Continuous-time Markov Chains

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Definition

- Continuous-time positive variable $t \in [0, \infty)$
- Time-dependent random state $X(t)$ takes values on a countable set
  - In general denote states as $i = 0, 1, 2, \ldots$, i.e., here the state space is $\mathbb{N}$
  - If $X(t) = i$ we say "the process is in state i at time t"
- **Def:** Process $X(t)$ is a continuous-time Markov chain (CTMC) if

$$\begin{aligned} &\mathrm{P}\left(X(t+s) = j \,\middle|\, X(s) = i, X(u) = x(u), u < s\right) \\ &= \mathrm{P}\left(X(t+s) = j \,\middle|\, X(s) = i\right) \end{aligned}$$

- Markov property $\Rightarrow$ Given the present state $X(s)$

  $\Rightarrow$ Future $X(t+s)$ is independent of the past $X(u) = x(u), u < s$
- In principle need to specify functions $\mathrm{P}\left(X(t+s) = j \,\middle|\, X(s) = i\right)$

  $\Rightarrow$ For all times t and s, for all pairs of states (i, j)

Notation and homogeneity

- **Notation**
  - $X[s:t]$ state values for all times $s \leq u \leq t$, includes borders
  - $X(s:t)$ values for all times $s < u < t$, borders excluded
  - $X(s:t]$ values for all times $s < u \leq t$, exclude left, include right
  - $X[s:t)$ values for all times $s \leq u < t$, include left, exclude right
- Homogeneous CTMC if $\mathrm{P}\left(X(t+s)=j \,\middle|\, X(s)=i\right)$ invariant for all s

  $\Rightarrow$ We restrict consideration to homogeneous CTMCs
- Still need $P_{ij}(t) := \mathrm{P}\left(X(t+s)=j \,\middle|\, X(s)=i\right)$ for all t and pairs (i,j)

  $\Rightarrow$ $P_{ij}(t)$ is known as the transition probability function. More later
- Markov property and homogeneity make description somewhat simpler

Transition times

- T_i = time until transition out of state i into any other state j
- **Def:** T_i is a random variable called transition time with ccdf

$$\mathrm{P}\left(T_i > t\right) = \mathrm{P}\left(X(0:t] = i \,\middle|\, X(0) = i\right)$$

- Probability of $T_i > t + s$ given that $T_i > s$? Use cdf expression

$$\begin{aligned}
\mathrm{P}\left(T_i > t+s \,\middle|\, T_i > s\right) &= \mathrm{P}\left(X(0:t+s] = i \,\middle|\, X[0:s] = i\right) \\
&= \mathrm{P}\left(X(s:t+s] = i \,\middle|\, X[0:s] = i\right) \\
&= \mathrm{P}\left(X(s:t+s] = i \,\middle|\, X(s) = i\right) \\
&= \mathrm{P}\left(X(0:t] = i \,\middle|\, X(0) = i\right)
\end{aligned}$$

- Used that $X[0:s] = i$ given, Markov property, and homogeneity
- From definition of T_i $\Rightarrow \mathrm{P}\left(T_i > t+s \,\middle|\, T_i > s\right) = \mathrm{P}\left(T_i > t\right)$

  $\Rightarrow$ Transition times are exponential random variables

Alternative definition

- Exponential transition times is a fundamental property of CTMCs

  $\Rightarrow$ Can be used as "algorithmic" definition of CTMCs

- Continuous-time random process $X(t)$ is a CTMC if

  (a) Transition times T_i are exponential random variables with mean $1/\nu_i$

  (b) When they occur, transition from state i to j with probability P_{ij}

$$\sum_{j=1}^{\infty} P_{ij} = 1, \qquad P_{ii} = 0$$

  (c) Transition times T_i and transitioned state j are independent

- Define matrix $\mathbf{P}$ grouping transition probabilities P_{ij}

- CTMC states evolve as in a discrete-time Markov chain

  $\Rightarrow$ State transitions occur at exponential intervals $T_i \sim \exp(\nu_i)$

  $\Rightarrow$ As opposed to occurring at fixed intervals

Embedded discrete-time Markov chain

- Consider a CTMC with transition matrix **P** and rates ν_i
- **Def:** CTMC's embedded discrete-time MC has transition matrix **P**
- Transition probabilities **P** describe a discrete-time MC
  - $\Rightarrow$ No self-transitions ($P_{ii} = 0$, **P**'s diagonal null)
  - $\Rightarrow$ Can use underlying discrete-time MCs to study CTMCs
- **Def:** State j accessible from i if accessible in the embedded MC
- **Def:** States i and j communicate if they do so in the embedded MC
  - $\Rightarrow$ Communication is a class property
- Recurrence, transience, ergodicity. Class properties . . . More later

Transition rates

- Expected value of transition time T_i is $\mathbb{E}[T_i] = 1/\nu_i$
  - $\Rightarrow$ Can interpret ν_i as the rate of transition out of state i
  - $\Rightarrow$ Of these transitions, a fraction P_{ij} are into state j
- **Def:** Transition rate from i to j is $q_{ij} := \nu_i P_{ij}$
- Transition rates offer yet another specification of CTMCs
- If q_{ij} are given can recover ν_i as

$$\nu_i = \nu_i \sum_{j=1}^{\infty} P_{ij} = \sum_{j=1}^{\infty} \nu_i P_{ij} = \sum_{j=1}^{\infty} q_{ij}$$

