Continuous-time Markov Chains

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Exponential distribution

- Exponential RVs often model times at which events occur

  $\Rightarrow$ Or time elapsed between occurrence of random events

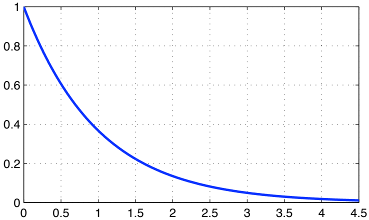
- RV $T \sim \exp(\lambda)$ is exponential with parameter λ if its pdf is

$$f_T(t) = \lambda e^{-\lambda t}, \qquad \text{for all } t \geq 0$$

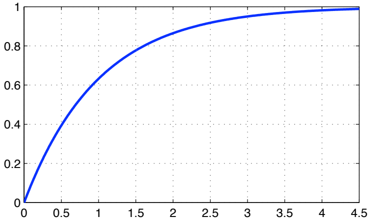
- Cdf, integral of the pdf, is $\Rightarrow F_T(t) = \mathrm{P}\left(T \leq t\right) = 1 - e^{-\lambda t}$

  $\Rightarrow$ Complementary (c)cdf is $\Rightarrow P(T \geq t) = 1 - F_T(t) = e^{-\lambda t}$

pdf ($\lambda = 1$)



cdf ($\lambda = 1$)



Expected value

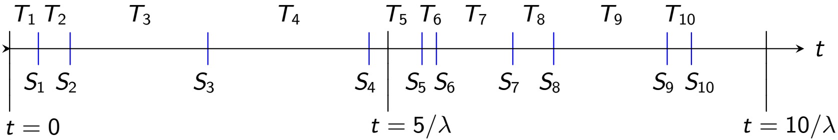
- Expected value of time $T \sim \exp(\lambda)$ is

$$\mathbb{E}\left[T\right] = \int_0^\infty t\lambda e^{-\lambda t}dt = -te^{-\lambda t}\Big|_0^\infty + \int_0^\infty e^{-\lambda t}dt = 0 + \frac{1}{\lambda}$$

$\Rightarrow$ Integrated by parts with $u = t$, $dv = \lambda e^{-\lambda t}dt$

- Mean time is inverse of parameter λ
  - $\Rightarrow$ λ is rate/frequency of events happening at intervals T
  - $\Rightarrow$ **Interpret:** Average of λt events by time t

- Bigger λ $\Rightarrow$ smaller expected times, larger frequency of events



Second moment and variance

- For second moment also integrate by parts ($u = t^2$, $dv = \lambda e^{-\lambda t} dt$)

$$\mathbb{E}\left[T^2\right] = \int_0^\infty t^2 \lambda e^{-\lambda t} dt = -t^2 e^{-\lambda t}\Big|_0^\infty + \int_0^\infty 2te^{-\lambda t} dt$$

- First term is 0, second is $(2/\lambda)\mathbb{E}[T]$

$$\mathbb{E}\left[T^2\right] = \frac{2}{\lambda}\int_0^\infty t\lambda e^{-\lambda t} = \frac{2}{\lambda^2}$$

- The variance is computed from the mean and second moment

$$\text{var}[T] = \mathbb{E}\left[T^2\right] - \mathbb{E}^2[T] = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$\Rightarrow$ Parameter λ controls mean and variance of exponential RV

Memoryless random times

- **Def:** Consider random time T. We say time T is memoryless if

$$\mathrm{P}\left(T > s + t \,\middle|\, T > t\right) = \mathrm{P}\left(T > s\right)$$

- Probability of waiting s extra units of time (e.g., seconds) given that we waited t seconds, is just the probability of waiting s seconds
  - $\Rightarrow$ System does not remember it has already waited t seconds
  - $\Rightarrow$ Same probability irrespectively of time already elapsed

Ex: Chemical reaction $A + B \rightarrow AB$ occurs when molecules A and B "collide". A, B move around randomly. Time T until reaction

Exponential RVs are memoryless

- Write memoryless property in terms of joint pdf

$$\mathrm{P}\left(T > s + t \,\middle|\, T > t\right) = \frac{\mathrm{P}\left(T > s + t, T > t\right)}{\mathrm{P}\left(T > t\right)} = \mathrm{P}\left(T > s\right)$$

- Notice event $\{T > s + t, T > t\}$ is equivalent to $\{T > s + t\}$

  $\Rightarrow$ Replace $\mathrm{P}\left(T > s + t, T > t\right) = \mathrm{P}\left(T > s + t\right)$ and reorder

$$\mathrm{P}\left(T > s + t\right) = \mathrm{P}\left(T > t\right)\mathrm{P}\left(T > s\right)$$

