PageRank: Ranking of nodes in graphs

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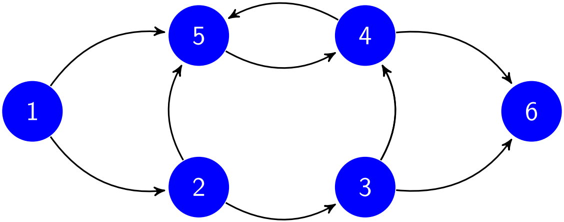
October 15, 2019

PageRank: Random walk

Ranking of nodes in graphs: Random walk

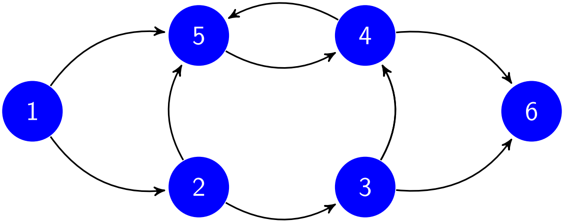
Ranking of nodes in graphs: Probability propagation

Graphs



- Graph $\Rightarrow$ A set of V of vertices or nodes $j = 1, \ldots, J$
  $\Rightarrow$ Connected by a set of edges E defined as ordered pairs (i, j)
- In figure $\Rightarrow$ Nodes are $V = \{1, 2, 3, 4, 5, 6\}$
  $\Rightarrow$ Edges $E = \{(1, 2), (1, 5), (2, 3), (2, 5), (3, 4), \ldots$
  $(3, 6), (4, 5), (4, 6), (5, 4)\}$
- Ex. 1: Websites and hyperlinks $\Rightarrow$ World Wide Web (WWW)
- Ex. 2: People and friendship $\Rightarrow$ Social network

How well connected nodes are?



- Q: Which node is the most connected? A: Define most connected
  - $\Rightarrow$ Can define "most connected" in different ways
- Two important connectivity indicators
  1) How many links point to a node (outgoing links irrelevant)
  2) How important are the links that point to a node
- Node rankings to measure website relevance, social influence

Connectivity ranking

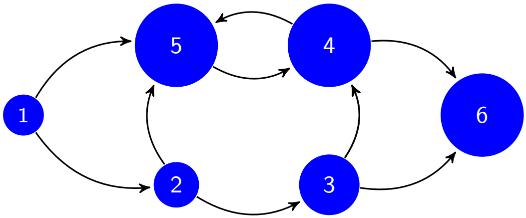
- Key insight: There is information in the structure of the network
- Knowledge is distributed through the network

  $\Rightarrow$ The network (not the nodes) knows the rankings

- Idea exploited by Google's PageRank© to rank webpages

  ... by social scientists to study trust & reputation in social networks

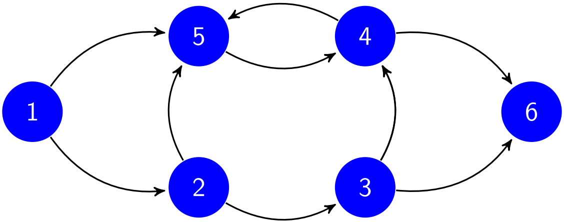
  ... by ISI to rank scientific papers, transactions & magazines ...



- No one points to 1
- Only 1 points to 2
- Only 2 points to 3, but 2 more important than 1
- 4 as high as 5 with less links
- Links to 5 have lower rank
- Same for 6

Preliminary definitions

- Graph $G = (V, E) \Rightarrow$ vertices $V = \{1, 2, \dots, J\}$ and edges E



- Outgoing neighborhood of i is the set of nodes j to which i points

$$n(i) := \{j : (i, j) \in E\}$$

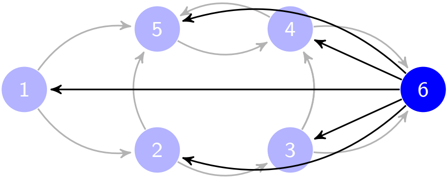
- Incoming neighborhood, $n^{-1}(i)$ is the set of nodes that point to i:

$$n^{-1}(i) := \{j : (j, i) \in E\}$$

- Strongly connected $G \Rightarrow$ directed path joining any pair of nodes

Definition of rank

- Agent A chooses node i, e.g., web page, at random for initial visit
- Next visit randomly chosen between links in the neighborhood $n(i)$
  - $\Rightarrow$ All neighbors chosen with equal probability
- If reach a dead end because node i has no neighbors
  - $\Rightarrow$ Chose next visit at random equiprobably among all nodes
- Redefine graph $\mathcal{G} = (V, E)$ adding edges from dead ends to all nodes
  - $\Rightarrow$ Restrict attention to connected (modified) graphs