- Can also recover P_{ij} as $\Rightarrow P_{ij} = q_{ij}/\nu_i = q_{ij}\left(\sum_{j=1}^{\infty} q_{ij}\right)^{-1}$

Birth and death process example

- State $X(t) = 0, 1, \ldots$ Interpret as number of individuals
- Birth and deaths occur at state-dependent rates. When $X(t) = i$
- Births $\Rightarrow$ Individuals added at exponential times with mean $1/\lambda_i$
  $\Rightarrow$ Birth or arrival rate $= \lambda_i$ births per unit of time
- Deaths $\Rightarrow$ Individuals removed at exponential times with rate $1/\mu_i$
  $\Rightarrow$ Death or departure rate $= \mu_i$ deaths per unit of time
- Birth and death times are independent
- Birth and death (BD) processes are then CTMCs

Transition times and probabilities

- Q: Transition times T_i? Leave state $i \neq 0$ when birth or death occur
- If T_B and T_D are times to next birth and death, $T_i = \min(T_B, T_D)$

  $\Rightarrow$ Since T_B and T_D are exponential, so is T_i with rate

$$\nu_i = \lambda_i + \mu_i$$

- When leaving state i can go to $i+1$ (birth first) or $i-1$ (death first)

  $\Rightarrow$ Birth occurs before death with probability $\dfrac{\lambda_i}{\lambda_i + \mu_i} = P_{i,i+1}$

  $\Rightarrow$ Death occurs before birth with probability $\dfrac{\mu_i}{\lambda_i + \mu_i} = P_{i,i-1}$

- Leave state 0 only if a birth occurs, then

$$\nu_0 = \lambda_0, \qquad P_{01} = 1$$

  $\Rightarrow$ If CTMC leaves 0, goes to 1 with probability 1

  $\Rightarrow$ Might not leave 0 if $\lambda_0 = 0$ (e.g., to model extinction)

Transition rates

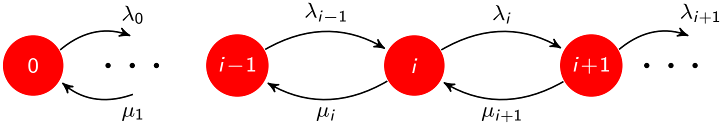
- Rate of transition from i to $i+1$ is (recall definition $q_{ij} = \nu_i P_{ij}$)

$$q_{i,i+1} = \nu_i P_{i,i+1} = (\lambda_i + \mu_i)\frac{\lambda_i}{\lambda_i + \mu_i} = \lambda_i$$

- Likewise, rate of transition from i to $i-1$ is

$$q_{i,i-1} = \nu_i P_{i,i-1} = (\lambda_i + \mu_i)\frac{\mu_i}{\lambda_i + \mu_i} = \mu_i$$

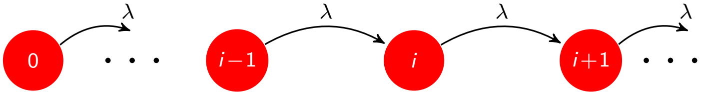
- For $i = 0 \Rightarrow q_{01} = \nu_0 P_{01} = \lambda_0$



- Somewhat more natural representation. Similar to discrete-time MCs

Poisson process example

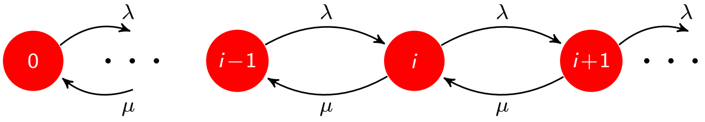
- A Poisson process is a BD process with $\lambda_i = \lambda$ and $\mu_i = 0$ constant
- State $N(t)$ counts the total number of events (arrivals) by time t
  - $\Rightarrow$ Arrivals occur a rate of λ per unit time
  - $\Rightarrow$ Transition times are the i.i.d. exponential interarrival times



- The Poisson process is a CTMC

M/M/1 queue example

- An M/M/1 queue is a BD process with $\lambda_i = \lambda$ and $\mu_i = \mu$ constant
- State $Q(t)$ is the number of customers in the system at time t
  - $\Rightarrow$ Customers arrive for service at a rate of λ per unit time
  - $\Rightarrow$ They are serviced at a rate of μ customers per unit time



- The M/M is for Markov arrivals/Markov departures
  - $\Rightarrow$ Implies a Poisson arrival process, exponential services times
  - $\Rightarrow$ The 1 is because there is only one server

Transition probability function

Transition probability function

- Two equivalent ways of specifying a CTMC

1) Transition time averages $1/\nu_i$ + transition probabilities P_{ij}
   - $\Rightarrow$ Easier description
   - $\Rightarrow$ Typical starting point for CTMC modeling

