Introduction to Random Processes

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Introductions

Who we are, where to find me, lecture times

- Gonzalo Mateos
- Associate Professor, Dept. of Electrical and Computer Engineering
- CSB 726, gmateosb@ece.rochester.edu
- http://www.hajim.rochester.edu/ece/sites/gmateos/
- Where? We meet in Gavett Hall 202
- When? Mondays and Wednesdays 4:50 pm to 6:05 pm
  - Due to travel, make-up lectures on Fridays 4:50 pm to 6:05 pm
- My office hours, Tuesdays at 10:30 am
  - Anytime, as long as you have something interesting to tell me
- Class website

  http://www.hajim.rochester.edu/ece/sites/gmateos/ECE440.html

Teaching assistant

- A great TA to help you with your homework

- Hamed Ajorlou
- CSB 701, hajorlou@ur.rochester.edu
- His office hours, Fridays at 2:30 pm

Prerequisites

(I) Probability theory

- Random (Stochastic) processes are collections of random variables
- Basic knowledge expected. Will review in the first six lectures

(II) Calculus and linear algebra

- Integrals, limits, infinite series, differential equations
- Vector/matrix notation, systems of linear equations, eigenvalues

(III) Programming in Matlab

- Needed for homework
  `https://tech.rochester.edu/software/matlab/`
- If you know programming you can learn Matlab in one afternoon
  - $\Rightarrow$ But it has to be one of this week's afternoons

Homework, exams and grading

(I) Homework sets (10 in 15 weeks) worth 28 points

- Important and demanding part of this class
- Collaboration accepted, welcomed, and encouraged

(II) Midterm examination on Wednesday October 30 worth 36 points

(III) Final take-home examination on December 15-17 worth 36 points

- Work independently. This time no collaboration, no discussion
- ECE 271 students get 10 free points
- At least 60 points are required for passing (C grade)
- B requires at least 75 points. A at least 92. No curve
  - $\Rightarrow$ Goal is for everyone to earn an A

Textbooks

- Good general reference for the class

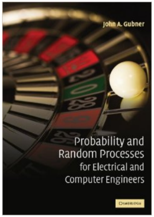
  John A. Gubner, *"Probability and Random Processes for Electrical and Computer Engineers,"* Cambridge University Press

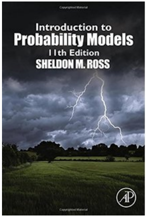
  ⇒ Available online: `http://www.library.rochester.edu/`

- Also nice for topics including Markov chains, queuing models

  Sheldon M. Ross, *"Introduction to Probability Models,"* 13th ed., Academic Press (previous editions are fine)

- Both on reserve for the class in Carlson Library





Be nice

- I work hard for this course, expect you to do the same
- ✓ Please come to class, be on time, pay attention, ask
- ✓ Do all of your homework
- × Do not hand in as yours the solution of others (or mine)
- × Do not collaborate in the exams
- A little bit of (conditional) probability ...
- Probability of getting an E in this class is 0.04
- Probability of getting an E given you skip 4 homework sets is 0.7

  $\Rightarrow$ I'll give you three notices, afterwards, I'll give up on you
- Come and learn. Useful down the road

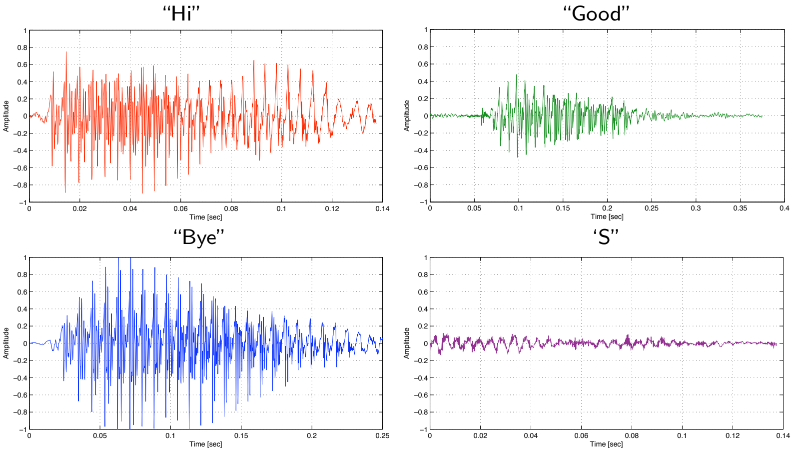
Class contents

Stochastic systems

- Stochastic system: Anything random that evolves in time
  - $\Rightarrow$ Time can be discrete $n = 0, 1, 2 \ldots$, or continuous $t \in [0, \infty)$
- More formally, random processes assign a function to a random event
- Compare with "random variable assigns a value to a random event"
- Can interpret a random process as a collection of random variables
  - $\Rightarrow$ Generalizes concept of random vector to functions
  - $\Rightarrow$ Or generalizes the concept of function to random settings

