Probability Review

Gonzalo Mateos
Dept. of ECE and Goergen Institute for Data Science
University of Rochester
gmateosb@ece.rochester.edu
http://www.ece.rochester.edu/~gmateosb/

August 28, 2024

Sigma-algebras and probability spaces

Probability

- An event is something that happens
- A random event has an uncertain outcome
  - $\Rightarrow$ The probability of an event measures how likely it is to occur

Example

- I've written a student's name in a piece of paper. Who is she/he?
- Event: Student x's name is written in the paper
- Probability: $P(x)$ measures how likely it is that x's name was written

- Probability is a measurement tool
  - $\Rightarrow$ Mathematical language for quantifying uncertainty

Sigma-algebra

- Given a sample space or universe S
  - Ex: All students in the class $S = \{x_1, x_2, \ldots, x_N\}$ (x_n denote names)
- **Def:** An outcome is an element or point in S, e.g., x_3
- **Def:** An event E is a subset of S
  - Ex: $\{x_1\}$, student with name x_1
  - Ex: Also $\{x_1, x_4\}$, students with names x_1 and x_4

  $\Rightarrow$ Outcome x_3 and event $\{x_3\}$ are different, the latter is a set
- **Def:** A sigma-algebra $\mathcal{F}$ is a collection of events $E \subseteq S$ such that
  - (i) The empty set $\emptyset$ belongs to $\mathcal{F}$: $\emptyset \in \mathcal{F}$
  - (ii) Closed under complement: If $E \in \mathcal{F}$, then $E^c \in \mathcal{F}$
  - (iii) Closed under countable unions: If $E_1, E_2, \ldots \in \mathcal{F}$, then $\cup_{i=1}^{\infty} E_i \in \mathcal{F}$
- $\mathcal{F}$ is a set of sets

Examples of sigma-algebras

Example

- No student and all students, i.e., $\mathcal{F}_0 := \{\emptyset, S\}$

Example

- Empty set, women, men, everyone, i.e., $\mathcal{F}_1 := \{\emptyset, \text{Women}, \text{Men}, S\}$

Example

- $\mathcal{F}_2$ including the empty set $\emptyset$ plus

  All events (sets) with one student $\{x_1\}, \ldots, \{x_N\}$ plus

  All events with two students $\{x_1, x_2\}$, $\{x_1, x_3\}$, ..., $\{x_1, x_N\}$,
  $\{x_2, x_3\}$, ..., $\{x_2, x_N\}$,
  ...
  $\{x_{N-1}, x_N\}$ plus

  All events with three, four, ..., N students

  $\Rightarrow$ $\mathcal{F}_2$ is known as the power set of S, denoted 2^S

Axioms of probability

- Define a function $P(E)$ from a sigma-algebra $\mathcal{F}$ to the real numbers
- $P(E)$ qualifies as a probability if
  - A1) Non-negativity: $P(E) \geq 0$
  - A2) Probability of universe: $P(S) = 1$
  - A3) Additivity: Given sequence of disjoint events $E_1, E_2, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P\left(E_i\right)$$

$\Rightarrow$ Disjoint (mutually exclusive) events means $E_i \cap E_j = \emptyset$, $i \neq j$

$\Rightarrow$ Union of countably infinite many disjoint events

- Triplet $(S, \mathcal{F}, P(\cdot))$ is called a probability space

Consequences of the axioms

- Implications of the axioms A1)-A3)
  - $\Rightarrow$ Impossible event: $P(\emptyset) = 0$
  - $\Rightarrow$ Monotonicity: $E_1 \subset E_2 \;\Rightarrow P(E_1) \leq P(E_2)$
  - $\Rightarrow$ Range: $0 \leq P(E) \leq 1$
  - $\Rightarrow$ Complement: $P(E^c) = 1 - P(E)$
  - $\Rightarrow$ Finite disjoint union: For disjoint events $E_1, \ldots, E_N$

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P\left(E_i\right)$$

  - $\Rightarrow$ Inclusion-exclusion: For any events E_1 and E_2

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Probability example

- Let's construct a probability space for our running example
- Universe of all students in the class $S = \{x_1, x_2, \ldots, x_N\}$
- Sigma-algebra with all combinations of students, i.e., $\mathcal{F} = 2^S$
- Suppose names are equiprobable $\Rightarrow P(\{x_n\}) = 1/N$ for all n

  $\Rightarrow$ Have to specify probability for all $E \in \mathcal{F} \Rightarrow$ Define $P(E) = \frac{|E|}{|S|}$
- Q: Is this function a probability?

  $\Rightarrow$ A1): $P(E) = \frac{|E|}{|S|} \geq 0$ ✓ $\Rightarrow$ A2): $P(S) = \frac{|S|}{|S|} = 1$ ✓

  $\Rightarrow$ A3): $P\left(\bigcup_{i=1}^{N} E_i\right) = \frac{\left|\bigcup_{i=1}^{N} E_i\right|}{|S|} = \frac{\sum_{i=1}^{N} |E_i|}{|S|} = \sum_{i=1}^{N} P(E_i)$ ✓
- The $P(\cdot)$ just defined is called uniform probability distribution

Conditional probability, total probability, Bayes' rule

Sigma-algebras and probability spaces

Conditional probability, total probability, Bayes' rule

Independence

Random variables

Discrete random variables

Continuous random variables

Expected values

Joint probability distributions

Joint expectations

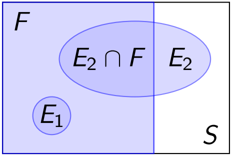
Conditional probability

- Consider events E and F, and suppose we know F occurred
- Q: What does this information imply about the probability of E?
- **Def:** Conditional probability of E given F is (need $P(F) > 0$)

$$P(E \,\big|\, F) = \frac{P(E \cap F)}{P(F)}$$

$\Rightarrow$ In general $P(E|F) \neq P(F|E)$

- Renormalize probabilities to the set F
  - Discard a piece of S
  - May discard a piece of E as well



- For given F with $P(F) > 0$, $P(\cdot|F)$ satisfies the axioms of probability

Conditional probability example

- The name I wrote is male. What is the probability of name x_n?
- Assume male names are $F = \{x_1, \ldots, x_M\}$ $\Rightarrow P(F) = \frac{M}{N}$
- If name x_n is male, $x_n \in F$ and we have for event $E = \{x_n\}$

$$P(E \cap F) = P(\{x_n\}) = \frac{1}{N}$$

$\Rightarrow$ Conditional probability is as you would expect

$$P(E \,\big|\, F) = \frac{P(E \cap F)}{P(F)} = \frac{1/N}{M/N} = \frac{1}{M}$$

- If name is female $x_n \notin F$, then $P(E \cap F) = P(\emptyset) = 0$

  $\Rightarrow$ As you would expect, then $P(E \,\big|\, F) = 0$

Law of total probability

- Consider event E and events F and F^c
  - F and F^c form a partition of the space S ($F \cup F^c = S$, $F \cap F^c = \emptyset$)
- Because $F \cup F^c = S$ cover space S, can write the set E as

$$E = E \cap S = E \cap [F \cup F^c] = [E \cap F] \cup [E \cap F^c]$$

- Because $F \cap F^c = \emptyset$ are disjoint, so is $[E \cap F] \cap [E \cap F^c] = \emptyset$

  $\Rightarrow P(E) = P([E \cap F] \cup [E \cap F^c]) = P(E \cap F) + P(E \cap F^c)$
- Use definition of conditional probability

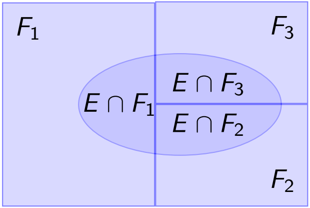
$$P(E) = P(E \mid F)P(F) + P(E \mid F^c)P(F^c)$$

- Translate conditional information $P(E \mid F)$ and $P(E \mid F^c)$

  $\Rightarrow$ Into unconditional information $P(E)$

Law of total probability (continued)

- In general, consider (possibly infinite) partition F_i, $i = 1, 2, \ldots$ of S
- Sets are disjoint $\Rightarrow F_i \cap F_j = \emptyset$ for $i \neq j$
- Sets cover the space $\Rightarrow \cup_{i=1}^{\infty} F_i = S$