2) Transition probability function $P_{ij}(t) := \text{P}\left(X(t+s) = j \,\middle|\, X(s) = i\right)$
   - $\Rightarrow$ More complete description for all $t \geq 0$
   - $\Rightarrow$ Similar in spirit to P_{ij}^n for discrete-time Markov chains

- Goal: compute $P_{ij}(t)$ from transition times and probabilities
   - $\Rightarrow$ Notice two obvious properties $P_{ij}(0) = 0$, $P_{ii}(0) = 1$

Roadmap to determine $P_{ij}(t)$

- Goal is to obtain a differential equation whose solution is $P_{ij}(t)$
  $\Rightarrow$ Study change in $P_{ij}(t)$ when time changes slightly
- Separate in two subproblems (divide and conquer)
  $\Rightarrow$ Transition probabilities for small time h, $P_{ij}(h)$
  $\Rightarrow$ Transition probabilities in $t+h$ as function of those in t and h
- We can combine both results in two different ways

1) Jump from 0 to t then to $t+h$ $\Rightarrow$ Process runs a little longer
   $\Rightarrow$ Changes where the process is going to $\Rightarrow$ Forward equations

2) Jump from 0 to h then to $t+h$ $\Rightarrow$ Process starts a little later
   $\Rightarrow$ Changes where the process comes from $\Rightarrow$ Backward equations

Transition probability in infinitesimal time

Theorem

The transition probability functions $P_{ii}(t)$ and $P_{ij}(t)$ satisfy the following limits as t approaches 0

$$\lim_{t\to 0} \frac{P_{ij}(t)}{t} = q_{ij}, \qquad \lim_{t\to 0} \frac{1 - P_{ii}(t)}{t} = \nu_i$$

- Since $P_{ij}(0) = 0$, $P_{ii}(0) = 1$ above limits are derivatives at $t = 0$

$$\left.\frac{\partial P_{ij}(t)}{\partial t}\right|_{t=0} = q_{ij}, \qquad \left.\frac{\partial P_{ii}(t)}{\partial t}\right|_{t=0} = -\nu_i$$

- Limits also imply that for small h (recall Taylor series)

$$P_{ij}(h) = q_{ij}h + o(h), \qquad P_{ii}(h) = 1 - \nu_i h + o(h)$$

- Transition rates q_{ij} are "instantaneous transition probabilities"
  - $\Rightarrow$ Transition probability coefficient for small time h

Probability of event in infinitesimal time (reminder)

- Q: Probability of an event happening in infinitesimal time h?
- Want $\mathrm{P}(T < h)$ for small h

$$\mathrm{P}(T < h) = \int_0^h \lambda e^{-\lambda t}\, dt \approx \lambda h$$

$\Rightarrow$ Equivalent to $\left.\frac{\partial \mathrm{P}(T < t)}{\partial t}\right|_{t=0} = \lambda$

- Sometimes also write $\mathrm{P}(T < h) = \lambda h + o(h)$

$\Rightarrow$ $o(h)$ implies $\lim_{h \to 0} \frac{o(h)}{h} = 0$

$\Rightarrow$ Read as "negligible with respect to h"

- Q: Two independent events in infinitesimal time h?

$$\mathrm{P}(T_1 \leq h, T_2 \leq h) \approx (\lambda_1 h)(\lambda_2 h) = \lambda_1 \lambda_2 h^2 = o(h)$$

Transition probability in infinitesimal time (proof)

Proof.

- Consider a small time h, and recall $T_i \sim \exp(\nu_i)$
- Since $1 - P_{ii}(h)$ is the probability of transitioning out of state i

$$1 - P_{ii}(h) = \mathrm{P}\left(T_i < h\right) = \nu_i h + o(h)$$

$\Rightarrow$ Divide by h and take limit to establish the second identity

- For $P_{ij}(t)$ notice that since two or more transitions have $o(h)$ prob.

$$P_{ij}(h) = \mathrm{P}\left(X(h) = j \,\middle|\, X(0) = i\right) = P_{ij}\mathrm{P}\left(T_i < h\right) + o(h)$$

- Again, since T_i is exponential $\mathrm{P}\left(T_i < h\right) = \nu_i h + o(h)$. Then

$$P_{ij}(h) = \nu_i P_{ij} h + o(h) = q_{ij} h + o(h)$$

$\Rightarrow$ Divide by h and take limit to establish the first identity $\square$

Chapman-Kolmogorov equations

Theorem

For all times s and t the transition probability functions $P_{ij}(t+s)$ are obtained from $P_{ik}(t)$ and $P_{kj}(s)$ as

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s)$$

- As for discrete-time MCs, to go from i to j in time $t+s$
  - $\Rightarrow$ Go from i to some state k in time t $\Rightarrow P_{ik}(t)$
  - $\Rightarrow$ In the remaining time s go from k to j $\Rightarrow P_{kj}(s)$
  - $\Rightarrow$ Sum over all possible intermediate states k

Chapman-Kolmogorov equations (proof)

Proof.

