = 1 \times \frac{4}{9} + 2 \times \frac{5}{9} = \frac{14}{9}$

(d) (6 points) $\mathrm{P}[X_0 = 2, X_2 = 1] =?$

0

Since $\mathrm{P}[X_0 = 2] = 0$, it follows that $\mathrm{P}[X_0 = 2, X_2 = 1] = 0$.

2. (10 points) Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables and that A and B are sets such that $\mathrm{P}[X_1 \in A] = p$, $\mathrm{P}[X_1 \in B] = q$, and $\mathrm{P}[X_1 \in A \cap B] = r$. Compute

$$\mathbb{E}\left[\sum_{j=1}^{n}\sum_{i=1}^{n}\mathbb{I}\{X_j \in A\}\mathbb{I}\{X_i \in B\}\right]$$

in terms of $n, p, q,$ and r. Show your work.

$n(n-1)pq + nr$

From the linearity of expectation, and pulling out the $i = j$ terms

$$\begin{aligned}\mathbb{E}\left[\sum_{j=1}^{n}\sum_{i=1}^{n}\mathbb{I}\{X_j \in A\}\mathbb{I}\{X_i \in B\}\right] &= \sum_{j=1}^{n}\sum_{i=1}^{n}\mathbb{E}\left[\mathbb{I}\{X_j \in A\}\mathbb{I}\{X_i \in B\}\right] \\ &= \sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq j}}^{n}\mathbb{E}\left[\mathbb{I}\{X_j \in A\}\mathbb{I}\{X_i \in B\}\right] \\ &\quad + \sum_{i=1}^{n}\mathbb{E}\left[\mathbb{I}\{X_i \in A\}\mathbb{I}\{X_i \in B\}\right].\end{aligned}$$

Note that for $i \neq j$ the random variables $\mathbb{I}\{X_j \in A\}$ and $\mathbb{I}\{X_i \in B\}$ are independent, so we can exchange product with the expectation operator in the first summands

$$\begin{aligned}\mathbb{E}\left[\sum_{j=1}^{n}\sum_{i=1}^{n}\mathbb{I}\{X_j \in A\}\mathbb{I}\{X_i \in B\}\right] &= \sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq j}}^{n}\mathbb{E}\left[\mathbb{I}\{X_j \in A\}\right]\mathbb{E}\left[\mathbb{I}\{X_i \in B\}\right] \\ &\quad + \sum_{i=1}^{n}\mathbb{E}\left[\mathbb{I}\{X_i \in A\}\mathbb{I}\{X_i \in B\}\right].\end{aligned}$$

Since expected values of indicator functions are the probabilities of the indicated events, then

$$\begin{aligned}\mathbb{E}\left[\mathbb{I}\{X_1 \in A\}\right] &= \mathrm{P}[X_1 \in A] = p, \\ \mathbb{E}\left[\mathbb{I}\{X_1 \in B\}\right] &= \mathrm{P}[X_1 \in B] = q, \\ \mathbb{E}\left[\mathbb{I}\{X_1 \in A\}\mathbb{I}\{X_1 \in B\}\right] &= \mathrm{P}[X_1 \in A \cap B] = r.\end{aligned}$$

Substituting these values, the final answer is

$$\mathbb{E}\left[\sum_{j=1}^{n}\sum_{i=1}^{n}\mathbb{I}\{X_j \in A\}\mathbb{I}\{X_i \in B\}\right] = \sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq j}}^{n} pq + \sum_{i=1}^{n} r = n(n-1)pq + nr.$$

3. (10 points) Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space S, and transition probability matrix $\mathbf{P}$ with entries P_{ij}, $i, j \in S$. For $n = 0, 1, 2 \ldots$ define $Y_n = X_{2n}$. Is $Y_{\mathbb{N}} = Y_0, Y_1, \ldots, Y_n, \ldots$ a Markov chain? If so, provide an expression for the transition probabilities of $Y_{\mathbb{N}}$. Justify your answer.

To determine whether $Y_{\mathbb{N}}$ is a MC, introduce the state history vectors $\mathbf{Y}_{n-1} = [Y_{n-1}, \ldots, Y_0]$ and $\mathbf{y} = [i_{n-1}, \ldots, i_0]$ for notational compactness, and check the Markov property, i.e.,

$$\begin{aligned}\mathrm{P}\left[Y_{n+1} = j \,\middle|\, Y_n = i, \mathbf{Y}_{n-1} = \mathbf{y}\right] &= \mathrm{P}\left[X_{2(n+1)} = j \,\middle|\, X_{2n} = i, X_{2(n-1)} = i_{n-1}, \ldots, X_0 = i_0\right] \\ &= \mathrm{P}\left[X_{2(n+1)} = j \,\middle|\, X_{2n} = j\right] = P_{ij}^2.\end{aligned}$$

The second equality follows because $X_{\mathbb{N}}$ is a MC, and the third one is by definition of two-step transition probabilities. Notice $\mathrm{P}\left[Y_{n+1} = j \,\middle|\, Y_n = i, \mathbf{Y}_{n-1} = \mathbf{y}\right]$ only depends on i and j.