- As before, because $\cup_{i=1}^{\infty} F_i = S$ cover the space, can write set E as

$$E = E \cap S = E \cap \left[\bigcup_{i=1}^{\infty} F_i\right] = \bigcup_{i=1}^{\infty} [E \cap F_i]$$

- Because $F_i \cap F_j = \emptyset$ are disjoint, so is $[E \cap F_i] \cap [E \cap F_j] = \emptyset$. Thus

$$P(E) = P\left(\bigcup_{i=1}^{\infty} [E \cap F_i]\right) = \sum_{i=1}^{\infty} P(E \cap F_i) = \sum_{i=1}^{\infty} P(E \,|\, F_i) P(F_i)$$

Total probability example

- Consider a probability class in some university
  - $\Rightarrow$ Seniors get an A with probability (w.p.) 0.9, juniors w.p. 0.8
  - $\Rightarrow$ An exchange student is a senior w.p. 0.7, and a junior w.p. 0.3
- Q: What is the probability of the exchange student scoring an A?
- Let $A =$ "exchange student gets an A," S denote senior, and J junior
  - $\Rightarrow$ Use the law of total probability

$$P(A) = P(A \,\big|\, S)P(S) + P(A \,\big|\, J)P(J)$$

$$= 0.9 \times 0.7 + 0.8 \times 0.3 = 0.87$$

Bayes' rule

- From the definition of conditional probability

$$P(E \mid F)P(F) = P(E \cap F)$$

- Likewise, for F conditioned on E we have

$$P(F \mid E)P(E) = P(F \cap E)$$

- Quantities above are equal, giving Bayes' rule

$$P(E \mid F) = \frac{P(F \mid E)P(E)}{P(F)}$$

- Bayes' rule allows time reversion. If F (future) comes after E (past),
  - $\Rightarrow P(E \mid F)$, probability of past (E) having seen the future (F)
  - $\Rightarrow P(F \mid E)$, probability of future (F) having seen past (E)
- Models often describe future | past. Interest is often in past | future

Bayes' rule example

- Consider the following partition of my email
  - $\Rightarrow E_1 =$ "spam" w.p. $P(E_1) = 0.7$
  - $\Rightarrow E_2 =$ "low priority" w.p. $P(E_2) = 0.2$
  - $\Rightarrow E_3 =$ "high priority" w.p. $P(E_3) = 0.1$
- Let $F =$ "an email contains the word *free*"
  - $\Rightarrow$ From experience know $P(F \mid E_1) = 0.9$, $P(F \mid E_2) = P(F \mid E_3) = 0.01$
- I got an email containing "free". What is the probability that it is spam?
- Apply Bayes' rule

$$P(E_1 \mid F) = \frac{P(F \mid E_1)P(E_1)}{P(F)} = \frac{P(F \mid E_1)P(E_1)}{\sum_{i=1}^{3} P(F \mid E_i)P(E_i)} = 0.995$$

$\Rightarrow$ Law of total probability very useful when applying Bayes' rule

Independence

Sigma-algebras and probability spaces

Conditional probability, total probability, Bayes' rule

Independence

Random variables

Discrete random variables

Continuous random variables

Expected values

Joint probability distributions

Joint expectations

Independence

- **Def:** Events E and F are independent if $P(E \cap F) = P(E)P(F)$

  $\Rightarrow$ Events that are not independent are dependent

- According to definition of conditional probability

$$P(E \,|\, F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)P(F)}{P(F)} = P(E)$$

  $\Rightarrow$ Intuitive, knowing F does not alter our perception of E

  $\Rightarrow$ F bears no information about E

  $\Rightarrow$ The symmetric is also true $P(F \,|\, E) = P(F)$

- Whether E and F are independent relies strongly on $P(\cdot)$
- Avoid confusing with disjoint events, meaning $E \cap F = \emptyset$
- Q: Can disjoint events with $P(E) > 0$, $P(F) > 0$ be independent? No

Independence example

- Wrote one name, asked a friend to write another (possibly the same)
- Probability space $(S, \mathcal{F}, P(\cdot))$ for this experiment
  - $\Rightarrow$ S is the set of all pairs of names $[x_n(1), x_n(2)]$, $|S| = N^2$
  - $\Rightarrow$ Sigma-algebra is (cartesian product) power set $\mathcal{F} = 2^S$
  - $\Rightarrow$ Define $P(E) = \frac{|E|}{|S|}$ as the uniform probability distribution
- Consider the events $E_1 =$'I wrote x_1' and $E_2 =$'My friend wrote x_2'
  Q: Are they independent? Yes, since

$$P(E_1 \cap E_2) = P\big(\{(x_1, x_2)\}\big) = \frac{|\{(x_1, x_2)\}|}{|S|} = \frac{1}{N^2} = P(E_1)P(E_2)$$

- Dependent events: $E_1 =$'I wrote x_1' and $E_3 =$'Both names are male'

Independence for more than two events

- **Def:** Events E_i, $i = 1, 2, \ldots$ are called mutually independent if

$$P\left(\bigcap_{i \in I} E_i\right) = \prod_{i \in I} P(E_i)$$

for every finite subset I of at least two integers

- Ex: Events E_1, E_2, and E_3 are mutually independent if all the following hold

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3)$$
$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$
$$P(E_1 \cap E_3) = P(E_1)P(E_3)$$
$$P(E_2 \cap E_3) = P(E_2)P(E_3)$$

- If $P(E_i \cap E_j) = P(E_i)P(E_j)$ for all (i, j), the E_i are pairwise independent

  $\Rightarrow$ Mutual independence $\rightarrow$ pairwise independence. Not the other way

Random variables

Sigma-algebras and probability spaces

Conditional probability, total probability, Bayes' rule

Independence

Random variables

Discrete random variables

Continuous random variables

Expected values

Joint probability distributions

Joint expectations

Random variable (RV) definition

- **Def:** RV $X(s)$ is a function that assigns a value to an outcome $s \in S$

  $\Rightarrow$ Think of RVs as measurements associated with an experiment

Example

- Throw a ball inside a $1m \times 1m$ square. Interested in ball position
- Uncertain outcome is the place $s \in [0,1]^2$ where the ball falls
- Random variables are $X(s)$ and $Y(s)$ position coordinates

- RV probabilities inferred from probabilities of underlying outcomes

$$P(X(s) = x) = P(\{s \in S : X(s) = x\})$$

$$P(X(s) \in (-\infty, x]) = P(\{s \in S : X(s) \in (-\infty, x]\})$$

- $X(s)$ is the random variable and x a particular value of $X(s)$

Example 1

- Throw coin for head (H) or tails (T). Coin is fair $P(H) = 1/2$, $P(T) = 1/2$. Pay \$1 for H, charge \$1 for T. Earnings?
- Possible outcomes are H and T
- To measure earnings define RV X with values

$$X(H) = 1, \qquad X(T) = -1$$

- Probabilities of the RV are

$$\begin{aligned} P(X = 1) &= P(H) = 1/2, \\ P(X = -1) &= P(T) = 1/2 \end{aligned}$$

$\Rightarrow$ Also have $P(X = x) = 0$ for all other $x \neq \pm 1$

Example 2

- Throw 2 coins. Pay \$1 for each H, charge \$1 for each T. Earnings?
- Now the possible outcomes are HH, HT, TH, and TT
- To measure earnings define RV Y with values

$$Y(HH) = 2, \quad Y(HT) = 0, \quad Y(TH) = 0, \quad Y(TT) = -2$$

- Probabilities of the RV are

$$\begin{aligned} P(Y=2) &= P(HH) &&= 1/4, \\ P(Y=0) &= P(HT) + P(TH) &&= 1/2, \\ P(Y=-2) &= P(TT) &&= 1/4 \end{aligned}$$

About Examples 1 and 2

- RVs are easier to manipulate than events
- Let $s_1 \in \{H, T\}$ be outcome of coin 1 and $s_2 \in \{H, T\}$ of coin 2