$$
\begin{aligned}
P_{ij}(t+s) & \\
&= \mathrm{P}\left(X(t+s)=j \,\middle|\, X(0)=i\right) && \text{Definition of } P_{ij}(t+s) \\
&= \sum_{k=0}^{\infty} \mathrm{P}\left(X(t+s)=j \,\middle|\, X(t)=k, X(0)=i\right) \mathrm{P}\left(X(t)=k \,\middle|\, X(0)=i\right) \\
& && \text{Law of total probability} \\
&= \sum_{k=0}^{\infty} \mathrm{P}\left(X(t+s)=j \,\middle|\, X(t)=k\right) P_{ik}(t) && \text{Markov property of CTMC} \\
& && \text{and definition of } P_{ik}(t) \\
&= \sum_{k=0}^{\infty} P_{kj}(s) P_{ik}(t) && \text{Definition of } P_{kj}(s)
\end{aligned}
$$

□

Combining both results

- Let us combine the last two results to express $P_{ij}(t+h)$
- Use Chapman-Kolmogorov's equations for $0 \to t \to h$

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(h) = P_{ij}(t)P_{jj}(h) + \sum_{k=0,k\neq j}^{\infty} P_{ik}(t)P_{kj}(h)$$

- Substitute infinitesimal time expressions for $P_{jj}(h)$ and $P_{kj}(h)$

$$P_{ij}(t+h) = P_{ij}(t)(1-\nu_j h) + \sum_{k=0,k\neq j}^{\infty} P_{ik}(t)q_{kj}h + o(h)$$

- Subtract $P_{ij}(t)$ from both sides and divide by h

$$\frac{P_{ij}(t+h) - P_{ij}(t)}{h} = -\nu_j P_{ij}(t) + \sum_{k=0,k\neq j}^{\infty} P_{ik}(t)q_{kj} + \frac{o(h)}{h}$$

- Right-hand side equals a "derivative" ratio. Let $h \to 0$ to prove ...

Kolmogorov's forward equations

Theorem

The transition probability functions $P_{ij}(t)$ of a CTMC satisfy the system of differential equations (for all pairs i, j)

$$\frac{\partial P_{ij}(t)}{\partial t} = \sum_{k=0, k\neq j}^{\infty} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t)$$

- Interpret each summand in Kolmogorov's forward equations
  - $\partial P_{ij}(t)/\partial t$ = rate of change of $P_{ij}(t)$
  - $q_{kj} P_{ik}(t)$ = (transition into k in $0 \to t$) $\times$
    (rate of moving into j in next instant)
  - $\nu_j P_{ij}(t)$ = (transition into j in $0 \to t$) $\times$
    (rate of leaving j in next instant)
- Change in $P_{ij}(t) = \sum_k$ (moving into j from k) $-$ (leaving j)
- Kolmogorov's forward equations valid in most cases, but not always

Kolmogorov's backward equations

- For forward equations used Chapman-Kolmogorov's for $0 \to t \to h$
- For backward equations we use $0 \to h \to t$ to express $P_{ij}(t+h)$ as

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(h)P_{kj}(t) = P_{ii}(h)P_{ij}(t) + \sum_{k=0, k\neq i}^{\infty} P_{ik}(h)P_{kj}(t)$$

- Substitute infinitesimal time expression for $P_{ii}(h)$ and $P_{ik}(h)$

$$P_{ij}(t+h) = (1-\nu_i h)P_{ij}(t) + \sum_{k=0, k\neq i}^{\infty} q_{ik} h P_{kj}(t) + o(h)$$

- Subtract $P_{ij}(t)$ from both sides and divide by h

$$\frac{P_{ij}(t+h) - P_{ij}(t)}{h} = -\nu_i P_{ij}(t) + \sum_{k=0, k\neq i}^{\infty} q_{ik} P_{kj}(t) + \frac{o(h)}{h}$$

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$$\frac{\partial P_{ij}(t)}{\partial t} = \sum_{k=0, k\neq i}^{\infty} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$$

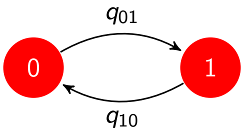
- Interpret each summand in Kolmogorov's backward equations
  - $\partial P_{ij}(t)/\partial t$ = rate of change of $P_{ij}(t)$
  - $q_{ik} P_{kj}(t)$ = (transition into j in $h \to t$) $\times$ (rate of transition into k in initial instant)
  - $\nu_i P_{ij}(t)$ = (transition into j in $h \to t$) $\times$ (rate of leaving i in initial instant)
- Forward equations $\Rightarrow$ change in $P_{ij}(t)$ if finish h later
- Backward equations $\Rightarrow$ change in $P_{ij}(t)$ if start h earlier
- Where process goes (forward) vs. where process comes from (backward)