- Rank of node i is the average number of visits of agent A to i

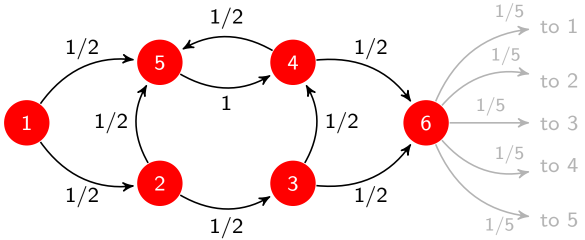
Equiprobable random walk

- Formally, let A_n be the node visited at time n
- Define transition probability P_{ij} from node i into node j

$$P_{ij} := \mathrm{P}\left(A_{n+1} = j \,\middle|\, A_n = i\right)$$

- Next visit equiprobable among i's $N_i := |n(i)|$ neighbors

$$P_{ij} = \frac{1}{|n(i)|} = \frac{1}{N_i}, \qquad \text{for all } j \in n(i)$$



- Still have a graph
- But also a MC
- Red (not blue) circles

Formal definition of rank

- **Def:** Rank r_i of i-th node is the time average of number of visits

$$r_i := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{A_m = i\}$$

$\Rightarrow$ Define vector of ranks $\mathbf{r} := [r_1, r_2, \ldots, r_J]^T$

- Rank r_i can be approximated by average r_{ni} at time n

$$r_{ni} := \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{A_m = i\}$$

$\Rightarrow$ Since $\lim_{n\to\infty} r_{ni} = r_i$, it holds $r_{ni} \approx r_i$ for n sufficiently large

$\Rightarrow$ Define vector of approximate ranks $\mathbf{r}_n := [r_{n1}, r_{n2}, \ldots, r_{nJ}]^T$

- If modified graph is connected, rank independent of initial visit

Ranking algorithm

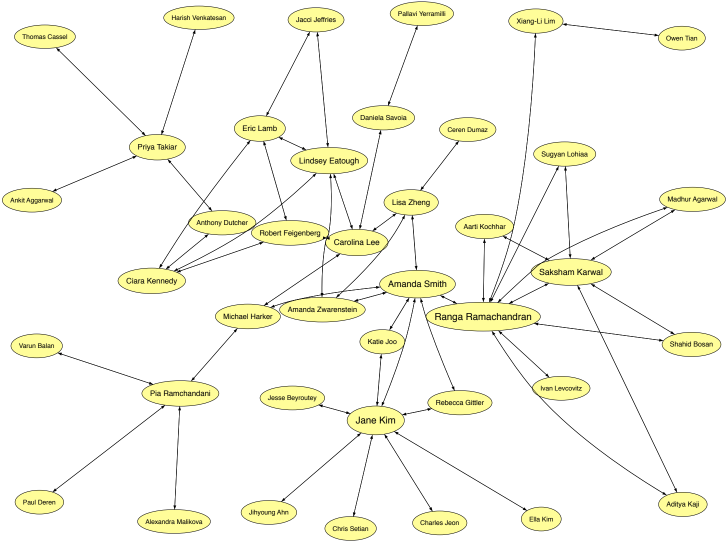
Output : Vector $\mathbf{r}(i)$ with ranking of node i
Input : Scalar n indicating maximum number of iterations
Input : Vector $N(i)$ containing number of neighbors of i
Input : Matrix $\mathbf{N}(i,j)$ containing indices j of neighbors of i

```
m = 1; r=zeros(J,1); % Initialize time and ranks
A_0 = random('unid',J); % Draw first visit uniformly at random
while m < n do
    jump = random('unid',N_{A_{m-1}}); % Neighbor uniformly at random
    A_m = N(A_{m-1}, jump); % Jump to selected neighbor
    r(A_m) = r(A_m) + 1; % Update ranking for A_m
    m = m + 1;
end
r = r/n; % Normalize by number of iterations n
```

Social graph example

- Asked probability students about homework collaboration
- Created (crude) graph of the social network of students in the class
  $\Rightarrow$ Used ranking algorithm to understand connectedness
- Ex: I want to know how well students are coping with the class
  $\Rightarrow$ Best to ask people with higher connectivity ranking
- 2009 data from "UPenn's ECE440"