- If $T \sim \exp(\lambda)$, ccdf is $\mathrm{P}\left(T > t\right) = e^{-\lambda t}$ so that

$$\mathrm{P}\left(T > s + t\right) = e^{-\lambda(s+t)} = e^{-\lambda t}e^{-\lambda s} = \mathrm{P}\left(T > t\right)\mathrm{P}\left(T > s\right)$$

- If random time T is exponential $\Rightarrow$ T is memoryless

Continuous memoryless RVs are exponential

- Consider a function $g(t)$ with the property $g(t+s) = g(t)g(s)$
- Q: Functional form of $g(t)$? Take logarithms

$$\log g(t+s) = \log g(t) + \log g(s)$$

$\Rightarrow$ Only holds for all t and s if $\log g(t) = ct$ for some constant c

$\Rightarrow$ Which in turn, can only hold if $g(t) = e^{ct}$ for some constant c

- Compare observation with statement of memoryless property

$$\mathrm{P}\left(T > s+t\right) = \mathrm{P}\left(T > t\right)\mathrm{P}\left(T > s\right)$$

$\Rightarrow$ It must be $\mathrm{P}\left(T > t\right) = e^{ct}$ for some constant c

- T continuous: only true for exponential $T \sim \exp(-c)$
- T discrete: only geometric $\mathrm{P}\left(T > t\right) = (1-p)^t$ with $(1-p) = e^c$
- If continuous random time T is memoryless $\Rightarrow$ T is exponential

Main memoryless property result

Theorem
A continuous random variable T is memoryless if and only if it is exponentially distributed. That is

$$\mathrm{P}\left(T > s + t \,\middle|\, T > t\right) = \mathrm{P}\left(T > s\right)$$

if and only if $f_T(t) = \lambda e^{-\lambda t}$ for some $\lambda > 0$

- Exponential RVs are memoryless. Do not remember elapsed time
  - $\Rightarrow$ Only type of continuous memoryless RVs
- Discrete RV T is memoryless if and only of it is geometric
  - $\Rightarrow$ Geometrics are discrete approximations of exponentials
  - $\Rightarrow$ Exponentials are continuous limits of geometrics
- Exponential = time until success $\Leftrightarrow$ Geometric = nr. trials until success

Exponential times example

- First customer's arrival to a store takes $T \sim \exp(1/10)$ minutes
  $\Rightarrow$ Suppose 5 minutes have passed without an arrival
- Q: How likely is it that the customer arrives within the next 3 mins.?
- Use memoryless property of exponential T

$$\begin{aligned} \mathrm{P}\left(T \leq 8 \,\middle|\, T > 5\right) &= 1 - \mathrm{P}\left(T > 8 \,\middle|\, T > 5\right) \\ &= 1 - \mathrm{P}\left(T > 3\right) = 1 - e^{-3\lambda} = 1 - e^{-0.3} \end{aligned}$$

- Q: How likely is it that the customer arrives after time $T = 9$?

$$\mathrm{P}\left(T > 9 \,\middle|\, T > 5\right) = \mathrm{P}\left(T > 4\right) = e^{-4\lambda} = e^{-0.4}$$

- Q: What is the expected additional time until the first arrival?
- Additional time is $T - 5$, and $\mathrm{P}\left(T - 5 > t \,\middle|\, T > 5\right) = \mathrm{P}\left(T > t\right)$

$$\mathbb{E}\left[T - 5 \,\middle|\, T > 5\right] = \mathbb{E}\left[T\right] = 1/\lambda = 10$$

Time to first event

- Independent exponential RVs T_1, T_2 with parameters λ_1, λ_2
- Q: Prob. distribution of time to first event, i.e., $T := \min(T_1, T_2)$?

  $\Rightarrow$ To have $T > t$ we need both $T_1 > t$ and $T_2 > t$
- Using independence of T_1 and T_2 we can write

$$\mathrm{P}(T > t) = \mathrm{P}(T_1 > t, T_2 > t) = \mathrm{P}(T_1 > t)\,\mathrm{P}(T_2 > t)$$

- Substituting expressions of exponential ccdfs

$$\mathrm{P}(T > t) = e^{-\lambda_1 t} e^{-\lambda_2 t} = e^{-(\lambda_1+\lambda_2)t}$$

- $T := \min(T_1, T_2)$ is exponentially distributed with parameter $\lambda_1 + \lambda_2$
- In general, for n independent RVs $T_i \sim \exp(\lambda_i)$ define $T := \min_i T_i$