  $\Rightarrow$ Can relate Y and Xs as

$$Y(s_1, s_2) = X_1(s_1) + X_2(s_2)$$

- Throw N coins. Earnings? Enumeration becomes cumbersome
- Alternatively, let $s_n \in \{H, T\}$ be outcome of n-th toss and define

$$Y(s_1, s_2, \ldots, s_N) = \sum_{n=1}^{N} X_n(s_n)$$

$\Rightarrow$ Will usually abuse notation and write $Y = \sum_{n=1}^{N} X_n$

Example 3

- Throw a coin until landing heads for the first time. $P(H) = p$
- Number of throws until the first head?
- Outcomes are H, TH, TTH, $TTTH$, $\ldots$ Note that $|S| = \infty$

  $\Rightarrow$ Stop tossing after first H (thus THT not a possible outcome)
- Let N be a RV counting the number of throws

  $\Rightarrow$ $N = n$ if we land T in the first $n-1$ throws and H in the n-th

$$
\begin{aligned}
P(N=1) &= P(H) &&= p \\
P(N=2) &= P(TH) &&= (1-p)p \\
&\vdots \\
P(N=n) &= P(\underbrace{TT\ldots T}_{n-1 \text{ tails}}H) &&= (1-p)^{n-1}p
\end{aligned}
$$

Example 3 (continued)

- From A2) we should have $P(S) = \sum_{n=1}^{\infty} P(N = n) = 1$
- Holds because $\sum_{n=1}^{\infty}(1-p)^{n-1}$ is a geometric series

$$\sum_{n=1}^{\infty}(1-p)^{n-1} = 1 + (1-p) + (1-p)^2 + \ldots = \frac{1}{1-(1-p)} = \frac{1}{p}$$

- Plug the sum of the geometric series in the expression for $P(S)$

$$\sum_{n=1}^{\infty} P(N = n) = p\sum_{n=1}^{\infty}(1-p)^{n-1} = p \times \frac{1}{p} = 1 \checkmark$$

Indicator function

- The indicator function of an event is a random variable
- Let $s \in S$ be an outcome, and $E \subset S$ be an event

$$\mathbb{I}\{E\}(s) = \begin{cases} 1, & \text{if } s \in E \\ 0, & \text{if } s \notin E \end{cases}$$

$\Rightarrow$ Indicates that outcome s belongs to set E, by taking value 1

Example

- Number of throws N until first H. Interested on N exceeding N_0
  - $\Rightarrow$ Event is $\{N : N > N_0\}$. Possible outcomes are $N = 1, 2, \ldots$
  - $\Rightarrow$ Denote indicator function as $\mathbb{I}_{N_0} = \mathbb{I}\{N : N > N_0\}$
- Probability $P(\mathbb{I}_{N_0} = 1) = P(N > N_0) = (1-p)^{N_0}$
  - $\Rightarrow$ For N to exceed N_0 need N_0 consecutive tails
  - $\Rightarrow$ Doesn't matter what happens afterwards

Discrete random variables

Sigma-algebras and probability spaces

Conditional probability, total probability, Bayes' rule

Independence

Random variables

Discrete random variables

Continuous random variables

Expected values

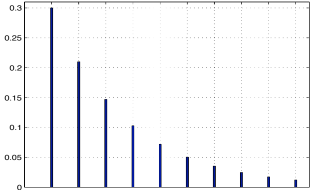
Joint probability distributions

Joint expectations

Probability mass and cumulative distribution functions

- Discrete RV takes on, at most, a countable number of values
- Probability mass function (pmf) $p_X(x) = P(X = x)$
  - If RV is clear from context, just write $p_X(x) = p(x)$

- If X supported in $\{x_1, x_2, \ldots\}$, pmf satisfies
  - (i) $p(x_i) > 0$ for $i = 1, 2, \ldots$
  - (ii) $p(x) = 0$ for all other $x \neq x_i$
  - (iii) $\sum_{i=1}^{\infty} p(x_i) = 1$
  - Pmf for "throw to first heads" $(p = 0.3)$

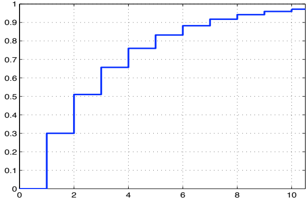


- Cumulative distribution function (cdf)

$$F_X(x) = P(X \leq x) = \sum_{i: x_i \leq x} p(x_i)$$

$\Rightarrow$ Staircase function with jumps at x_i

  - Cdf for "throw to first heads" $(p = 0.3)$

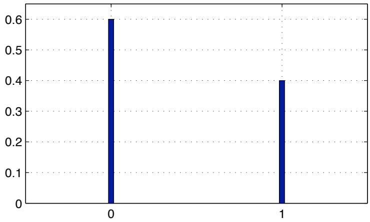


Bernoulli

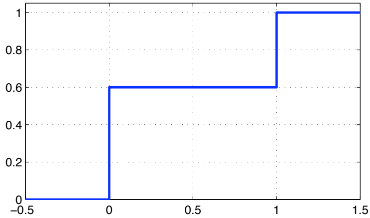
- A trial/experiment/bet can succeed w.p. p or fail w.p. $q := 1 - p$
  - $\Rightarrow$ Ex: coin throws, any indication of an event
- Bernoulli X can be 0 or 1. Pmf is $p(x) = p^x q^{1-x}$
- Cdf is

$$F(x) = \begin{cases} 0, & x < 0 \\ q, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

pmf ($p = 0.4$)



cdf ($p = 0.4$)



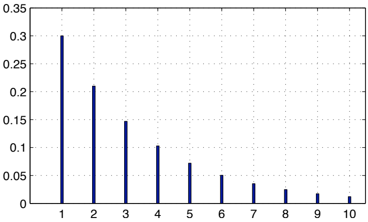
Geometric

- Count number of Bernoulli trials needed to register first success

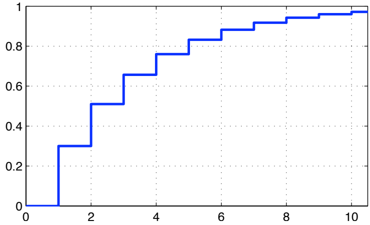
  $\Rightarrow$ Trials succeed w.p. p and are independent

- Number of trials X until success is geometric with parameter p
- Pmf is $p(x) = p(1-p)^{x-1}$
  - One success after $x-1$ failures, trials are independent
- Cdf is $F(x) = 1-(1-p)^x$
  - Recall $\mathrm{P}(X > x) = (1-p)^x$; or just sum the geometric series

pmf ($p = 0.3$)



cdf ($p = 0.3$)



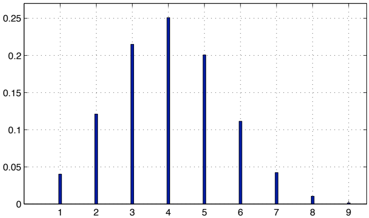
Binomial

- Count number of successes X in n Bernoulli trials
  - $\Rightarrow$ Trials succeed w.p. p and are independent
- Number of successes X is binomial with parameters (n, p). Pmf is

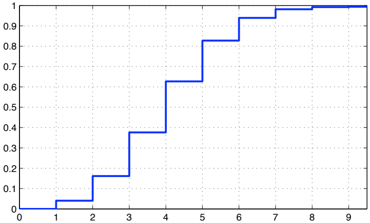
$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

  - $\Rightarrow$ $X = x$ for x successes (p^x) and $n - x$ failures $((1-p)^{n-x})$.
  - $\Rightarrow$ $\binom{n}{x}$ ways of drawing x successes and $n - x$ failures

pmf $(n = 9,\ p = 0.4)$



cdf $(n = 9,\ p = 0.4)$



Binomial (continued)

- Let Y_i, $i = 1, \ldots n$ be Bernoulli RVs with parameter p
  $\Rightarrow$ Y_i associated with independent events
- Can write binomial X with parameters (n, p) as $\Rightarrow X = \sum_{i=1}^{n} Y_i$