Ranked class graph



Convergence metrics

- Recall $\mathbf{r}$ is vector of ranks and $\mathbf{r}_n$ of rank iterates
- By definition $\lim_{n\to\infty} \mathbf{r}_n = \mathbf{r}$. How fast $\mathbf{r}_n$ converges to $\mathbf{r}$ ($\mathbf{r}$ given)?
- Can measure by ℓ_2 distance between $\mathbf{r}$ and $\mathbf{r}_n$

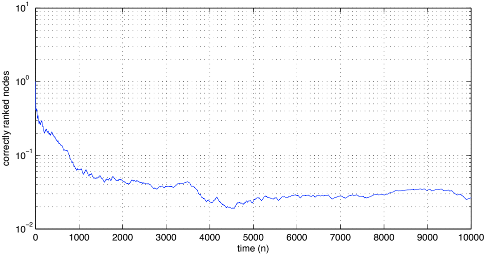
$$\zeta_n := \|\mathbf{r} - \mathbf{r}_n\|_2 = \left(\sum_{i=1}^{J} (r_{ni} - r_i)^2 \right)^{1/2}$$

- If interest is only on highest ranked nodes, e.g., a web search
  - $\Rightarrow$ Denote $r^{(i)}$ as the index of the i-th highest ranked node
  - $\Rightarrow$ Let $r_n^{(i)}$ be the index of the i-th highest ranked node at time n
- First element wrongly ranked at time n

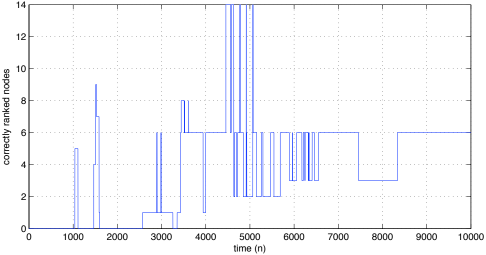
$$\xi_n := \arg\min_i \{ r^{(i)} \neq r_n^{(i)} \}$$

Evaluation of convergence metrics

Distance



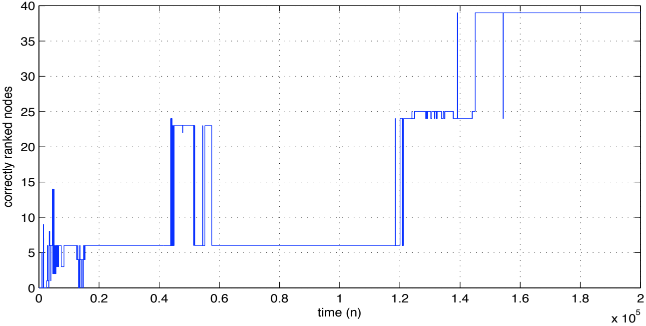
First element wrongly ranked



- Distance close to 10^{-2} in $\approx 5 \times 10^3$ iterations
- Bad: Two highest ranks in $\approx 4 \times 10^3$ iterations
- Awful: Six best ranks in $\approx 8 \times 10^3$ iterations
- (Very) slow convergence

When does this algorithm converge?

- Cannot confidently claim convergence until 10^5 iterations
  - $\Rightarrow$ Beyond particular case, slow convergence inherent to algorithm



- Example has 40 nodes, want to use in network with 10^9 nodes!
  - $\Rightarrow$ Leverage properties of MCs to obtain a faster algorithm

PageRank: Probability propagation

Ranking of nodes in graphs: Random walk

Ranking of nodes in graphs: Probability propagation

Limit probabilities

- Recall definition of rank $\Rightarrow r_i := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{A_m = i\}$
- Rank is time average of number of state visits in a MC
  $\Rightarrow$ Can be as well obtained from limiting probabilities
- Recall transition probabilities $\Rightarrow P_{ij} = \frac{1}{N_i}$, for all $j \in n(i)$
- Stationary distribution $\boldsymbol{\pi} = [\pi_1, \pi_1, \ldots, \pi_J]^T$ solution of

$$\pi_i = \sum_{j \in n^{-1}(i)} P_{ji}\pi_j = \sum_{j \in n^{-1}(i)} \frac{\pi_j}{N_j} \qquad \text{for all } i$$

  $\Rightarrow$ Plus normalization equation $\sum_{i=1}^{J} \pi_i = 1$
- As per ergodicity of MC (strongly connected G) $\Rightarrow \mathbf{r} = \boldsymbol{\pi}$