  $\Rightarrow$ T is exponentially distributed with parameter $\sum_{i=1}^{n} \lambda_i$

First event to happen

- Q: Prob. $\mathrm{P}(T_1 < T_2)$ of $T_1 \sim \exp(\lambda_1)$ happening before $T_2 \sim \exp(\lambda_2)$?
- Condition on $T_2 = t$, integrate over the pdf of T_2, independence

$$\mathrm{P}(T_1 < T_2) = \int_0^\infty \mathrm{P}\left(T_1 < t \,\middle|\, T_2 = t\right) f_{T_2}(t)\, dt = \int_0^\infty F_{T_1}(t) f_{T_2}(t)\, dt$$

- Substitute expressions for exponential pdf and cdf

$$\mathrm{P}(T_1 < T_2) = \int_0^\infty (1 - e^{-\lambda_1 t})\lambda_2 e^{-\lambda_2 t}\, dt = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

- Either T_1 comes before T_2 or vice versa, hence

$$\mathrm{P}(T_2 < T_1) = 1 - \mathrm{P}(T_1 < T_2) = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

$\Rightarrow$ Probabilities are relative values of rates (parameters)

- Larger rate $\Rightarrow$ smaller average $\Rightarrow$ higher prob. happening first

Additional properties of exponential RVs

- Consider n independent RVs $T_i \sim \exp(\lambda_i)$. T_i time to i-th event
- Probability of j-th event happening first

$$\mathrm{P}\left(T_j = \min_i T_i\right) = \frac{\lambda_j}{\sum_{i=1}^n \lambda_i}, \; j = 1, \ldots, n$$

- Time to first event and rank ordering of events are independent

$$\mathrm{P}\left(\min_i T_i \geq t, T_{i_1} < \ldots < T_{i_n}\right) = \mathrm{P}\left(\min_i T_i \geq t\right) \mathrm{P}\left(T_{i_1} < \ldots < T_{i_n}\right)$$

- Suppose $T \sim \exp(\lambda)$, independent of non-negative RV X
- Strong memoryless property asserts

$$\mathrm{P}\left(T > X + t \,\middle|\, T > X\right) = \mathrm{P}\left(T > t\right)$$

$\Rightarrow$ Also forgets random but independent elapsed times

Strong memoryless property example

- Independent customer arrival times $T_i \sim \exp(\lambda_i)$, $i = 1, \ldots, 3$

  $\Rightarrow$ Suppose customer 3 arrives first, i.e., $\min(T_1, T_2) > T_3$

- Q: Probability that time between first and second arrival exceeds t?

- We want to compute

$$\mathrm{P}\left(\min(T_1, T_2) - T_3 > t \,\middle|\, \min(T_1, T_2) > T_3\right)$$

- Denote $T_{i_2} := \min(T_1, T_2)$ the time to second arrival

  $\Rightarrow$ Recall $T_{i_2} \sim \exp(\lambda_1 + \lambda_2)$, T_{i_2} independent of $T_{i_1} = T_3$

- Apply the strong memoryless property

$$\mathrm{P}\left(T_{i_2} - T_3 > t \,\middle|\, T_{i_2} > T_3\right) = \mathrm{P}\left(T_{i_2} > t\right) = e^{-(\lambda_1+\lambda_2)t}$$

Probability of event in infinitesimal time

- Q: Probability of an event happening in infinitesimal time h?
- Want $\mathrm{P}(T < h)$ for small h

$$\mathrm{P}(T < h) = \int_0^h \lambda e^{-\lambda t}\, dt \approx \lambda h$$

$\Rightarrow$ Equivalent to $\left.\dfrac{\partial \mathrm{P}(T < t)}{\partial t}\right|_{t=0} = \lambda$

- Sometimes also write $\mathrm{P}(T < h) = \lambda h + o(h)$

$\Rightarrow$ $o(h)$ implies $\displaystyle\lim_{h\to 0} \frac{o(h)}{h} = 0$

$\Rightarrow$ Read as "negligible with respect to h"

- Q: Two independent events in infinitesimal time h?

$$\mathrm{P}(T_1 \le h, T_2 \le h) \approx (\lambda_1 h)(\lambda_2 h) = \lambda_1 \lambda_2 h^2 = o(h)$$