Example

- Consider binomials Y and Z with parameters (n_Y, p) and (n_Z, p)
  $\Rightarrow$ Q: Probability distribution of $X = Y + Z$?
- Write $Y = \sum_{i=1}^{n_Y} Y_i$ and $Z = \sum_{i=1}^{n_Z} Z_i$, thus

$$X = \sum_{i=1}^{n_Y} Y_i + \sum_{i=1}^{n_Z} Z_i$$

$\Rightarrow$ X is binomial with parameter $(n_Y + n_Z, p)$

Poisson

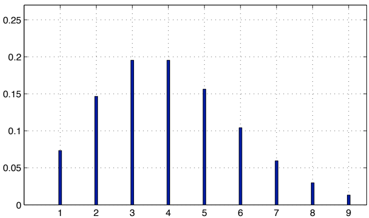
- Counts of rare events (radioactive decay, packet arrivals, accidents)
- Usually modeled as Poisson with parameter λ and pmf

$$p(x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

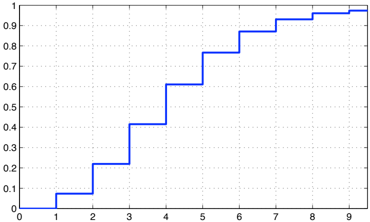
- Q: Is this a properly defined pmf? Yes
- Taylor's expansion of $e^x = 1 + x + x^2/2 + \ldots + x^i/i! + \ldots$. Then

$$P(S) = \sum_{i=0}^{\infty} p(i) = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1 \checkmark$$

pmf ($\lambda = 4$)

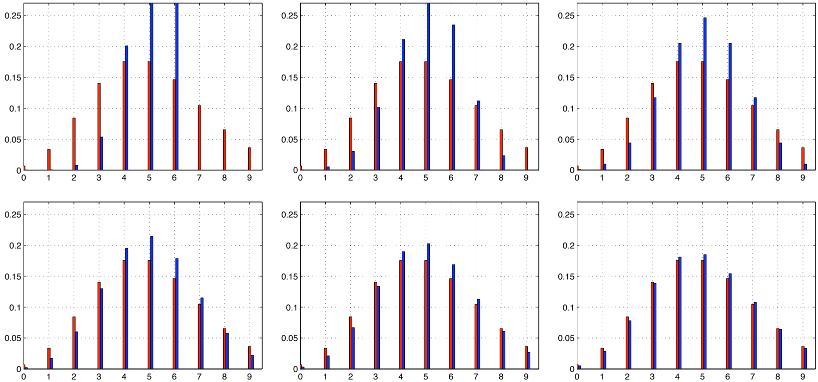


cdf ($\lambda = 4$)



Poisson approximation of binomial

- X is binomial with parameters (n, p)
- Let $n \to \infty$ while maintaining a constant product $np = \lambda$
  - If we just let $n \to \infty$ number of successes diverges. Boring
- Compare with Poisson distribution with parameter λ
  - $\lambda = 5, \; n = 6, 8, 10, 15, 20, 50$



Poisson and binomial (continued)

- This is, in fact, the motivation for the definition of a Poisson RV
- Substituting $p = \lambda/n$ in the pmf of a binomial RV

$$p_n(x) = \frac{n!}{(n-x)!x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$
$$= \frac{n(n-1)\ldots(n-x+1)}{n^x}\frac{\lambda^x}{x!}\frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$$

$\Rightarrow$ Used factorials' defs., $(1-\lambda/n)^{n-x} = \frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$, and reordered terms

- In the limit, red term is $\lim_{n\to\infty}(1-\lambda/n)^n = e^{-\lambda}$
- Black and blue terms converge to 1. From both observations

$$\lim_{n\to\infty} p_n(x) = 1\frac{\lambda^x}{x!}\frac{e^{-\lambda}}{1} = e^{-\lambda}\frac{\lambda^x}{x!}$$

$\Rightarrow$ Limit is the pmf of a Poisson RV

Closing remarks

- Binomial distribution is motivated by counting successes
- The Poisson is an approximation for large number of trials n
  - $\Rightarrow$ Poisson distribution is more tractable (compare pmfs)
- Sometimes called "law of rare events"
  - Individual events (successes) happen with small probability $p = \lambda/n$
  - Aggregate event (number of successes), though, need not be rare
- Notice that all four RVs seen so far are related to "coin tosses"

Where did the probability space go?

- Random variables are mappings $X(s) : S \mapsto \mathbb{R}$
  - $\Rightarrow$ The underlying probability space often "disappears"
  - $\Rightarrow$ This is for notational convenience, but it's still there

Example

- Let's construct a probability space for a Bernoulli RV
  - Let $S = [0, 1]$, $\mathcal{F}$ the Borel sigma-field and $\mathrm{P}([a, b]) = b - a$, $a \leq b$
- Fix a parameter $p \in [0, 1]$ and define

$$X(s) = \begin{cases} 1, & s \leq p, \\ 0, & s > p. \end{cases}$$

$\Rightarrow \mathrm{P}(X = 1) = \mathrm{P}(s \leq p) = \mathrm{P}([0, p]) = p$ and $\mathrm{P}(X = 0) = 1 - p$

- Can do a similar construction for all distributions consider so far

Continuous random variables

Sigma-algebras and probability spaces

Conditional probability, total probability, Bayes' rule

Independence

Random variables

Discrete random variables

Continuous random variables

Expected values

Joint probability distributions

Joint expectations

Continuous RVs, probability density function

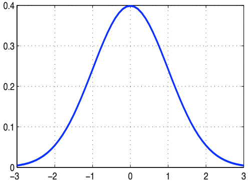
- Possible values for continuous RV X form a dense subset $\mathcal{X} \subseteq \mathbb{R}$

  $\Rightarrow$ Uncountably infinite number of possible values

- Probability density function (pdf) $f_X(x) \geq 0$ is such that for any subset $\mathcal{X} \subseteq \mathbb{R}$ (Normal pdf to the right)

$$P(X \in \mathcal{X}) = \int_{\mathcal{X}} f_X(x)dx$$

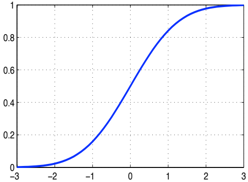
  $\Rightarrow$ Will have $P(X = x) = 0$ for all $x \in \mathcal{X}$



- Cdf defined as before and related to the pdf (Normal cdf to the right)

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)\, du$$

$$\Rightarrow P(X \leq \infty) = F_X(\infty) = \lim_{x \to \infty} F_X(x) = 1$$



More on cdfs and pdfs

- When the set $\mathcal{X} = [a, b]$ is an interval of $\mathbb{R}$

$$\mathrm{P}\left(X \in [a, b]\right) = \mathrm{P}\left(X \leq b\right) - \mathrm{P}\left(X \leq a\right) = F_X(b) - F_X(a)$$

- In terms of the pdf it can be written as

$$\mathrm{P}\left(X \in [a, b]\right) = \int_a^b f_X(x)\, dx$$

- For small interval $[x_0, x_0 + \delta x]$, in particular

$$\mathrm{P}\left(X \in [x_0, x_0 + \delta x]\right) = \int_{x_0}^{x_0+\delta x} f_X(x)\, dx \approx f_X(x_0)\delta x$$

$\Rightarrow$ Probability is the "area under the pdf" (thus "density")

- Another relationship between pdf and cdf is $\Rightarrow \dfrac{\partial F_X(x)}{\partial x} = f_X(x)$

$\Rightarrow$ Fundamental theorem of calculus ("derivative inverse of integral")

Uniform

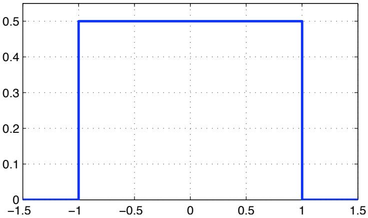
- Model problems with equal probability of landing on an interval $[a, b]$
- Pdf of uniform RV is $f(x) = 0$ outside the interval $[a, b]$ and

$$f(x) = \frac{1}{b-a}, \quad \text{for } a \leq x \leq b$$

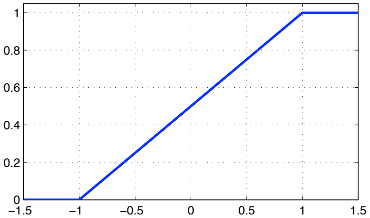
- Cdf is $F(x) = (x-a)/(b-a)$ in the interval $[a, b]$ (0 before, 1 after)
- Prob. of interval $[\alpha, \beta] \subseteq [a, b]$ is $\int_\alpha^\beta f(x)dx = (\beta - \alpha)/(b-a)$