Matrix notation, eigenvalue problem

- As always, can define matrix $\mathbf{P}$ with elements P_{ij}

$$\pi_i = \sum_{j \in n^{-1}(i)} P_{ji}\pi_j = \sum_{j=1}^{J} P_{ji}\pi_j \qquad \text{for all } i$$

- Right hand side is just definition of a matrix product leading to

$$\boldsymbol{\pi} = \mathbf{P}^T\boldsymbol{\pi}, \qquad \boldsymbol{\pi}^T\mathbf{1} = 1$$

$\Rightarrow$ Also added normalization equation

- Idea: solve system of linear equations or eigenvalue problem on $\mathbf{P}^T$

$\Rightarrow$ Requires matrix $\mathbf{P}$ available at a central location

$\Rightarrow$ Computationally costly (sparse matrix $\mathbf{P}$ with 10^{18} entries)

What are limit probabilities?

- Let $p_i(n)$ denote probability of agent A visiting node i at time n

$$p_i(n) := \mathrm{P}\left(A_n = i\right)$$

- Probabilities at time $n+1$ and n can be related

$$\mathrm{P}\left(A_{n+1} = i\right) = \sum_{j \in n^{-1}(i)} \mathrm{P}\left(A_{n+1} = i \,\middle|\, A_n = j\right) \mathrm{P}\left(A_n = j\right)$$

- Which is, of course, probability propagation in a MC

$$p_i(n+1) = \sum_{j \in n^{-1}(i)} P_{ji} p_j(n)$$

- By definition limit probabilities are (let $\mathbf{p}(n) = [p_1(n), \ldots, p_J(n)]^T$)

$$\lim_{n\to\infty} \mathbf{p}(n) = \boldsymbol{\pi} = \mathbf{r}$$

$\Rightarrow$ Compute ranks from limit of propagated probabilities

Probability propagation

- Can also write probability propagation in matrix form

$$p_i(n+1) = \sum_{j \in n^{-1}(i)} P_{ji} p_j(n) = \sum_{j=1}^{J} P_{ji} p_j(n) \qquad \text{for all } i$$

- Right hand side is just definition of a matrix product leading to

$$\mathbf{p}(n+1) = \mathbf{P}^T \mathbf{p}(n)$$

- Idea: can approximate rank by large n probability distribution

  $\Rightarrow \mathbf{r} = \lim_{n \to \infty} \mathbf{p}(n) \approx \mathbf{p}(n)$ for n sufficiently large

Ranking algorithm

- Algorithm is just a recursive matrix product, a power iteration

Output : Vector $\mathbf{r}(i)$ with ranking of node i
Input : Scalar n indicating maximum number of iterations
Input : Matrix $\mathbf{P}$ containing transition probabilities

```
m = 1; % Initialize time
r=(1/J)ones(J,1); % Initial distribution uniform across all nodes
while m < n do
    r = P^T r;  % Probability propagation
    m = m + 1;
end
```

Interpretation of probability propagation

- Q: Why does the random walk converge so slow?
- A: Need to register a large number of agent visits to every state

  Ex: 40 nodes, say 100 visits to each $\Rightarrow 4 \times 10^3$ iters.

- Smart idea: Unleash a large number of agents K

$$r_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}\{A_{km} = i\}$$

- Visits are now spread over time and space

  $\Rightarrow$ Converges "K times faster"

  $\Rightarrow$ But haven't changed computational cost

Interpretation of prob. propagation (continued)

- Q: What happens if we unleash infinite number of agents K?

$$r_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}\left\{A_{km} = i\right\}$$

- Using law of large numbers and expected value of indicator function

$$r_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{E}\left[\mathbb{I}\left\{A_m = i\right\}\right] = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathrm{P}\left(A_m = i\right)$$

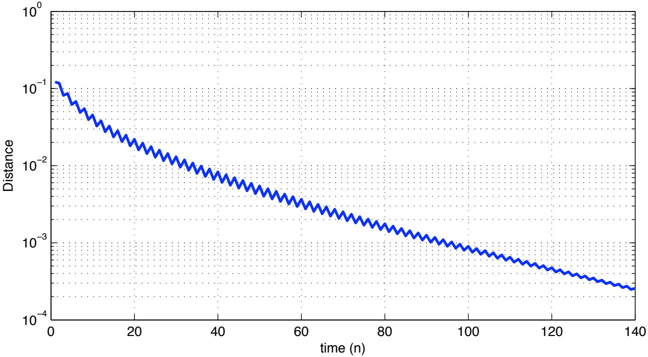
- Graph walk is an ergodic MC, then $\lim_{m\to\infty} \mathrm{P}\left(A_m = i\right)$ exists, and

$$r_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_i(m) = \lim_{n\to\infty} p_i(n)$$