Counting and Poisson processes

Exponential random variables

Counting processes and definition of Poisson processes

Properties of Poisson processes

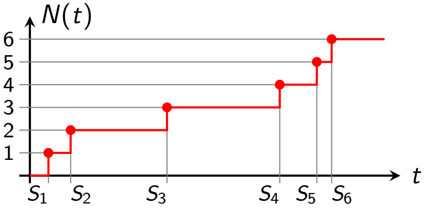
Counting processes

- Random process $N(t)$ in continuous time $t \in \mathbb{R}_+$
- **Def:** Counting process $N(t)$ counts number of events by time t
- Nonnegative integer valued: $N(0) = 0$, $N(t) \in \{0, 1, 2, \ldots\}$
- Nondecreasing: $N(s) \leq N(t)$ for $s < t$
- Event counter: $N(t) - N(s)$ = number of events in interval $(s, t]$
  - $N(t)$ continuous from the right
  - $N(S_4) - N(S_2) = 2$, while $N(S_4) - N(S_2 - \epsilon) = 3$ for small $\epsilon > 0$

Ex.1: # text messages (SMS) typed since beginning of class

Ex.2: # economic crises since 1900

Ex.3: # customers at Wegmans since 8 am this morning



Independent increments

- Consider times $s_1 < t_1 < s_2 < t_2$ and intervals $(s_1, t_1]$ and $(s_2, t_2]$
  - $\Rightarrow$ $N(t_1) - N(s_1)$ events occur in $(s_1, t_1]$
  - $\Rightarrow$ $N(t_2) - N(s_2)$ events occur in $(s_2, t_2]$
- **Def:** Independent increments implies latter two are independent

$$\mathrm{P}\left(N(t_1) - N(s_1) = k, N(t_2) - N(s_2) = l\right)$$
$$= \mathrm{P}\left(N(t_1) - N(s_1) = k\right)\mathrm{P}\left(N(t_2) - N(s_2) = l\right)$$

- Number of events in disjoint time intervals are independent

Ex.1: Likely true for SMS, except for "have to send" messages

Ex.2: Most likely not true for economic crises (business cycle)

Ex.3: Likely true for Wegmans, except for unforeseen events (storms)

- Does not mean $N(t)$ independent of $N(s)$, say for $t > s$
  - $\Rightarrow$ These events are clearly dependent, since $N(t)$ is at least $N(s)$

Stationary increments

- Consider time intervals $(0, t]$ and $(s, s+t]$
  $\Rightarrow N(t)$ events occur in $(0, t]$
  $\Rightarrow N(s+t) - N(s)$ events in $(s, s+t]$
- **Def:** Stationary increments implies latter two have same prob. dist.

$$\mathrm{P}\left(N(s+t) - N(s) = k\right) = \mathrm{P}\left(N(t) = k\right)$$

- Prob. dist. of number of events depends on length of interval only

Ex.1: Likely true if lecture is good and you keep interest in the class

Ex.2: Maybe true if you do not believe we become better at preventing crises

Ex.3: Most likely not true because of, e.g., rush hours and slow days

Poisson process

- **Def:** A counting process $N(t)$ is a Poisson process if
  (a) The process has stationary and independent increments
  (b) The number of events in $(0, t]$ has Poisson distribution with mean λt

$$\mathrm{P}\left(N(t) = n\right) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

- An equivalent definition is the following
  (i) The process has stationary and independent increments
  (ii) Single event in infinitesimal time $\Rightarrow \mathrm{P}\left(N(h) = 1\right) = \lambda h + o(h)$
  (iii) Multiple events in infinitesimal time $\Rightarrow \mathrm{P}\left(N(h) > 1\right) = o(h)$

  $\Rightarrow$ A more intuitive definition (even hard to grasp now)

- Conditions (i) and (a) are the same
- That (b) implies (ii) and (iii) is obvious
  - Substitute small h in Poisson pmf's expression for $\mathrm{P}\left(N(t) = n\right)$
- To see that (ii) and (iii) imply (b) requires some work

Explanation of model (i)-(iii)

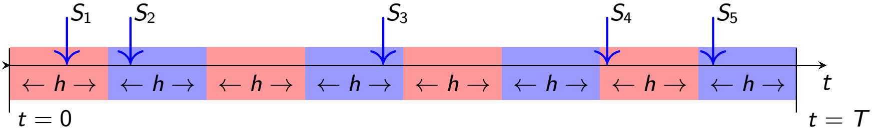
- Consider time T and divide interval $(0, T]$ in n subintervals
- Subintervals are of duration $h = T/n$, h vanishes as n increases
  - $\Rightarrow$ The m-th subinterval spans $\big((m-1)h, mh\big]$
- Define A_m as the number of events that occur in m-th subinterval

$$A_m = N\big(mh\big) - N\big((m-1)h\big)$$

- The total number of events in $(0, T]$ is the sum of A_m, $m = 1, \ldots, n$

$$N(T) = \sum_{m=1}^{n} A_m = \sum_{m=1}^{n} N\big(mh\big) - N\big((m-1)h\big)$$

- In figure, $N(T) = 5$, A_1, A_2, A_4, A_7, A_8 are 1 and A_3, A_5, A_6 are 0



Probability distribution of A_m (intuitive arg.)