Determination of transition probability function

A CTMC with two states

Ex: Simplest possible CTMC has only two states. Say 0 and 1

- Transition rates are q_{01} and q_{10}
- Given q_{01} and q_{10} can find rates of transitions out of $\{0, 1\}$



$$\nu_0 = \sum_j q_{0j} = q_{01}, \qquad \nu_1 = \sum_j q_{1j} = q_{10}$$

- Use Kolmogorov's equations to find transition probability functions

$$P_{00}(t), \quad P_{01}(t), \quad P_{10}(t), \quad P_{11}(t)$$

- Transition probabilities out of each state sum up to one

$$P_{00}(t) + P_{01}(t) = 1, \qquad P_{10}(t) + P_{11}(t) = 1$$

Kolmogorov's forward equations

- Kolmogorov's forward equations (process runs a little longer)

$$P'_{ij}(t) = \sum_{k=0, k\neq j}^{\infty} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t)$$

- For the two state CTMC

$$P'_{00}(t) = q_{10} P_{01}(t) - \nu_0 P_{00}(t), \qquad P'_{01}(t) = q_{01} P_{00}(t) - \nu_1 P_{01}(t)$$
$$P'_{10}(t) = q_{10} P_{11}(t) - \nu_0 P_{10}(t), \qquad P'_{11}(t) = q_{01} P_{10}(t) - \nu_1 P_{11}(t)$$

- Probabilities out of 0 sum up to 1 $\Rightarrow$ eqs. in first row are equivalent
- Probabilities out of 1 sum up to 1 $\Rightarrow$ eqs. in second row are equivalent

  $\Rightarrow$ Pick the equations for $P'_{00}(t)$ and P'_{11}(t)

Solution of forward equations

- Use $\Rightarrow$ Relation between transition rates: $\nu_0 = q_{01}$ and $\nu_1 = q_{10}$
  $\Rightarrow$ Probs. sum 1: $P_{01}(t) = 1 - P_{00}(t)$ and $P_{10}(t) = 1 - P_{11}(t)$

$$P'_{00}(t) = q_{10}\big[1 - P_{00}(t)\big] - q_{01}P_{00}(t) = q_{10} - (q_{10} + q_{01})P_{00}(t)$$
$$P'_{11}(t) = q_{01}\big[1 - P_{11}(t)\big] - q_{10}P_{11}(t) = q_{01} - (q_{10} + q_{01})P_{11}(t)$$

- Can obtain exact same pair of equations from backward equations
- First-order linear differential equations $\Rightarrow$ Solutions are exponential
- For $P_{00}(t)$ propose candidate solution (just differentiate to check)

$$P_{00}(t) = \frac{q_{10}}{q_{10} + q_{01}} + ce^{-(q_{10}+q_{01})t}$$

$\Rightarrow$ To determine c use initial condition $P_{00}(0) = 1$

Solution of forward equations (continued)

- Evaluation of candidate solution at initial condition $P_{00}(0) = 1$ yields

$$1 = \frac{q_{10}}{q_{10} + q_{01}} + c \Rightarrow c = \frac{q_{01}}{q_{10} + q_{01}}$$

- Finally transition probability function $P_{00}(t)$

$$P_{00}(t) = \frac{q_{10}}{q_{10} + q_{01}} + \frac{q_{01}}{q_{10} + q_{01}} e^{-(q_{10}+q_{01})t}$$

- Repeat for $P_{11}(t)$. Same exponent, different constants

$$P_{11}(t) = \frac{q_{01}}{q_{10} + q_{01}} + \frac{q_{10}}{q_{10} + q_{01}} e^{-(q_{10}+q_{01})t}$$

- As time goes to infinity, $P_{00}(t)$ and $P_{11}(t)$ converge exponentially
  - $\Rightarrow$ Convergence rate depends on magnitude of $q_{10} + q_{01}$

Convergence of transition probabilities

- Recall $P_{01}(t) = 1 - P_{00}(t)$ and $P_{10}(t) = 1 - P_{11}(t)$
- Limiting (steady-state) probabilities are

$$\lim_{t\to\infty} P_{00}(t) = \frac{q_{10}}{q_{10}+q_{01}}, \qquad \lim_{t\to\infty} P_{01}(t) = \frac{q_{01}}{q_{10}+q_{01}}$$
$$\lim_{t\to\infty} P_{11}(t) = \frac{q_{01}}{q_{10}+q_{01}}, \qquad \lim_{t\to\infty} P_{10}(t) = \frac{q_{10}}{q_{10}+q_{01}}$$

- Limit distribution exists and is independent of initial condition

  $\Rightarrow$ Compare across diagonals

Kolmogorov's forward equations in matrix form

- Restrict attention to finite CTMCs with N states
  - $\Rightarrow$ Define matrix $\mathbf{R} \in \mathbb{R}^{N \times N}$ with elements $r_{ij} = q_{ij}$, $r_{ii} = -\nu_i$
- Rewrite Kolmogorov's forward eqs. as (process runs a little longer)

$$P_{ij}^{'}(t) = \sum_{k=1, k \neq j}^{N} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t) = \sum_{k=1}^{N} r_{kj} P_{ik}(t)$$