The preceding argument establishes that $Y_{\mathbb{N}}$ is a MC, with transition probabilities $\hat{P}_{ij} = P_{ij}^2$.

4. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables, where $\mathrm{P}[X_1 = 1] = \mathrm{P}[X_1 = 2] = \mathrm{P}[X_1 = 3] = 1/3$. Define

$$T = \min\{n \geq 1 : X_n \notin \{2, 3\}\}.$$

(a) (6 points) Compute $\mathbb{E}\left[X_i \,\middle|\, T = t\right]$, for $i = 1, \ldots, t-1$.

$\frac{5}{2}$

From the definition of T, then for $i = 1, \ldots, t-1$ one has

$$\begin{aligned}\mathbb{E}\left[X_i \,\middle|\, T = t\right] &= \mathbb{E}\left[X_i \,\middle|\, X_1 \in \{2,3\}, \ldots, X_i \in \{2,3\}, \ldots, X_{t-1} \in \{2,3\}, X_t = 1\right] \\ &= \mathbb{E}\left[X_i \,\middle|\, X_i \in \{2,3\}\right]\end{aligned}$$

where the last equality follows from the independence of the sequence $X_{\mathbb{N}}$. The relevant conditional pmf $\mathrm{P}\left[X_i = x \,\middle|\, X_i \in \{2,3\}\right]$ is

$$\begin{aligned}\mathrm{P}\left[X_i = 1 \,\middle|\, X_i \in \{2,3\}\right] &= 0 \\ \mathrm{P}\left[X_i = 2 \,\middle|\, X_i \in \{2,3\}\right] &= \frac{\mathrm{P}[X_i = 2, X_i \in \{2,3\}]}{\mathrm{P}[X_i \in \{2,3\}]} = \frac{\mathrm{P}[X_i = 2]}{\mathrm{P}[X_i = 2] + \mathrm{P}[X_i = 3]} = \frac{1}{2} \\ \mathrm{P}\left[X_i = 3 \,\middle|\, X_i \in \{2,3\}\right] &= \frac{\mathrm{P}[X_i = 3, X_i \in \{2,3\}]}{\mathrm{P}[X_i \in \{2,3\}]} = \frac{\mathrm{P}[X_i = 3]}{\mathrm{P}[X_i = 2] + \mathrm{P}[X_i = 3]} = \frac{1}{2}.\end{aligned}$$

Hence, the conditional expectation is $\mathbb{E}\left[X_i \,\middle|\, T = t\right] = 2 \times \frac{1}{2} + 3 \times \frac{1}{2} = \frac{5}{2}$, for $i = 1, \ldots, t-1$.

(b) (5 points) Compute $\mathbb{E}\left[X_t \,\middle|\, T = t\right]$.

1

Reasoning as in the previous part, from the definition of T and using the independence of the sequence $X_{\mathbb{N}}$

$$\begin{aligned}\mathbb{E}\left[X_t \,\middle|\, T = t\right] &= \mathbb{E}\left[X_t \,\middle|\, X_1 \in \{2,3\}, \ldots, X_i \in \{2,3\}, \ldots, X_{t-1} \in \{2,3\}, X_t = 1\right] \\ &= \mathbb{E}\left[X_t \,\middle|\, X_t = 1\right] = 1.\end{aligned}$$

(c) (5 points) Compute $\mathbb{E}\left[X_{t+1} \,\middle|\, T=t\right]$.

2

Notice that $T=t$ does not provide any information about $X_{t+1}=x$ (they are independent events), which formally follows from

$$\begin{aligned}\mathbb{E}\left[X_{t+1} \,\middle|\, T=t\right] &= \mathbb{E}\left[X_{t+1} \,\middle|\, X_1 \in \{2,3\}, \ldots, X_i \in \{2,3\}, \ldots, X_{t-1} \in \{2,3\}, X_t = 1\right] \\ &= \mathbb{E}\left[X_{t+1}\right] = 1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} = 2.\end{aligned}$$

5. (10 points) Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables, where $\mathrm{P}[X_n = 1] = 1/3$, $\mathrm{P}[X_n = 2] = 1/6$, $\mathrm{P}[X_n = 3] = 1/2$. Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{X_{2i-1} = X_{2i}\}$$

and provide justification for the existence of the limit.