- Note first that since increments are stationary as per (i), it holds

$$P\left(A_m = k\right) = P\Big(N\big(mh\big) - N\big((m-1)h\big) = k\Big) = P\left(N(h) = k\right)$$

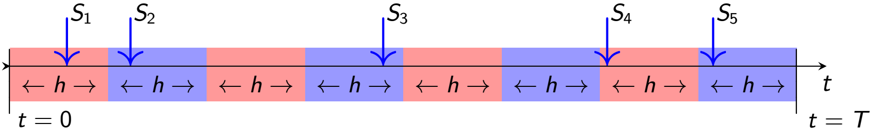
- In particular, using (ii) and (iii)

$$\begin{aligned} P\left(A_m = 1\right) &= P\left(N(h) = 1\right) = \lambda h + o(h) \\ P\left(A_m > 1\right) &= P\left(N(h) > 1\right) = o(h) \end{aligned}$$

- Set aside $o(h)$ probabilities – They're negligible with respect to λh

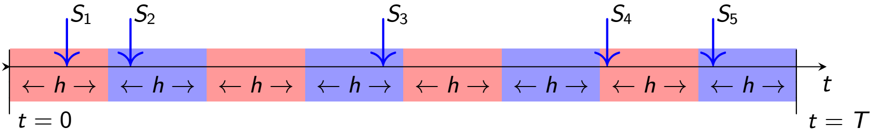
$$P\left(A_m = 1\right) = \lambda h \qquad P\left(A_m = 0\right) = 1 - \lambda h$$

$\Rightarrow$ A_m is Bernoulli with parameter λh



Probability distribution of $N(T)$ (intuitive arg.)

- Since increments are also independent as per (i), A_m are independent
- $N(T)$ is sum of n independent Bernoulli RVs with parameter λh
  $\Rightarrow$ $N(T)$ is binomial with parameters $(n, \lambda h) = (n, \lambda T/n)$
- As interval length $h \to 0$, number of intervals $n \to \infty$
  $\Rightarrow$ The product $n \times \lambda h = \lambda T$ stays constant
  $\Rightarrow$ $N(T)$ is Poisson with parameter λT (Law of rare events)
- Then (ii)-(iii) imply (b) and definitions are equivalent
  $\Rightarrow$ Not a proof because we neglected $o(h)$ terms
  $\Rightarrow$ But explains what a Poisson process is



What is a Poisson process?

- Fundamental defining properties of a Poisson process
  - Events happen in small interval h with probability λh proportional to h
  - Whether event happens in an interval has no effect on other intervals
- Modeling questions

Q1: Expect probability of event proportional to length of interval?

Q2: Expect subsequent intervals to behave independently?

$\Rightarrow$ If positive, then a Poisson process model is appropriate

- Typically arise in a large population of agents acting independently

$\Rightarrow$ Larger interval, larger chance an agent takes an action

$\Rightarrow$ Action of one agent has no effect on action of other agents

$\Rightarrow$ Has therefore negligible effect on action of group

Examples of Poisson processes

Ex.1: Number of people arriving at subway station. Number of cars arriving at a highway entrance. Number of customers entering a store ... Large number of agents (people, drivers, customers) acting independently

Ex.2: SMS generated by all students in the class. Once you send an SMS you are likely to stay silent for a while. But in a large population this has a minimal effect in the probability of someone generating a SMS

Ex.3: Count of molecule reactions. Molecules are "removed" from pool of reactants once they react. But effect is negligible in large population. Eventually reactants are depleted, but in small time scale process is approximately Poisson

Formal argument to show equivalence

- Define $A_{\max} := \max_{m=1,\ldots,n} (A_m)$, maximum nr. of events in one interval
- If $A_{\max} \leq 1$ all intervals have 0 or 1 events. Consider probability

$$\mathrm{P}\left(N(T) = k \,\middle|\, A_{\max} \leq 1\right)$$

$\Rightarrow$ For given h, $N(T)$ conditioned on $A_{\max} \leq 1$ is binomial

$\Rightarrow$ Parameters are $n = T/h$ and $p = \lambda h + o(h)$

- Interval length $h \to 0$ $\Rightarrow$ Parameter $p \to 0$, nr. of intervals $n \to \infty$