$\Rightarrow$ Probability propagation $\approx$ Unleashing infinitely many agents

Distance to rank

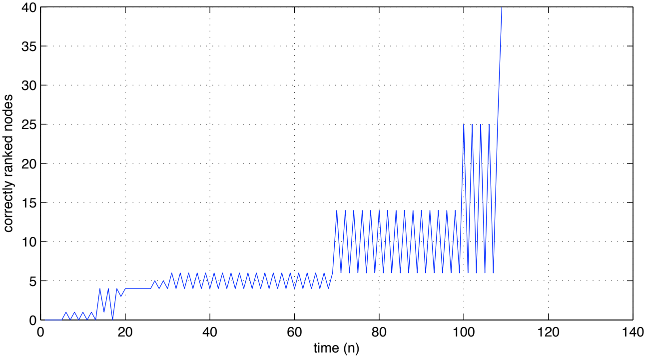
- Initialize with uniform probability distribution $\Rightarrow \mathbf{p}(0) = (1/J)\mathbf{1}$
  - $\Rightarrow$ Plot distance between $\mathbf{p}(n)$ and $\mathbf{r}$



- Distance is 10^{-2} in ≈ 30 iters., 10^{-4} in ≈ 140 iters.
  - $\Rightarrow$ Convergence two orders of magnitude faster than random walk

Number of nodes correctly ranked

- Rank of highest ranked node that is wrongly ranked by time n



- Not bad: All nodes correctly ranked in 120 iterations
- Good: Ten best ranks in 70 iterations
- Great: Four best ranks in 20 iterations

Distributed algorithm to compute ranks

- Nodes want to compute their rank r_i
  - $\Rightarrow$ Can communicate with neighbors only (incoming + outgoing)
  - $\Rightarrow$ Access to neighborhood information only
- Recall probability update

$$p_i(n+1) = \sum_{j \in n^{-1}(i)} P_{ji} p_j(n) = \sum_{j \in n^{-1}(i)} \frac{1}{N_j} p_j(n)$$

  - $\Rightarrow$ Uses local information only
- Distributed algorithm. Nodes keep local rank estimates $r_i(n)$
  - Receive rank (probability) estimates $r_j(n)$ from neighbors $j \in n^{-1}(i)$
  - Update local rank estimate $r_i(n+1) = \sum_{j \in n^{-1}(i)} r_j(n)/N_j$
  - Communicate rank estimate $r_i(n+1)$ to outgoing neighbors $j \in n(i)$
- Only need to know the number of neighbors of my neighbors

Distributed implementation of random walk

- Can communicate with neighbors only (incoming + outgoing)
  - $\Rightarrow$ But cannot access neighborhood information
  - $\Rightarrow$ Pass agent ('hot potato') around
- Local rank estimates $r_i(n)$ and counter with number of visits V_i
- Algorithm run by node i at time n

if *Agent received from neighbor* **then**
  $V_i = V_i + 1$
  Choose random neighbor
  Send agent to chosen neighbor
end
$n = n + 1$; $r_i(n) = V_i/n$;

- Speed up convergence by generating many agents to pass around

Comparison of different algorithms

- Random walk (RW) implementation
  - $\Rightarrow$ Most secure. No information shared with other nodes
  - $\Rightarrow$ Implementation can be distributed
  - $\Rightarrow$ Convergence exceedingly slow
- System of linear equations
  - $\Rightarrow$ Least security. Graph in central server
  - $\Rightarrow$ Distributed implementation not clear
  - $\Rightarrow$ Convergence not an issue
  - $\Rightarrow$ But computationally costly to obtain approximate solutions
- Probability propagation
  - $\Rightarrow$ Somewhat secure. Information shared with neighbors only
  - $\Rightarrow$ Implementation can be distributed
  - $\Rightarrow$ Convergence rate acceptable (orders of magnitude faster than RW)

Glossary

- Graph, nodes and edges
- Connectivity indicators
- Node ranking
- Google's PageRank
- Node's neighborhood
- Strong connectivity
- Random walk on a graph
- Long-run fraction of state visits
- Ranking algorithm
- Convergence metrics
- Computational cost
- Probability propagation
- Power method
- Distributed algorithm
- Security