  $\Rightarrow$ Depends on interval's width $\beta - \alpha$ only, not on its position

pdf $(a = -1,\ b = 1)$



cdf $(a = -1,\ b = 1)$



Exponential

- Model duration of phone calls, lifetime of electronic components
- Pdf of exponential RV is

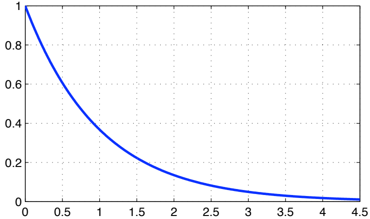
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$\Rightarrow$ As parameter λ increases, "height" increases and "width" decreases

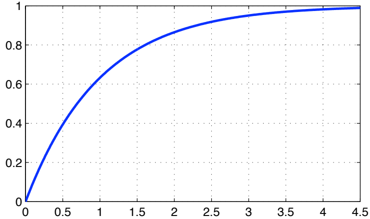
- Cdf obtained by integrating pdf

$$F(x) = \int_{-\infty}^{x} f(u)\, du = \int_{0}^{x} \lambda e^{-\lambda u}\, du = -e^{-\lambda u}\Big|_{0}^{x} = 1 - e^{-\lambda x}$$

pdf ($\lambda = 1$)



cdf ($\lambda = 1$)



Normal / Gaussian

- Model randomness arising from large number of random effects
- Pdf of normal RV is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

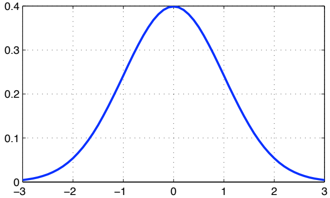
$\Rightarrow$ μ is the mean (center), σ^2 is the variance (width)

$\Rightarrow$ 0.68 prob. between $\mu \pm \sigma$, 0.997 prob. in $\mu \pm 3\sigma$

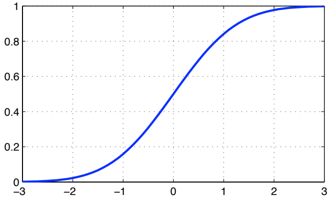
$\Rightarrow$ Standard normal RV has $\mu = 0$ and $\sigma^2 = 1$

- Cdf $F(x)$ cannot be expressed in terms of elementary functions

pdf ($\mu = 0$, $\sigma^2 = 1$)



cdf ($\mu = 0$, $\sigma^2 = 1$)



Expected values

Sigma-algebras and probability spaces

Conditional probability, total probability, Bayes' rule

Independence

Random variables

Discrete random variables

Continuous random variables

Expected values

Joint probability distributions

Joint expectations

Expected values

- We are asked to summarize information about a RV in a single value
  - $\Rightarrow$ What should this value be?
- If we are allowed a description with a few values
  - $\Rightarrow$ What should they be?
- Expected (mean) values are convenient answers to these questions
- **Beware:** Expectations are condensed descriptions
  - $\Rightarrow$ They overlook some aspects of the random phenomenon
  - $\Rightarrow$ Whole story told by the probability distribution (cdf)

Definition for discrete RVs

- Discrete RV X taking on values x_i, $i = 1, 2, \ldots$ with pmf $p(x)$
- **Def:** The expected value of the discrete RV X is

$$\mathbb{E}[X] := \sum_{i=1}^{\infty} x_i p(x_i) = \sum_{x:p(x)>0} x p(x)$$

- Weighted average of possible values x_i. Probabilities are weights
- Common average if RV takes values x_i, $i = 1, \ldots, N$ equiprobably

$$\mathbb{E}[X] = \sum_{i=1}^{N} x_i p(x_i) = \sum_{i=1}^{N} x_i \frac{1}{N} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Expected value of Bernoulli and geometric RVs

Ex: For a Bernoulli RV $p(x) = p^x q^{1-x}$, for $x \in \{0, 1\}$

$$\mathbb{E}[X] = 1 \times p + 0 \times q = p$$

Ex: For a geometric RV $p(x) = p(1-p)^{x-1} = pq^{x-1}$, for $x \geq 1$

- Note that $\partial q^x / \partial q = xq^{x-1}$ and that derivatives are linear operators

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} xpq^{x-1} = p \sum_{x=1}^{\infty} \frac{\partial q^x}{\partial q} = p \frac{\partial}{\partial q} \left(\sum_{x=1}^{\infty} q^x \right)$$

- Sum inside derivative is geometric. Sums to $q/(1-q)$, thus

$$\mathbb{E}[X] = p \frac{\partial}{\partial q} \left(\frac{q}{1-q} \right) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

- Time to first success is inverse of success probability. Reasonable

Expected value of Poisson RV

Ex: For a Poisson RV $p(x) = e^{-\lambda}(\lambda^x/x!)$, for $x \geq 0$

- First summand in definition is 0, pull λ out, and use $\frac{x}{x!} = \frac{1}{(x-1)!}$

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x e^{-\lambda}\frac{\lambda^x}{x!} = \lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

- Sum is Taylor's expansion of $e^{\lambda} = 1 + \lambda + \lambda^2/2! + \ldots + \lambda^x/x!$

$$\mathbb{E}[X] = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

- Poisson is limit of binomial for large number of trials n, with $\lambda = np$

  $\Rightarrow$ Counts number of successes in n trials that succeed w.p. p

- Expected number of successes is $\lambda = np$

  $\Rightarrow$ Number of trials $\times$ probability of individual success. Reasonable

Definition for continuous RVs

- Continuous RV X taking values on $\mathbb{R}$ with pdf $f(x)$
- **Def:** The expected value of the continuous RV X is

$$\mathbb{E}[X] := \int_{-\infty}^{\infty} x f(x)\, dx$$

- Compare with $\mathbb{E}[X] := \sum_{x:p(x)>0} xp(x)$ in the discrete RV case
- Note that the integral or sum are assumed to be well defined

  $\Rightarrow$ Otherwise we say the expectation does not exist

Expected value of normal RV

Ex: For a normal RV add and subtract μ, separate integrals

$$
\begin{aligned}
\mathbb{E}[X] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x+\mu-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\
&= \mu\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx + \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx
\end{aligned}
$$

- First integral is 1 because it integrates a pdf in all $\mathbb{R}$
- Second integral is 0 by symmetry. Both observations yield

$$\mathbb{E}[X] = \mu$$

- The mean of a RV with a symmetric pdf is the point of symmetry

Expected value of uniform and exponential RVs

Ex: For a uniform RV $f(x) = 1/(b-a)$, for $a \leq x \leq b$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_a^b \frac{x}{b-a}\, dx = \frac{b^2 - a^2}{2(b-a)} = \frac{(a+b)}{2}$$

- Makes sense, since pdf is symmetric around midpoint $(a+b)/2$

Ex: For an exponential RV (non symmetric) integrate by parts

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^{\infty} x \lambda e^{-\lambda x}\, dx \\
&= -x e^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\, dx \\
&= -x e^{-\lambda x}\Big|_0^{\infty} - \frac{e^{-\lambda x}}{\lambda}\Big|_0^{\infty} = \frac{1}{\lambda}
\end{aligned}$$

Expected value of a function of a RV

- Consider a function $g(X)$ of a RV X. Expected value of $g(X)$?
- g(X) is also a RV, then it also has a pmf $p_{g(X)}(g(x))$

$$\mathbb{E}[g(X)] = \sum_{g(x):p_{g(X)}(g(x))>0} g(x)p_{g(X)}(g(x))$$

$\Rightarrow$ Requires calculating the pmf of $g(X)$. There is a simpler way

Theorem

Consider a function $g(X)$ of a discrete RV X with pmf $p_X(x)$. Then

$$\mathbb{E}[g(X)] = \sum_{i=1}^{\infty} g(x_i)p_X(x_i)$$

- Weighted average of functional values. No need to find pmf of $g(X)$
- Same can be proved for a continuous RV