$\Rightarrow$ Product np $\Rightarrow \lim_{h \to 0} np = \lim_{h \to 0} (T/h)(\lambda h + o(h)) = \lambda T$

- $N(T)$ conditioned on $A_{\max} \leq 1$ is Poisson with parameter λT

$$\mathrm{P}\left(N(T) = k \,\middle|\, A_{\max} \leq 1\right) = e^{-\lambda T} \frac{(\lambda T)^k}{k!}$$

Formal argument to show equivalence (continued)

- Separate study in $A_{\max} \leq 1$ and $A_{\max} > 1$. That is, condition

$$\begin{aligned}
\mathrm{P}\left(N(T) = k\right) = \; & \mathrm{P}\left(N(T) = k \,\middle|\, A_{\max} \leq 1\right) \mathrm{P}\left(A_{\max} \leq 1\right) \\
& + \mathrm{P}\left(N(T) = k \,\middle|\, A_{\max} > 1\right) \mathrm{P}\left(A_{\max} > 1\right)
\end{aligned}$$

- Property (iii) implies that $\mathrm{P}\left(A_{\max} > 1\right)$ vanishes as $h \to 0$

$$\mathrm{P}\left(A_{\max} > 1\right) \leq \sum_{m=1}^{n} \mathrm{P}\left(A_m > 1\right) = no(h) = T \frac{o(h)}{h} \to 0$$

- Thus, as $h \to 0$, $\mathrm{P}\left(A_{\max} > 1\right) \to 0$ and $\mathrm{P}\left(A_{\max} \leq 1\right) \to 1$. Then

$$\lim_{h \to 0} \mathrm{P}\left(N(T) = k\right) = \lim_{h \to 0} \mathrm{P}\left(N(T) = k \,\middle|\, A_{\max} \leq 1\right)$$

- Right-hand side is Poisson $\Rightarrow$ $N(T)$ Poisson with parameter λT

Properties of Poisson processes

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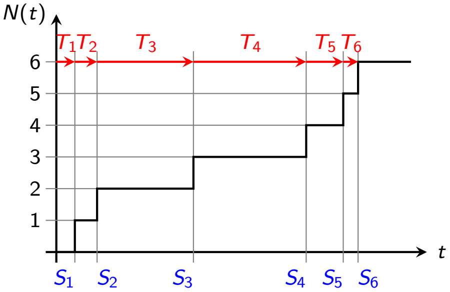
Properties of Poisson processes

Arrival times and interarrival times



- Let $S_1, S_2, \ldots$ be the sequence of absolute times of events (arrivals)
- **Def:** S_i is known as the i-th arrival time

  $\Rightarrow$ Notice that $S_i = \min_t(N(t) \geq i)$
- Let $T_1, T_2, \ldots$ be the sequence of times between events
- **Def:** T_i is known as the i-th interarrival time
- Useful identities: $S_i = \sum_{k=1}^{i} T_k$ and $T_i = S_i - S_{i-1}$, where $S_0 = 0$

Interarrival times are i.i.d. exponential RVs

- Ccdf of $T_1 \Rightarrow \mathrm{P}(T_1 > t) = \mathrm{P}(N(t) = 0) = e^{-\lambda t}$

  $\Rightarrow$ T_1 has exponential distribution with parameter λ

- Since increments are stationary and independent, likely T_i are i.i.d.

Theorem

Interarrival times T_i of a Poisson process are independent identically distributed exponential random variables with parameter λ, i.e.,

$$\mathrm{P}(T_i > t) = e^{-\lambda t}$$

- Have already proved for T_1. Let us see the rest

Interarrival times are i.i.d. exponential RVs

Proof.

- Recall S_i is i-th arrival time. Condition on S_i

$$P\left(T_{i+1} > t\right) = \int P\left(T_{i+1} > t \,\middle|\, S_i = s\right) f_{S_i}(s)\, ds$$

- To have $T_{i+1} > t$ given that $S_i = s$ it must be $N(s+t) = N(s)$

$$P\left(T_{i+1} > t \,\middle|\, S_i = s\right) = P\left(N(t+s) - N(s) = 0 \,\middle|\, N(s) = i\right)$$

- Since increments are independent, drop conditioning on $N(s) = i$

$$P\left(T_{i+1} > t \,\middle|\, S_i = s\right) = P\left(N(t+s) - N(s) = 0\right)$$