$\frac{7}{18}$

Because $X_{\mathbb{N}}$ is i.i.d., then $Y_{\mathbb{N}} = \mathbb{I}\{X_1 = X_2\}, \mathbb{I}\{X_3 = X_4\}, \ldots, \mathbb{I}\{X_{2n-1} = X_{2n}\}, \ldots$ is also i.i.d. By the strong law of large numbers the limit exists and is equal to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{X_{2i-1} = X_{2i}\} = \mathbb{E}\left[\mathbb{I}\{X_1 = X_2\}\right] = \mathrm{P}[X_1 = X_2], \quad \text{w.p. } 1.$$

Since the $X_{\mathbb{N}}$ are i.i.d., $\mathrm{P}[X_1 = X_2]$ is given by

$$\begin{aligned}\mathrm{P}[X_1 = X_2] &= \mathrm{P}[X_1 = 1]\,\mathrm{P}[X_2 = 1] + \mathrm{P}[X_1 = 2]\,\mathrm{P}[X_2 = 2] + \mathrm{P}[X_1 = 3]\,\mathrm{P}[X_2 = 3] \\ &= \left(\frac{1}{3}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{7}{18}.\end{aligned}$$

6. (14 points) Suppose that days are either rainy (r) or sunny (s). If on any particular day it is rainy, the the next day will be rainy with probability $2/3$ and sunny with probability $1/3$. Similarly, if on any particular day it is sunny, then the next day will be rainy with probability $1/4$ and sunny with probability $3/4$. What is the long-run fraction of days that will be rainy?

$\frac{3}{7}$

The process is a Markov chain with state space $S = \{r, s\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 2/3 & 1/3 \\ 1/4 & 3/4 \end{pmatrix}.$$

The long-run fraction of days that will be rainy is π_r, where $\boldsymbol{\pi} = [\pi_r, \pi_s]^T$ is the unique (the Markov chain is ergodic) stationary distribution which satisfies

$$\begin{pmatrix} \pi_r \\ \pi_s \end{pmatrix} = \begin{pmatrix} 2/3 & 1/4 \\ 1/3 & 3/4 \end{pmatrix} \begin{pmatrix} \pi_r \\ \pi_s \end{pmatrix}, \quad \pi_r + \pi_s = 1.$$

Solving the linear system yields $\pi_r = \frac{3}{7}$, so in the long run it will rain three days per week.

7. Consider a branching process and suppose that X_n is the number of individuals in generation n. Suppose the k-th individual in generation n creates $Q_{k,n+1}$ individuals in generation $n+1$, and that the $Q_{k,n}$ are i.i.d. across individuals and generations, and independent of X_0. Under the preceding assumptions, $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$ for which

$$X_{n+1} = \sum_{k=1}^{X_n} Q_{k,n+1} \quad \text{if } X_n > 0,$$

and $X_{n+1} = 0$ if $X_n = 0$. Let

$$p_1(x) = \mathrm{P}\left[Q_{1,1} = x\right] \text{ and}$$
$$p_k(x) = \sum_{y=0}^{x} p_1(y) p_{k-1}(x-y), \;\; k \geq 2.$$

You may want to recall that for independent, non-negative, integer-valued random variables U and V, the pmf $p_W(x)$ of $W = U + V$ is given by the discrete convolution of the pmfs $p_U(x)$ and $p_V(x)$ of U and V, that is

$$p_W(x) = \sum_{y=0}^{x} p_U(y) p_V(x-y).$$

(a) (10 points) Determine the transition probability matrix of $X_{\mathbb{N}}$ in terms of $p_k(\cdot)$, $k \geq 1$.

The key point to recognize here is that $p_k(x)$ is the probability that k individuals create a total of x individuals in the next generation. Specifically, $p_k(x)$, $k \geq 2$ is obtained by summing from $y = 0$ to x the probability that a single individual generates y individuals ($p_1(y)$) times the probability that $k-1$ individuals generate $x-y$ individuals ($p_{k-1}(x-y)$).

The transition probabilities are thus

$$P_{00} = 1 \text{ and } P_{0j} = 0, \;\; j \neq 0, \;\; \text{(state 0 is an absorbing state)},$$
$$P_{ij} = p_i(j), \;\; i \neq 0 \text{ and for all } j.$$

More formally,

$$P_{ij} = \mathrm{P}\left[X_{n+1} = j \,\middle|\, X_n = i\right] = \mathrm{P}\left[\sum_{k=1}^{X_n} Q_{k,n+1} = j \,\middle|\, X_n = i\right]$$
$$\mathrm{P}\left[\sum_{k=1}^{i} Q_{k,n+1} = j \,\middle|\, X_n = i\right] = \mathrm{P}\left[\sum_{k=1}^{i} Q_{k,n+1} = j\right] = p_i(j).$$

(b) (4 points) From the information given, can you determine whether any of the states is recurrent? Justify your answer.

State 0 is recurrent.

State 0 (extinction) is an absorbing state, hence a recurrent state.

(c) (4 points) From the information given, can you determine whether the Markov chain is irreducible? Justify your answer.

$X_{\mathbb{N}}$ is not irreducible.

State 0 (extinction) is an absorbing state, hence it does not communicate with any other state. Accordingly, it is the sole member of the recurrent communication class $\mathcal{R}_1 = \{0\}$. This implies the total number of classes is strictly greater than 1, meaning $X_{\mathbb{N}}$ is not irreducible.