A voice recognition system

- Random event ∼ word spoken. Random process ∼ the waveform
  - Try the file speech_signals.m



Four thematic blocks

(I) Probability theory review (6 lectures)

- Probability spaces, random variables, independence, expectation
- Conditional probability: time $n+1$ given time n, future given past ...
- Limits in probability, almost sure limits: behavior as $n \to \infty$ ...
- Common probability distributions (binomial, exponential, Poisson, Gaussian)

- Random processes are complicated entities

  $\Rightarrow$ Restrict attention to particular classes that are somewhat tractable

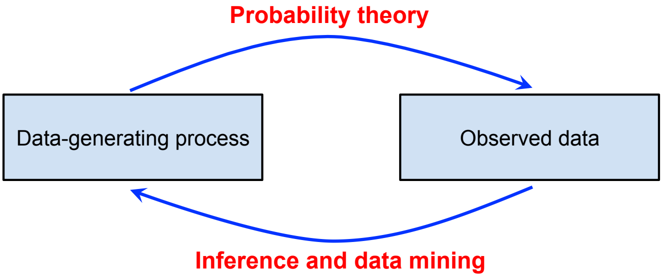
(II) Markov chains (6 lectures)

(III) Continuous-time Markov chains (7 lectures)

(IV) Stationary random processes (8 lectures)

- Midterm covers up to Markov chains

Probability and statistical inference

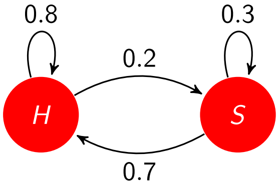


- Probability theory is a formalism to work with uncertainty
  - Given a data-generating process, what are properties of outcomes?
- Statistical inference deals with the inverse problem
  - Given outcomes, what can we say on the data-generating process?
  - ECE409 - Machine Learning, ECE442 - Network Science Analytics, CSC440 - Data Mining, ECE441 - Detection and Estimation Theory, . . .

Markov chains

- Countable set of states $1, 2, \ldots$. At discrete time n, state is X_n
- Memoryless (Markov) property
  - $\Rightarrow$ Probability of next state X_{n+1} depends on current state X_n
  - $\Rightarrow$ But not on past states X_{n-1}, X_{n-2}, ...

- Can be happy ($X_n = 0$) or sad ($X_n = 1$)
- Tomorrow's mood only affected by today's mood
- Whether happy or sad today, likely to be happy tomorrow
- But when sad, a little less likely so

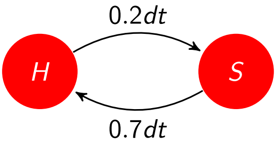


- Of interest: classification of states, ergodicity, limiting distributions
- Applications: Google's PageRank, communication networks, queues, reinforcement learning, ...

Continuous-time Markov chains

- Countable set of states $1, 2, \ldots$. Continuous-time index t, state $X(t)$
  - $\Rightarrow$ Transition between states can happen at any time
  - $\Rightarrow$ Markov: Future independent of the past given the present

- Probability of changing state in an infinitesimal time dt



- Of interest: Poisson processes, exponential distributions, transition probabilities, Kolmogorov equations, limit distributions
- Applications: Chemical reactions, queues, epidemic modeling, traffic engineering, weather forecasting, ...

Stationary random processes

- Continuous time t, continuous state $X(t)$, not necessarily Markov
- Prob. distribution of $X(t)$ constant or becomes constant as t grows

  $\Rightarrow$ System has a steady state in a random sense
- Of interest: Brownian motion, white noise, Gaussian processes, autocorrelation, power spectral density
- Applications: Black Scholes model for option pricing, radar, face recognition, noise in electric circuits, filtering and equalization, ...

Gambling

An interesting betting game

- There is a certain game in a certain casino in which ...

  $\Rightarrow$ Your chances of winning are $p > 1/2$
- You place \$1 bets
  (a) With probability p you gain \$1; and
  (b) With probability $1 - p$ you lose your \$1 bet
- The catch is that you either
  (a) Play until you go broke (lose all your money)
  (b) Keep playing forever
- You start with an initial wealth of $\$w_0$
- Q: Shall you play this game?