- Since increments are also stationary and $N(t)$ is Poisson, then

$$P\left(T_{i+1} > t \,\middle|\, S_i = s\right) = P\left(N(t) = 0\right) = e^{-\lambda t}$$

- Substituting into integral yields $\Rightarrow P\left(T_{i+1} > t\right) = e^{-\lambda t}$ $\square$

Interarrival times example

- Let $N_1(t)$ and $N_2(t)$ be Poisson processes with rates λ_1 and λ_2
  - $\Rightarrow$ Suppose $N_1(t)$ and $N_2(t)$ are independent
- Q: What is the expected time till the first arrival from either process?
- Denote as $S_1^{(i)}$ the first arrival time from process $i = 1, 2$
  - $\Rightarrow$ We are looking for $\mathbb{E}\left[\min\left(S_1^{(1)}, S_1^{(2)}\right)\right]$
- Note that $S_1^{(1)} = T_1^{(1)}$ and $S_1^{(2)} = T_1^{(2)}$
  - $\Rightarrow$ $S_1^{(1)} \sim \exp(\lambda_1)$ and $S_1^{(2)} \sim \exp(\lambda_2)$
  - $\Rightarrow$ Also, $S_1^{(1)}$ and $S_1^{(2)}$ are independent
- Recall that $\min\left(S_1^{(1)}, S_1^{(2)}\right) \sim \exp(\lambda_1 + \lambda_2)$, then

$$\mathbb{E}\left[\min\left(S_1^{(1)}, S_1^{(2)}\right)\right] = \frac{1}{\lambda_1 + \lambda_2}$$

Alternative definition of Poisson process

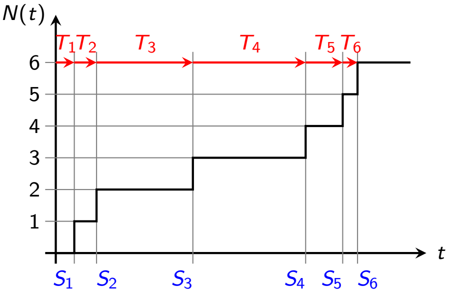
- Start with sequence of independent random times $T_1, T_2, \ldots$
- Times $T_i \sim \exp(\lambda)$ have exponential distribution with parameter λ
- Define i-th arrival time S_i

$$S_i = T_1 + T_2 + \ldots + T_i$$

- Define counting process of events occurred by time t

$$N(t) = \max_i (S_i \leq t)$$

- $N(t)$ is a Poisson process



- If $N(t)$ is a Poisson process interarrival times T_i are i.i.d. exponential
- To show that definition is equivalent have to show the converse

$\Rightarrow$ If interarrival times are i.i.d. exponential, process is Poisson

Alternative definition of Poisson process (cont.)

- Exponential i.i.d. interarrival times $\Rightarrow$ Q: Poisson process?
  $\Rightarrow$ Show that implies definition (i)-(iii)
- Stationarity true because all T_i have same distribution
- Independent increments true because
  - Interarrival times are independent
  - Exponential RVs are memoryless
- Can have more than one event in $(0, h]$ only if $T_1 < h$ and $T_2 < h$

$$\begin{aligned} \mathrm{P}\left(N(h) > 1\right) &\leq \mathrm{P}\left(T_1 \leq h\right) \mathrm{P}\left(T_2 \leq h\right) \\ &= (1 - e^{-\lambda h})^2 = (\lambda h)^2 + o(h^2) = o(h) \end{aligned}$$

- We have no event in $(0, h]$ if $T_1 > h$

$$\mathrm{P}\left(N(h) = 0\right) = \mathrm{P}\left(T_1 \geq h\right) = e^{-\lambda h} = 1 - \lambda h + o(h)$$

- The remaining case is $N(h) = 1$, whose probability is

$$\mathrm{P}\left(N(h) = 1\right) = 1 - \mathrm{P}\left(N(h) = 0\right) - \mathrm{P}\left(N(h) > 1\right) = \lambda h + o(h)$$

Three definitions of Poisson processes

Def. 1: Prob. of event proportional to interval width. Intervals independent

- Physical model definition
- Can a phenomenon be reasonably modeled as a Poisson process?
- The other two definitions are used for analysis and/or simulation

Def. 2: Prob. distribution of events in $(0, t]$ is Poisson

- Event centric definition. Nr. of events in given time intervals
- Allows analysis and simulation
- Used when information about nr. of events in given time is desired

Def. 3: Prob. distribution of interarrival times is exponential

- Time centric definition. Times at which events happen
- Allows analysis and simulation
- Used when information about event times is of interest