- Right-hand side defines elements of a matrix product

$$r_{1j}P_{i1}(t) \qquad r_{kj}P_{ik}(t) \qquad r_{Nj}P_{iN}(t)$$

$$\begin{pmatrix} r_{11} & \cdot & r_{1j} & \cdot & r_{1N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{k1} & \cdot & r_{kj} & \cdot & r_{kN} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{N1} & \cdot & r_{Nj} & \cdot & r_{NN} \end{pmatrix} = \mathbf{R}$$

$$\mathbf{P}(t) = \begin{pmatrix} P_{11}(t) & \cdot & P_{1k}(t) & \cdot & P_{1N}(t) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{i1}(t) & \cdot & P_{ik}(t) & \cdot & P_{iN}(t) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{N1}(t) & \cdot & P_{Nk}(t) & \cdot & P_{JN}(t) \end{pmatrix} \quad \begin{pmatrix} s_{11} & \cdot & s_{1j} & \cdot & s_{1N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{i1} & \cdot & s_{ij} & \cdot & s_{iN} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{N1} & \cdot & s_{Nk} & \cdot & s_{NN} \end{pmatrix} = \mathbf{P}(t)\mathbf{R} = \mathbf{P}^{'}(t)$$

Kolmogorov's backward equations in matrix form

- Similarly, Kolmogorov's backward eqs. (process starts a little later)

$$P'_{ij}(t) = \sum_{k=1, k\neq i}^{N} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t) = \sum_{k=1}^{N} r_{ik} P_{kj}(t)$$

- Right-hand side also defines a matrix product

$$\begin{pmatrix} P_{11}(t) & \cdot & P_{1j}(t) & \cdot & P_{1N}(t) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{k1}(t) & \cdot & P_{kj}(t) & \cdot & P_{kN}(t) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{N1}(t) & \cdot & P_{Nj}(t) & \cdot & P_{NN}(t) \end{pmatrix} = \mathbf{P}(t)$$

$r_{i1}P_{1j}(t)$, $r_{ik}P_{kj}(t)$, $r_{iN}P_{Nj}(t)$

$$\mathbf{R} = \begin{pmatrix} r_{11} & \cdot & r_{1k} & \cdot & r_{1N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{i1} & \cdot & r_{ik} & \cdot & r_{iN} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{N1} & \cdot & r_{Nk} & \cdot & r_{JN} \end{pmatrix} \quad \begin{pmatrix} s_{11} & \cdot & s_{1j} & \cdot & s_{1N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{i1} & \cdot & s_{ij} & \cdot & s_{iN} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{N1} & \cdot & s_{Nk} & \cdot & s_{NN} \end{pmatrix} = \mathbf{RP}(t) = \mathbf{P}'(t)$$

Kolmogorov's equations in matrix form

- Matrix form of Kolmogorov's forward equation $\Rightarrow \mathbf{P}'(t) = \mathbf{P}(t)\mathbf{R}$
- Matrix form of Kolmogorov's backward equation $\Rightarrow \mathbf{P}'(t) = \mathbf{R}\mathbf{P}(t)$
  - $\Rightarrow$ More similar than apparent
  - $\Rightarrow$ But not equivalent because matrix product not commutative
- Notwithstanding both equations have to accept the same solution

Matrix exponential

- Kolmogorov's equations are first-order linear differential equations
  - $\Rightarrow$ They are coupled, $P'_{ij}(t)$ depends on $P_{kj}(t)$ for all k
  - $\Rightarrow$ Accepts exponential solution $\Rightarrow$ Define matrix exponential
- **Def:** The matrix exponential $e^{\mathbf{A}t}$ of matrix $\mathbf{A}t$ is the series

$$e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{(\mathbf{A}t)^n}{n!} = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2} + \frac{(\mathbf{A}t)^3}{2 \times 3} + \ldots$$

- Derivative of matrix exponential with respect to t

$$\frac{\partial e^{\mathbf{A}t}}{\partial t} = \mathbf{0} + \mathbf{A} + \mathbf{A}^2 t + \frac{\mathbf{A}^3 t^2}{2} + \ldots = \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2} + \ldots\right) = \mathbf{A}e^{\mathbf{A}t}$$

- Putting $\mathbf{A}$ on right side of product shows that $\Rightarrow \dfrac{\partial e^{\mathbf{A}t}}{\partial t} = e^{\mathbf{A}t}\mathbf{A}$

Solution of Kolmogorov's equations

- Propose solution of the form $\mathbf{P}(t) = e^{\mathbf{R}t}$
- $\mathbf{P}(t)$ solves backward equations, since derivative is

$$\frac{\partial \mathbf{P}(t)}{\partial t} = \frac{\partial e^{\mathbf{R}t}}{\partial t} = \mathbf{R}e^{\mathbf{R}t} = \mathbf{R}\mathbf{P}(t)$$

- It also solves forward equations

$$\frac{\partial \mathbf{P}(t)}{\partial t} = \frac{\partial e^{\mathbf{R}t}}{\partial t} = e^{\mathbf{R}t}\mathbf{R} = \mathbf{P}(t)\mathbf{R}$$

- Notice that $\mathbf{P}(0) = \mathbf{I}$, as it should ($P_{ii}(0) = 1$, and $P_{ij}(0) = 0$)