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x)\,dx$$

Expected value of a linear transformation

- Consider a linear function (actually affine) $g(X) = aX + b$

$$\begin{aligned}\mathbb{E}[aX + b] &= \sum_{i=1}^{\infty}(ax_i + b)p_X(x_i) \\ &= \sum_{i=1}^{\infty} ax_i p_X(x_i) + \sum_{i=1}^{\infty} bp_X(x_i) \\ &= a\sum_{i=1}^{\infty} x_i p_X(x_i) + b\sum_{i=1}^{\infty} p_X(x_i) \\ &= a\mathbb{E}[X] + b1\end{aligned}$$

- Can interchange expectation with additive/multiplicative constants

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

$\Rightarrow$ Again, the same holds for a continuous RV

Expected value of an indicator function

- Let X be a RV and $\mathcal{X}$ be a set

$$\mathbb{I}\{X \in \mathcal{X}\} = \begin{cases} 1, & \text{if } x \in \mathcal{X} \\ 0, & \text{if } x \notin \mathcal{X} \end{cases}$$

- Expected value of $\mathbb{I}\{X \in \mathcal{X}\}$ in the discrete case

$$\mathbb{E}\left[\mathbb{I}\{X \in \mathcal{X}\}\right] = \sum_{x: p_X(x)>0} \mathbb{I}\{x \in \mathcal{X}\} p_X(x) = \sum_{x \in \mathcal{X}} p_X(x) = \mathrm{P}\left(X \in \mathcal{X}\right)$$

- Likewise in the continuous case

$$\mathbb{E}\left[\mathbb{I}\{X \in \mathcal{X}\}\right] = \int_{-\infty}^{\infty} \mathbb{I}\{x \in \mathcal{X}\} f_X(x) dx = \int_{x \in \mathcal{X}} f_X(x) dx = \mathrm{P}\left(X \in \mathcal{X}\right)$$

- Expected value of indicator RV = Probability of indicated event
  - $\Rightarrow$ Recall $\mathbb{E}[X] = p$ for Bernoulli RV (it "indicates success")

Moments, central moments and variance

- **Def:** The n-th moment $(n \geq 0)$ of a RV is

$$\mathbb{E}\left[X^n\right] = \sum_{i=1}^{\infty} x_i^n p(x_i)$$

- **Def:** The n-th central moment corrects for the mean, that is

$$\mathbb{E}\left[\left(X - \mathbb{E}\left[X\right]\right)^n\right] = \sum_{i=1}^{\infty} \left(x_i - \mathbb{E}\left[X\right]\right)^n p(x_i)$$

- 0-th order moment is $\mathbb{E}\left[X^0\right] = 1$; 1-st moment is the mean $\mathbb{E}\left[X\right]$
- 2-nd central moment is the variance. Measures width of the pmf

$$\text{var}\left[X\right] = \mathbb{E}\left[\left(X - \mathbb{E}\left[X\right]\right)^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X]$$

Ex: For affine functions

$$\text{var}\left[aX + b\right] = a^2 \text{var}\left[X\right]$$

Variance of Bernoulli and Poisson RVs

Ex: For a Bernoulli RV X with parameter p, $\mathbb{E}[X] = \mathbb{E}\left[X^2\right] = p$

$$\Rightarrow \text{var}[X] = \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X] = p - p^2 = p(1-p)$$

Ex: For Poisson RV Y with parameter λ, second moment is

$$\begin{aligned}
\mathbb{E}\left[Y^2\right] &= \sum_{y=0}^{\infty} y^2 e^{-\lambda}\frac{\lambda^y}{y!} = \sum_{y=1}^{\infty} y\frac{e^{-\lambda}\lambda^y}{(y-1)!} \\
&= \sum_{y=1}^{\infty}(y-1)\frac{e^{-\lambda}\lambda^y}{(y-1)!} + \sum_{y=1}^{\infty}\frac{e^{-\lambda}\lambda^y}{(y-1)!} \\
&= e^{-\lambda}\lambda^2\sum_{y=2}^{\infty}\frac{\lambda^{y-2}}{(y-2)!} + e^{-\lambda}\lambda\sum_{y=1}^{\infty}\frac{\lambda^{y-1}}{(y-1)!} \\
&= e^{-\lambda}\lambda^2 e^{\lambda} + e^{-\lambda}\lambda e^{\lambda} = \lambda^2 + \lambda
\end{aligned}$$

$$\Rightarrow \text{var}[Y] = \mathbb{E}\left[Y^2\right] - \mathbb{E}^2[Y] = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Joint probability distributions

Sigma-algebras and probability spaces

Conditional probability, total probability, Bayes' rule

Independence

Random variables

Discrete random variables

Continuous random variables

Expected values

Joint probability distributions

Joint expectations

Joint cdf

- Want to study problems with more than one RV. Say, e.g., X and Y
- Probability distributions of X and Y are not sufficient

  $\Rightarrow$ Joint probability distribution (cdf) of (X, Y) defined as

$$F_{XY}(x, y) = \mathrm{P}\left(X \leq x, Y \leq y\right)$$

- If X, Y clear from context omit subindex to write $F_{XY}(x, y) = F(x, y)$
- Can recover $F_X(x)$ by considering all possible values of Y

$$F_X(x) = \mathrm{P}\left(X \leq x\right) = \mathrm{P}\left(X \leq x, Y \leq \infty\right) = F_{XY}(x, \infty)$$

  $\Rightarrow$ $F_X(x)$ and $F_Y(y) = F_{XY}(\infty, y)$ are called marginal cdfs

Joint pmf

- Consider discrete RVs X and Y
  X takes values in $\mathcal{X} := \{x_1, x_2, \ldots\}$ and Y in $\mathcal{Y} := \{y_1, y_2, \ldots\}$
- Joint pmf of (X, Y) defined as

$$p_{XY}(x, y) = \mathrm{P}\left(X = x, Y = y\right)$$

- Possible values (x, y) are elements of the Cartesian product $\mathcal{X} \times \mathcal{Y}$
  - (x_1, y_1), (x_1, y_2), ..., (x_2, y_1), (x_2, y_2), ..., (x_3, y_1), (x_3, y_2), ...
- Marginal pmf $p_X(x)$ obtained by summing over all values of Y

$$p_X(x) = \mathrm{P}\left(X = x\right) = \sum_{y \in \mathcal{Y}} \mathrm{P}\left(X = x, Y = y\right) = \sum_{y \in \mathcal{Y}} p_{XY}(x, y)$$

$\Rightarrow$ Likewise $p_Y(y) = \sum_{x \in \mathcal{X}} p_{XY}(x, y)$. Marginalize by summing

Joint pdf

- Consider continuous RVs X, Y. Arbitrary set $\mathcal{A} \in \mathbb{R}^2$
- Joint pdf is a function $f_{XY}(x, y) : \mathbb{R}^2 \to \mathbb{R}^+$ such that

$$\mathrm{P}\left((X, Y) \in \mathcal{A}\right) = \iint_{\mathcal{A}} f_{XY}(x, y)\, dxdy$$

- Marginalization. There are two ways of writing $\mathrm{P}\left(X \in \mathcal{X}\right)$

$$\mathrm{P}\left(X \in \mathcal{X}\right) = \mathrm{P}\left(X \in \mathcal{X}, Y \in \mathbb{R}\right) = \int_{X \in \mathcal{X}} \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dy\, dx$$

$\Rightarrow$ Definition of $f_X(x)$ $\Rightarrow \mathrm{P}\left(X \in \mathcal{X}\right) = \int_{X \in \mathcal{X}} f_X(x)\, dx$

Joint pdf

- Consider continuous RVs X, Y. Arbitrary set $\mathcal{A} \in \mathbb{R}^2$
- Joint pdf is a function $f_{XY}(x, y) : \mathbb{R}^2 \to \mathbb{R}^+$ such that

$$\mathrm{P}\left((X, Y) \in \mathcal{A}\right) = \iint_{\mathcal{A}} f_{XY}(x, y)\, dxdy$$

- Marginalization. There are two ways of writing $\mathrm{P}\left(X \in \mathcal{X}\right)$

$$\mathrm{P}\left(X \in \mathcal{X}\right) = \mathrm{P}\left(X \in \mathcal{X}, Y \in \mathbb{R}\right) = \int_{X \in \mathcal{X}} \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dy\, dx$$