Modeling

- Let t be a time index (number of bets placed)
- Denote as $X(t)$ the outcome of the bet at time t
  - $\Rightarrow X(t) = 1$ if bet is won (w.p. p)
  - $\Rightarrow X(t) = 0$ if bet is lost (w.p. $1 - p$)
- $X(t)$ is called a Bernoulli random variable with parameter p
- Denote as $W(t)$ the player's wealth at time t. Initialize $W(0) = w_0$
- At times $t > 0$ wealth $W(t)$ depends on past wins and losses
  - $\Rightarrow$ When bet is won $W(t+1) = W(t) + 1$
  - $\Rightarrow$ When bet is lost $W(t+1) = W(t) - 1$
- More compactly can write $W(t+1) = W(t) + (2X(t) - 1)$
  - $\Rightarrow$ Only holds so long as $W(t) > 0$

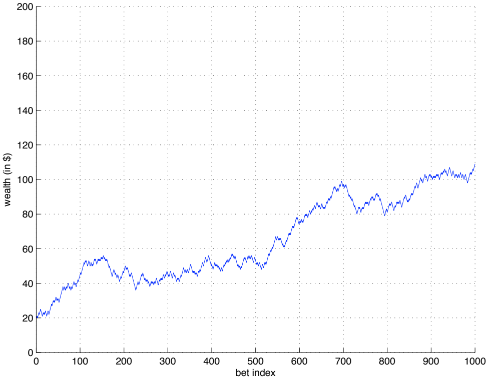
Coding

```
t = 0; w(t) = w_0; max_t = 10^3; // Initialize variables
% repeat while not broke up to time max_t
while (w(t) > 0) & (t < max_t) do
    x(t) = random('bino',1,p); % Draw Bernoulli random variable
    if x(t) == 1 then
        w(t+1) = w(t) + b; % If x = 1 wealth increases by b
    else
        w(t+1) = w(t) - b; % If x = 0 wealth decreases by b
    end
    t = t + 1;
end
```

- Initial wealth $w_0 = 20$, bet $b = 1$, win probability $p = 0.55$
- Q: Shall we play?

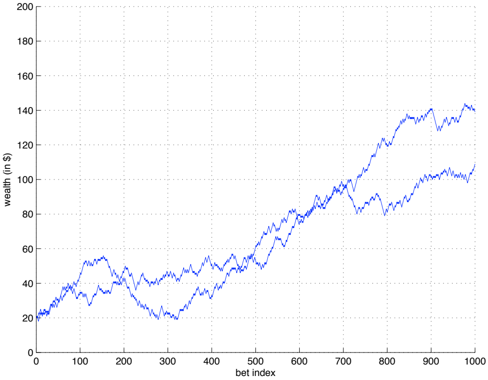
One lucky player

- She didn't go broke. After $t = 1000$ bets, her wealth is $W(t) = 109$
  - $\Rightarrow$ Less likely to go broke now because wealth increased



Two lucky players

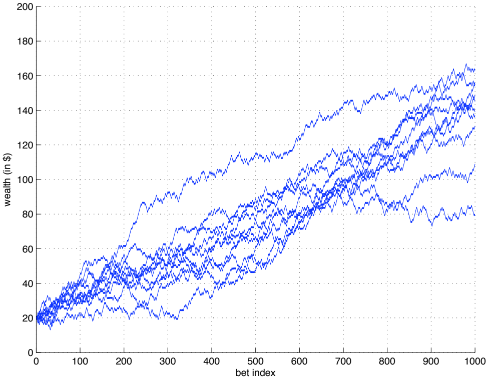
- After $t = 1000$ bets, wealths are $W_1(t) = 109$ and $W_2(t) = 139$
  - $\Rightarrow$ Increasing wealth seems to be a pattern



Ten lucky players

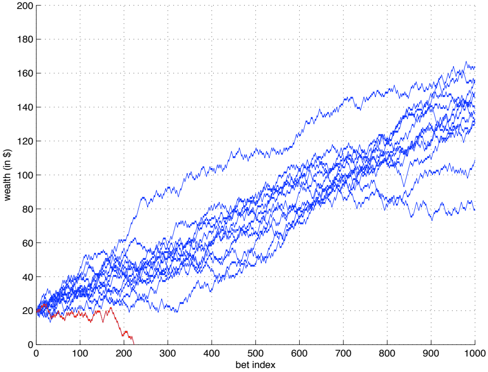
- Wealths $W_j(t)$ after $t = 1000$ bets between 78 and 139