Computing the matrix exponential

- Suppose $\mathbf{A} \in \mathbb{R}^{n \times n}$ is diagonalizable, i.e., $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$
  - $\Rightarrow$ Diagonal matrix $\mathbf{D} = \text{diag}(\lambda_1, \ldots, \lambda_n)$ collects eigenvalues λ_i
  - $\Rightarrow$ Matrix $\mathbf{U}$ has the corresponding eigenvectors as columns
- We have the following neat identity

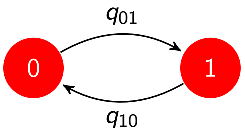
$$e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{(\mathbf{U}\mathbf{D}\mathbf{U}^{-1}t)^n}{n!} = \mathbf{U}\left(\sum_{n=0}^{\infty} \frac{(\mathbf{D}t)^n}{n!}\right)\mathbf{U}^{-1} = \mathbf{U}e^{\mathbf{D}t}\mathbf{U}^{-1}$$

- But since $\mathbf{D}$ is diagonal, then

$$e^{\mathbf{D}t} = \sum_{n=0}^{\infty} \frac{(\mathbf{D}t)^n}{n!} = \begin{pmatrix} e^{\lambda_1 t} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{\lambda_n t} \end{pmatrix}$$

Two state CTMC example

Ex: Simplest CTMC with two states 0 and 1



- Transition rates are $q_{01} = 3$ and $q_{10} = 1$
- Recall transition time rates are $\nu_0 = q_{01} = 3$, $\nu_1 = q_{10} = 1$, hence

$$\mathbf{R} = \begin{pmatrix} -\nu_0 & q_{01} \\ q_{10} & -\nu_1 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$$

- Eigenvalues of $\mathbf{R}$ are $0, -4$, eigenvectors $[1, 1]^T$ and $[-3, 1]^T$. Thus

$$\mathbf{U} = \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{U}^{-1} = \begin{pmatrix} 1/4 & 3/4 \\ -1/4 & 1/1 \end{pmatrix}, \quad e^{\mathbf{D}t} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-4t} \end{pmatrix}$$

- The solution to the forward equations is

$$\mathbf{P}(t) = e^{\mathbf{R}t} = \mathbf{U}e^{\mathbf{D}t}\mathbf{U}^{-1} = \begin{pmatrix} 1/4 + (3/4)e^{-4t} & 3/4 - (3/4)e^{-4t} \\ 1/4 - (1/4)e^{-4t} & 3/4 + (1/4)e^{-4t} \end{pmatrix}$$

Unconditional probabilities

- $\mathbf{P}(t)$ is transition prob. from states at time 0 to states at time t
- Define unconditional probs. at time t, $p_j(t) := \text{P}\left(X(t) = j\right)$
  - $\Rightarrow$ Group in vector $\mathbf{p}(t) = [p_1(t), p_2(t), \ldots, p_j(t), \ldots]^T$
- Given initial distribution $\mathbf{p}(0)$, find $p_j(t)$ conditioning on initial state

$$p_j(t) = \sum_{i=0}^{\infty} \text{P}\left(X(t) = j \,\middle|\, X(0) = i\right) \text{P}\left(X(0) = i\right) = \sum_{i=0}^{\infty} P_{ij}(t) p_i(0)$$

- Using compact matrix-vector notation $\Rightarrow \mathbf{p}(t) = \mathbf{P}^T(t)\mathbf{p}(0)$
  - $\Rightarrow$ Compare with discrete-time MC $\Rightarrow \mathbf{p}(n) = \left(\mathbf{P}^n\right)^T \mathbf{p}(0)$

Limit probabilities and ergodicity

Continuous-time Markov chains

Transition probability function

Determination of transition probability function

Limit probabilities and ergodicity

Recurrent and transient states

- Recall the embedded discrete-time MC associated with any CTMC
  - ⇒ Transition probs. of MC form the matrix **P** of the CTMC
  - ⇒ No self transitions ($P_{ii} = 0$, **P**'s diagonal null)
- States $i \leftrightarrow j$ communicate in the CTMC if $i \leftrightarrow j$ in the MC
  - ⇒ Communication partitions MC in classes
  - ⇒ Induces CTMC partition as well
- **Def:** CTMC is irreducible if embedded MC contains a single class
- State i is recurrent if it is recurrent in the embedded MC
  - ⇒ Likewise, define transience and positive recurrence for CTMCs
- Transience and recurrence shared by elements of a MC class
  - ⇒ Transience and recurrence are class properties of CTMCs
- Periodicity not possible in CTMCs

Limiting probabilities

Theorem

Consider irreducible, positive recurrent CTMC with transition rates ν_i and q_{ij}. Then, $\lim_{t\to\infty} P_{ij}(t)$ exists and is independent of the initial state i, i.e.,

$$P_j = \lim_{t\to\infty} P_{ij}(t) \quad \textit{exists for all } (i,j)$$

Furthermore, steady-state probabilities $P_j \geq 0$ are the unique nonnegative solution of the system of linear equations

$$\nu_j P_j = \sum_{k=0,k\neq j}^{\infty} q_{kj} P_k, \qquad \sum_{j=0}^{\infty} P_j = 1$$