$\Rightarrow$ Definition of $f_X(x)$ $\Rightarrow \mathrm{P}\left(X \in \mathcal{X}\right) = \int_{X \in \mathcal{X}} f_X(x)\, dx$

- Lipstick on a pig (same thing written differently is still same thing)

$$\Rightarrow f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dy, \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dx$$

Example

- Consider two Bernoulli RVs B_1, B_2, with the same parameter p

  $\Rightarrow$ Define $X = B_1$ and $Y = B_1 + B_2$
- The pmf of X is

$$p_X(0) = 1 - p, \quad p_X(1) = p$$

- Likewise, the pmf of Y is

$$p_Y(0) = (1-p)^2, \quad p_Y(1) = 2p(1-p), \quad p_Y(2) = p^2$$

- The joint pmf of X and Y is

$$\begin{array}{lll} p_{XY}(0,0) = (1-p)^2, & p_{XY}(0,1) = p(1-p), & p_{XY}(0,2) = 0 \\ p_{XY}(1,0) = 0, & p_{XY}(1,1) = p(1-p), & p_{XY}(1,2) = p^2 \end{array}$$

Random vectors

- For convenience often arrange RVs in a vector
  $\Rightarrow$ Prob. distribution of vector is joint distribution of its entries
- Consider, e.g., two RVs X and Y. Random vector is $\mathbf{X} = [X, Y]^\top$
- If X and Y are discrete, vector variable $\mathbf{X}$ is discrete with pmf

$$p_{\mathbf{X}}(\mathbf{x}) = p_{\mathbf{X}}\left([x, y]^\top\right) = p_{XY}(x, y)$$

- If X, Y continuous, $\mathbf{X}$ continuous with pdf

$$f_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{X}}\left([x, y]^\top\right) = f_{XY}(x, y)$$

- Vector cdf is $\Rightarrow F_{\mathbf{X}}(\mathbf{x}) = F_{\mathbf{X}}\left([x, y]^\top\right) = F_{XY}(x, y)$
- In general, can define n-dimensional RVs $\mathbf{X} := [X_1, X_2, \ldots, X_n]^\top$
  $\Rightarrow$ Just notation, definitions carry over from the $n = 2$ case

Joint expectations

Sigma-algebras and probability spaces

Conditional probability, total probability, Bayes' rule

Independence

Random variables

Discrete random variables

Continuous random variables

Expected values

Joint probability distributions

Joint expectations

Joint expectations

- RVs X and Y and function $g(X, Y)$. Function $g(X, Y)$ also a RV
- Expected value of $g(X, Y)$ when X and Y discrete can be written as

$$\mathbb{E}\left[g(X, Y)\right] = \sum_{x,y: p_{XY}(x,y)>0} g(x, y) p_{XY}(x, y)$$

- When X and Y are continuous

$$\mathbb{E}\left[g(X, Y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y)\, dxdy$$

$\Rightarrow$ Can have more than two RVs and use vector notation

Ex: Linear transformation of a vector RV $\mathbf{X} \in \mathbb{R}^n$: $g(\mathbf{X}) = \mathbf{a}^\top \mathbf{X}$

$$\Rightarrow \mathbb{E}\left[\mathbf{a}^\top \mathbf{X}\right] = \int_{\mathbb{R}^n} \mathbf{a}^\top \mathbf{x} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

Expected value of a sum of random variables

- Expected value of the sum of two continuous RVs

$$\mathbb{E}\left[X+Y\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f_{XY}(x,y)\,dxdy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,f_{XY}(x,y)\,dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\,f_{XY}(x,y)\,dxdy$$

- Remove x (y) from innermost integral in first (second) summand

$$\mathbb{E}\left[X+Y\right] = \int_{-\infty}^{\infty} x\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx + \int_{-\infty}^{\infty} y\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$$
$$= \int_{-\infty}^{\infty} x f_X(x)\,dx + \int_{-\infty}^{\infty} y f_Y(y)\,dy$$
$$= \mathbb{E}\left[X\right] + \mathbb{E}\left[Y\right]$$

$\Rightarrow$ Used marginal expressions

- Expectation $\leftrightarrow$ summation $\Rightarrow \mathbb{E}\left[\sum_i X_i\right] = \sum_i \mathbb{E}\left[X_i\right]$

Expected value is a linear operator

- Combining with earlier result $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ proves that

$$\mathbb{E}[a_x X + a_y Y + b] = a_x \mathbb{E}[X] + a_y \mathbb{E}[Y] + b$$

- Better yet, using vector notation (with $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{X} \in \mathbb{R}^n$, b a scalar)

$$\mathbb{E}\left[\mathbf{a}^\top \mathbf{X} + b\right] = \mathbf{a}^\top \mathbb{E}[\mathbf{X}] + b$$

- Also, if $\mathbf{A}$ is an $m \times n$ matrix with rows $\mathbf{a}_1^\top, \ldots, \mathbf{a}_m^\top$ and $\mathbf{b} \in \mathbb{R}^m$ a vector with elements $b_1, \ldots, b_m$, we can write

$$\mathbb{E}[\mathbf{A}\mathbf{X} + \mathbf{b}] = \begin{pmatrix} \mathbb{E}\left[\mathbf{a}_1^\top \mathbf{X} + b_1\right] \\ \mathbb{E}\left[\mathbf{a}_2^\top \mathbf{X} + b_2\right] \\ \vdots \\ \mathbb{E}\left[\mathbf{a}_m^\top \mathbf{X} + b_m\right] \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^\top \mathbb{E}[\mathbf{X}] + b_1 \\ \mathbf{a}_2^\top \mathbb{E}[\mathbf{X}] + b_2 \\ \vdots \\ \mathbf{a}_m^\top \mathbb{E}[\mathbf{X}] + b_m \end{pmatrix} = \mathbf{A}\mathbb{E}[\mathbf{X}] + \mathbf{b}$$

- Expected value operator can be interchanged with linear operations

Independence of RVs

- Events E and F are independent if $\mathrm{P}(E \cap F) = \mathrm{P}(E)\,\mathrm{P}(F)$
- **Def:** RVs X and Y are independent if events $X \leq x$ and $Y \leq y$ are independent for all x and y, i.e.

$$\mathrm{P}(X \leq x, Y \leq y) = \mathrm{P}(X \leq x)\,\mathrm{P}(Y \leq y)$$

$\Rightarrow$ By definition, equivalent to $F_{XY}(x, y) = F_X(x)F_Y(y)$

- For discrete RVs equivalent to analogous relation between pmfs

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

- For continuous RVs the analogous is true for pdfs

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

- Independence $\Leftrightarrow$ Joint distribution factorizes into product of marginals

Sum of independent Poisson RVs

- Independent Poisson RVs X and Y with parameters λ_x and λ_y
- Q: Probability distribution of the sum RV $Z := X + Y$?
- $Z = n$ only if $X = k$, $Y = n - k$ for some $0 \leq k \leq n$
(use independence, Poisson pmf, rearrange terms, binomial theorem)

$$p_Z(n) = \sum_{k=0}^{n} \mathrm{P}\left(X = k, Y = n - k\right) \quad = \sum_{k=0}^{n} \mathrm{P}\left(X = k\right) \mathrm{P}\left(Y = n - k\right)$$

$$= \sum_{k=0}^{n} e^{-\lambda_x} \frac{\lambda_x^k}{k!} e^{-\lambda_y} \frac{\lambda_y^{n-k}}{(n-k)!} \quad = \frac{e^{-(\lambda_x+\lambda_y)}}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \lambda_x^k \lambda_y^{n-k}$$

$$= \frac{e^{-(\lambda_x+\lambda_y)}}{n!} (\lambda_x + \lambda_y)^n$$

- Z is Poisson with parameter $\lambda_z := \lambda_x + \lambda_y$

  $\Rightarrow$ Sum of independent Poissons is Poisson (parameters added)

Expected value of a binomial RV

- Binomial RVs count number of successes in n Bernoulli trials

Ex: Let X_i, $i = 1, \ldots n$ be n independent Bernoulli RVs

- Can write binomial $X = \sum_{i=1}^{n} X_i \;\Rightarrow \mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = np$
- Expected nr. successes = nr. trials $\times$ prob. individual success
  - Same interpretation that we observed for Poisson RVs

Ex: Dependent Bernoulli trials. $Y = \sum_{i=1}^{n} X_i$, but X_i are not independent

- Expected nr. successes is still $\mathbb{E}[Y] = np$
  - Linearity of expectation does not require independence
  - Y is not binomial distributed

Expected value of a product of independent RVs

Theorem

For independent RVs X and Y, and arbitrary functions $g(X)$ and $h(Y)$:

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)]$$

The expected value of the product is the product of the expected values

- Can show that $g(X)$ and $h(Y)$ are also independent. Intuitive

Ex: Special case when $g(X) = X$ and $h(Y) = Y$ yields

$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$$

- Expectation and product can be interchanged if RVs are independent
- Different from interchange with linear operations (always possible)

Expected value of a product of independent RVs

Proof.