  $\Rightarrow$ Increasing wealth is definitely a pattern



One unlucky player

- But this does not mean that all players will turn out as winners
  - $\Rightarrow$ The twelfth player $j = 12$ goes broke



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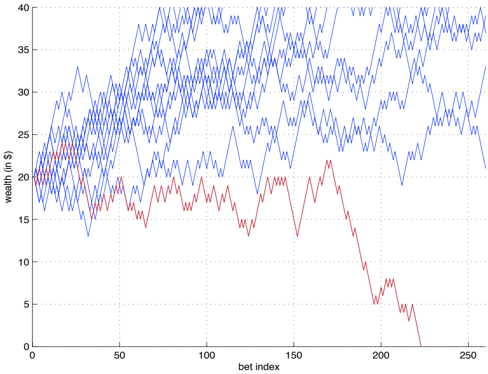
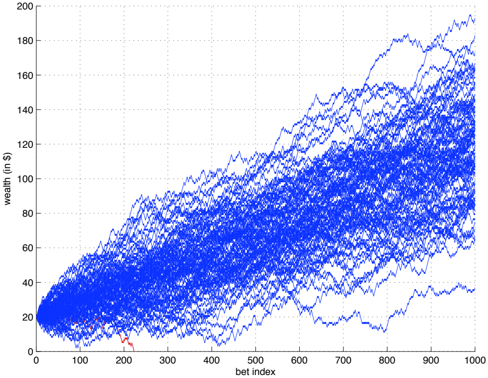


One hundred players

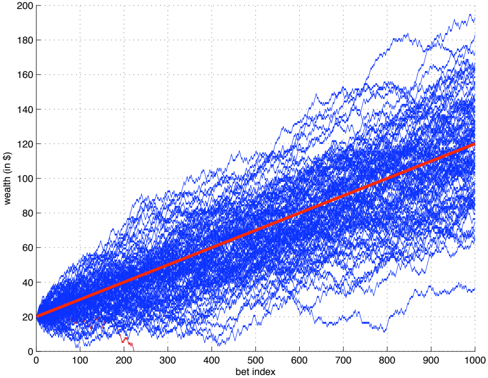
- All players (except for $j = 12$) end up with substantially more money



Average tendency

- It is not difficult to find a line estimating the average of $W(t)$

  $\Rightarrow \bar{w}(t) \approx w_0 + (2p-1)t \approx w_0 + 0.1t$ (recall $p = 0.55$)



Where does the average tendency come from?

- Assuming we do not go broke, we can write

$$W(t+1) = W(t) + \Big(2X(t) - 1\Big), \quad t = 0, 1, 2, \ldots$$

  - The assumption is incorrect as we saw, but suffices for simplicity
- Taking expectations on both sides and using linearity of expectation

$$\mathbb{E}\left[W(t+1)\right] = \mathbb{E}\left[W(t)\right] + \Big(2\mathbb{E}\left[X(t)\right] - 1\Big)$$

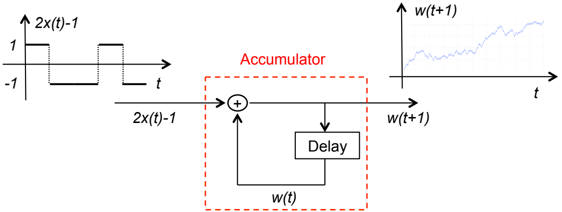
- The expected value of Bernoulli $X(t)$ is

$$\mathbb{E}\left[X(t)\right] \;=\; 1 \times \mathrm{P}\left(X(t) = 1\right) + 0 \times \mathrm{P}\left(X(t) = 0\right) \;=\; p$$

- Which yields $\Rightarrow \mathbb{E}\left[W(t+1)\right] = \mathbb{E}\left[W(t)\right] + (2p - 1)$
- Applying recursively $\Rightarrow \mathbb{E}\left[W(t+1)\right] = w_0 + (2p-1)(t+1)$

Gambling as LTI system with stochastic input

- Recall the evolution of wealth $W(t+1) = W(t) + \Big(2X(t) - 1\Big)$



- View $W(t+1)$ as output of LTI system with random input $2X(t) - 1$
- Recognize accumulator $\Rightarrow W(t+1) = w_0 + \sum_{\tau=0}^{t} \Big(2X(\tau) - 1\Big)$
  - Useful, a lot we can say about sums of random variables
- Filtering random processes in signal processing, communications, ...