Number of visitors to a web page example

Ex: Count number of visits to a webpage between 6:00pm to 6:10pm

Def 1: Q: Poisson process? Yes, seems reasonable to have

- Probability of visit proportional to time interval duration
- Independent visits over disjoint time intervals

- Model as Poisson process with rate λ visits/second (v/s)

Def 2: Arrivals in $(s, s+t]$ are Poisson with parameter λt

- Prob. of exactly 5 visits in 1 sec? $\Rightarrow \mathrm{P}\left(N(1)=5\right) = e^{-\lambda}\lambda^5/5!$
- Expected nr. of visits in 10 minutes? $\Rightarrow \mathbb{E}\left[N(600)\right] = 600\lambda$
- On average, data shows N visits in 10 minutes. Estimate $\hat{\lambda} = N/600$

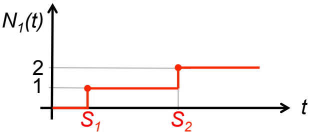
Def 3: Interarrival times are i.i.d. $T_i \sim \exp(\lambda)$

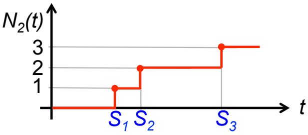
- Expected time between visits? $\Rightarrow \mathbb{E}\left[T_i\right] = 1/\lambda$
- Expected arrival time S_n of n-th visitor?

  $\Rightarrow$ Recall $S_n = \sum_{i=1}^n T_i$, hence $\mathbb{E}\left[S_n\right] = \sum_{i=1}^n \mathbb{E}\left[T_i\right] = n/\lambda$

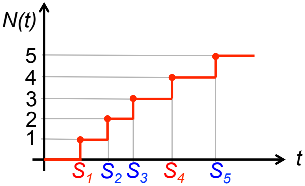
Superposition of Poisson processes

- Let $N_1(t), N_2(t)$ be Poisson processes with rates λ_1 and λ_2
  $\Rightarrow$ Suppose $N_1(t)$ and $N_2(t)$ are independent



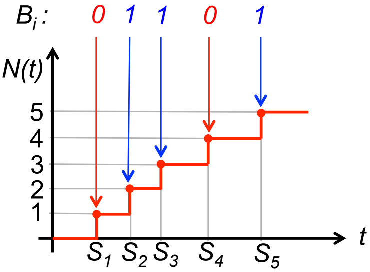


- Then $N(t) := N_1(t) + N_2(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$



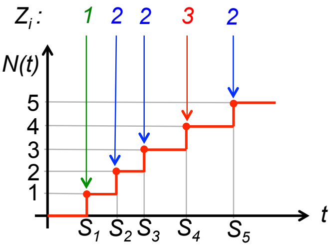
Thinning of a Poisson process

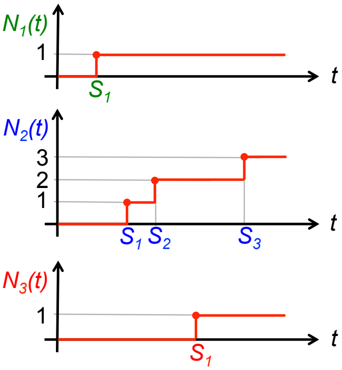
- Let $B_{\mathbb{N}} = B_1, B_2, \dots$ be an i.i.d. sequence of Bernoulli(p) RVs
- Let $N(t)$ be a Poisson process with rate λ, independent of $B_{\mathbb{N}}$
- Then $M(t) := \sum_{i=1}^{N(t)} B_i$ is a Poisson process with rate λp



Splitting of a Poisson process

- Let $Z_{\mathbb{N}} = Z_1, Z_2, \ldots$ be an i.i.d. sequence of RVs, $Z_i \in \{1, \ldots, m\}$
- Let $N(t)$ be a Poisson process with rate λ, independent of $Z_{\mathbb{N}}$
- Define $N_k(t) = \sum_{i=1}^{N(t)} \mathbb{I}\{Z_i = k\}$, for each $k = 1, \ldots, m$
- Then each $N_k(t)$ is a Poisson process with rate $\lambda \mathrm{P}(Z_1 = k)$





Glossary

- Random times
- Exponential distribution
- Memoryless random times
- Time to first event
- First event to happen
- Strong memoryless property
- Event in infinitesimal interval
- Continuous-time process
- Counting process
- Right-continuous function
- Poisson process
- Independent increments
- Stationary increments
- Equivalent definitions
- Arrival times
- Interarrival times
- Event and time centric
- Superposition of Poisson processes
- Thinning of a Poisson process
- Splitting of a Poisson process