- Suppose X and Y continuous RVs. Use definition of independence

$$\mathbb{E}\left[g(X)h(Y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_{XY}(x,y)\,dxdy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)\,dxdy$$

- Integrand is product of a function of x and a function of y

$$\mathbb{E}\left[g(X)h(Y)\right] = \int_{-\infty}^{\infty} g(x)f_X(x)\,dx \int_{-\infty}^{\infty} h(y)f_Y(y)\,dy$$
$$= \mathbb{E}\left[g(X)\right]\mathbb{E}\left[h(Y)\right]$$

$\square$

Variance of a sum of independent RVs

- Let X_n, $n = 1, \ldots N$ be independent with $\mathbb{E}[X_n] = \mu_n$, $\text{var}[X_n] = \sigma_n^2$
- Q: Variance of sum $X := \sum_{n=1}^{N} X_n$?
- Notice that mean of X is $\mathbb{E}[X] = \sum_{n=1}^{N} \mu_n$. Then

$$\text{var}[X] = \mathbb{E}\left[\left(\sum_{n=1}^{N} X_n - \sum_{n=1}^{N} \mu_n\right)^2\right] = \mathbb{E}\left[\left(\sum_{n=1}^{N} (X_n - \mu_n)\right)^2\right]$$

- Expand square and interchange summation and expectation

$$\text{var}[X] = \sum_{n=1}^{N} \sum_{m=1}^{N} \mathbb{E}\Big[(X_n - \mu_n)(X_m - \mu_m)\Big]$$

Variance of a sum of independent RVs (continued)

- Separate terms in sum. Then use independence and $\mathbb{E}(X_n - \mu_n) = 0$

$$\text{var}[X] = \sum_{n=1, n\neq m}^{N} \sum_{m=1}^{N} \mathbb{E}\Big[(X_n - \mu_n)(X_m - \mu_m)\Big] + \sum_{n=1}^{N} \mathbb{E}\Big[(X_n - \mu_n)^2\Big]$$
$$= \sum_{n=1, n\neq m}^{N} \sum_{m=1}^{N} \mathbb{E}(X_n - \mu_n)\mathbb{E}(X_m - \mu_m) + \sum_{n=1}^{N} \sigma_n^2 = \sum_{n=1}^{N} \sigma_n^2$$

- If RVs are independent $\Rightarrow$ Variance of sum is sum of variances
- Slightly more general result holds for independent X_i, $i = 1, \ldots, n$

$$\text{var}\left[\sum_i (a_i X_i + b_i)\right] = \sum_i a_i^2 \text{var}[X_i]$$

Variance of binomial RV and sample mean

Ex: Let X_i, $i = 1, \ldots n$ be independent Bernoulli RVs

$\Rightarrow$ Recall $\mathbb{E}[X_i] = p$ and $\text{var}[X_i] = p(1-p)$

- Write binomial X with parameters (n, p) as: $X = \sum_{i=1}^{n} X_i$
- Variance of binomial then $\Rightarrow \text{var}[X] = \sum_{i=1}^{n} \text{var}[X_i] = np(1-p)$

Ex: Let Y_i, $i = 1, \ldots n$ be independent RVs and $\mathbb{E}[Y_i] = \mu$, $\text{var}[Y_i] = \sigma^2$

- Sample mean is $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. What about $\mathbb{E}\left[\bar{Y}\right]$ and $\text{var}\left[\bar{Y}\right]$?
- Expected value $\Rightarrow \mathbb{E}\left[\bar{Y}\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[Y_i] = \mu$
- Variance $\Rightarrow \text{var}\left[\bar{Y}\right] = \frac{1}{n^2}\sum_{i=1}^{n} \text{var}[Y_i] = \frac{\sigma^2}{n}$ (used independence)

Covariance

- **Def:** The covariance of X and Y is (generalizes variance to pairs of RVs)

$$\text{cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$$

- If $\text{cov}(X, Y) = 0$ variables X and Y are said to be uncorrelated
- If X, Y independent then $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$ and $\text{cov}(X, Y) = 0$

  $\Rightarrow$ Independence implies uncorrelated RVs
- Opposite is not true, may have $\text{cov}(X, Y) = 0$ for dependent X, Y
  - Ex: X uniform in $[-a, a]$ and $Y = X^2$

  $\Rightarrow$ But uncorrelatedness implies independence if X, Y are normal
- If $\text{cov}(X, Y) > 0$ then X and Y tend to move in the same direction

  $\Rightarrow$ Positive correlation
- If $\text{cov}(X, Y) < 0$ then X and Y tend to move in opposite directions

  $\Rightarrow$ Negative correlation

Covariance example

- Let X be a zero-mean random signal and Z zero-mean noise

  $\Rightarrow$ Signal X and noise Z are independent

- Consider received signals $Y_1 = X + Z$ and $Y_2 = -X + Z$

(I) Y_1 and X are positively correlated (X, Y_1 move in same direction)

$$\begin{aligned}\text{cov}(X, Y_1) &= \mathbb{E}[XY_1] - \mathbb{E}[X]\,\mathbb{E}[Y_1] \\ &= \mathbb{E}[X(X+Z)] - \mathbb{E}[X]\,\mathbb{E}[X+Z]\end{aligned}$$

- Second term is 0 ($\mathbb{E}[X] = 0$). For first term independence of X, Z

$$\mathbb{E}[X(X+Z)] = \mathbb{E}\left[X^2\right] + \mathbb{E}[X]\,\mathbb{E}[Z] = \mathbb{E}\left[X^2\right]$$

- Combining observations $\Rightarrow \text{cov}(X, Y_1) = \mathbb{E}\left[X^2\right] > 0$

Covariance example (continued)

(II) Y_2 and X are negatively correlated (X, Y_2 move opposite direction)

- Same computations $\Rightarrow \text{cov}(X, Y_2) = -\mathbb{E}\left[X^2\right] < 0$

(III) Can also compute correlation between Y_1 and Y_2

$$\begin{aligned}\text{cov}(Y_1, Y_2) &= \mathbb{E}\left[(X+Z)(-X+Z)\right] - \mathbb{E}\left[(X+Z)\right]\mathbb{E}\left[(-X+Z)\right] \\ &= -\mathbb{E}\left[X^2\right] + \mathbb{E}\left[Z^2\right]\end{aligned}$$

$\Rightarrow$ Negative correlation if $\mathbb{E}\left[X^2\right] > \mathbb{E}\left[Z^2\right]$ (small noise)

$\Rightarrow$ Positive correlation if $\mathbb{E}\left[X^2\right] < \mathbb{E}\left[Z^2\right]$ (large noise)

- Correlation between X and Y_1 or X and Y_2 comes from causality
- Correlation between Y_1 and Y_2 does not. Latent variables X and Z

  $\Rightarrow$ Correlation does not imply causation

Plausible, indeed commonly used, model of a communication channel

Glossary

- Sample space
- Outcome and event
- Sigma-algebra
- Countable union
- Axioms of probability
- Probability space
- Conditional probability
- Law of total probability
- Bayes' rule
- Independent events
- Random variable (RV)
- Discrete RV
- Bernoulli, binomial, Poisson
- Continuous RV
- Uniform, Normal, exponential
- Indicator RV
- Pmf, pdf and cdf
- Law of rare events
- Expected value
- Variance and standard deviation
- Joint probability distribution
- Marginal distribution
- Random vector
- Independent RVs
- Covariance
- Uncorrelated RVs