Numerical analysis of simulation outcomes

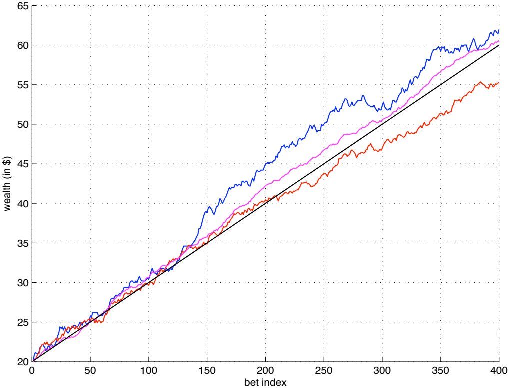
- For a more accurate approximation analyze simulation outcomes
- Consider J experiments. Each yields a wealth history $W_j(t)$
- Can estimate the average outcome via the sample average $\bar{W}_J(t)$

$$\bar{W}_J(t) := \frac{1}{J}\sum_{j=1}^{J} W_j(t)$$

- Do not confuse $\bar{W}_J(t)$ with $\mathbb{E}\left[W(t)\right]$
  - $\bar{W}_J(t)$ is computed from experiments, it is a random quantity in itself
  - $\mathbb{E}\left[W(t)\right]$ is a property of the random variable $W(t)$
  - We will see later that for large J, $\bar{W}_J(t) \to \mathbb{E}\left[W(t)\right]$

Analysis of simulation outcomes: mean

- Expected value $\mathbb{E}[W(t)]$ in black
- Sample average for $J = 10$ (blue), $J = 20$ (red), and $J = 100$ (magenta)



Analysis of simulation outcomes: distribution

- There is more information in the simulation's output
- Estimate the distribution function of $W(t)$ $\Rightarrow$ Histogram
- Consider a grid of points $w^{(0)}, \ldots, w^{(M)}$
- Indicator function of the event $w^{(m)} \leq W_j(t) < w^{(m+1)}$

$$\mathbb{I}\left\{w^{(m)} \leq W_j(t) < w^{(m+1)}\right\} = \begin{cases} 1, & \text{if } w^{(m)} \leq W_j(t) < w^{(m+1)} \\ 0, & \text{otherwise} \end{cases}$$

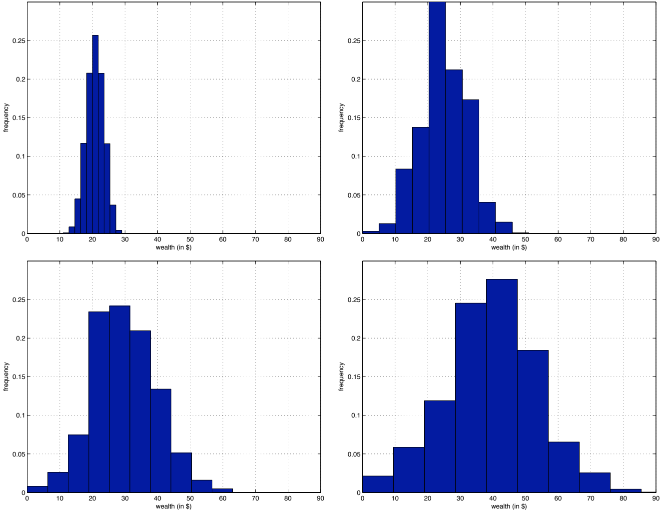
- Histogram is then defined as

$$H\left[t; w^{(m)}, w^{(m+1)}\right] = \frac{1}{J} \sum_{j=1}^{J} \mathbb{I}\left\{w^{(m)} \leq W_j(t) < w^{(m+1)}\right\}$$

- Fraction of experiments with wealth $W_j(t)$ between $w^{(m)}$ and $w^{(m+1)}$

Histogram

- Distribution broadens and shifts to the right ($t = 10, 50, 100, 200$)



What is this class about?

- Analysis and simulation of stochastic systems

  $\Rightarrow$ A system that evolves in time with some randomness

- They are usually quite complex $\Rightarrow$ Simulations

- We will learn how to model stochastic systems, e.g.,
  - $X(t)$ Bernoulli with parameter p
  - $W(t+1) = W(t) + 1$, when $X(t) = 1$
  - $W(t+1) = W(t) - 1$, when $X(t) = 0$

- ... how to analyze their properties, e.g., $\mathbb{E}\left[W(t)\right] = w_0 + (2p-1)t$

- ... and how to interpret simulations and experiments, e.g.,
  - Average tendency through sample average
  - Estimate probability distributions via histograms