- Limit distribution exists and is independent of initial condition
  - $\Rightarrow$ Obtained as solution of system of linear equations
  - $\Rightarrow$ Like discrete-time MCs, but equations slightly different

Algebraic relation to determine limit probabilities

- As with MCs difficult part is to prove that $P_j = \lim_{t\to\infty} P_{ij}(t)$ exists
- Algebraic relations obtained from Kolmogorov's forward equations

$$\frac{\partial P_{ij}(t)}{\partial t} = \sum_{k=0,k\neq j}^{\infty} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t)$$

- If limit distribution exists we have, independent of initial state i

$$\lim_{t\to\infty} \frac{\partial P_{ij}(t)}{\partial t} = 0, \qquad \lim_{t\to\infty} P_{ij}(t) = P_j$$

- Considering the limit of Kolomogorov's forward equations yields

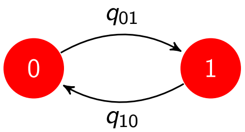
$$0 = \sum_{k=0,k\neq j}^{\infty} q_{kj} P_k - \nu_j P_j$$

- Reordering terms the limit distribution equations follow

Two state CTMC example

Ex: Simplest CTMC with two states 0 and 1

- Transition rates are q_{01} and q_{10}



- From transition rates find mean transition times $\nu_0 = q_{01}$, $\nu_1 = q_{10}$
- Stationary distribution equations

$$\nu_0 P_0 = q_{10} P_1, \qquad \nu_1 P_1 = q_{01} P_0, \qquad P_0 + P_1 = 1,$$
$$q_{01} P_0 = q_{10} P_1, \qquad q_{10} P_1 = q_{01} P_0$$

- Solution yields $\Rightarrow P_0 = \dfrac{q_{10}}{q_{10}+q_{01}}, \qquad P_1 = \dfrac{q_{01}}{q_{10}+q_{01}}$
- Larger rate q_{10} of entering 0 $\Rightarrow$ Larger prob. P_0 of being at 0
- Larger rate q_{01} of entering 1 $\Rightarrow$ Larger prob. P_1 of being at 1

Ergodicity

- **Def:** Fraction of time $T_i(t)$ spent in state i by time t

$$T_i(t) := \frac{1}{t}\int_0^t \mathbb{I}\{X(\tau) = i\} d\tau$$

$\Rightarrow$ $T_i(t)$ a time/ergodic average, $\lim_{t\to\infty} T_i(t)$ is an ergodic limit

- If CTMC is irreducible, positive recurrent, the ergodic theorem holds

$$P_i = \lim_{t\to\infty} T_i(t) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{I}\{X(\tau) = i\} d\tau \qquad \text{a.s.}$$

- Ergodic limit coincides with limit probabilities (almost surely)

Function's ergodic limit

- Consider function $f(i)$ associated with state i. Can write $f\big(X(t)\big)$ as

$$f\big(X(t)\big) = \sum_{i=1}^{\infty} f(i)\mathbb{I}\left\{X(t) = i\right\}$$

- Consider the time average of $f\big(X(t)\big)$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f\big(X(\tau)\big)d\tau = \lim_{t\to\infty} \frac{1}{t}\int_0^t \sum_{i=1}^{\infty} f(i)\mathbb{I}\left\{X(\tau) = i\right\}d\tau$$

- Interchange summation with integral and limit to say

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f\big(X(\tau)\big)d\tau = \sum_{i=1}^{\infty} f(i) \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{I}\left\{X(\tau) = i\right\}d\tau = \sum_{i=1}^{\infty} f(i)P_i$$

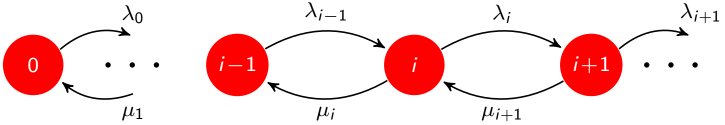
- Function's ergodic limit = Function's expectation under limiting dist.

Limit distribution equations as balance equations

- Recall limit distribution equations $\Rightarrow \nu_j P_j = \displaystyle\sum_{k=0,k\neq j}^{\infty} q_{kj} P_k$
- P_j = fraction of time spent in state j
- ν_j = rate of transition out of state j given CTMC is in state j
  $\Rightarrow \nu_j P_j$ = rate of transition out of state j (unconditional)
- q_{kj} = rate of transition from k to j given CTMC is in state k
  $\Rightarrow q_{kj} P_k$ = rate of transition from k to j (unconditional)
  $\Rightarrow \displaystyle\sum_{k=0,k\neq j}^{\infty} q_{kj} P_k$ = rate of transition into j, from all states
- Rate of transition out of state j = Rate of transition into state j
- Balance equations $\Rightarrow$ Balance nr. of transitions in and out of state j

Limit distribution for birth and death process

- Birth/deaths occur at state-dependent rates. When $X(t) = i$
- Births $\Rightarrow$ Individuals added at exponential times with mean $1/\lambda_i$
  $\Rightarrow$ Birth rate = upward transition rate = $q_{i,i+1} = \lambda_i$
- Deaths $\Rightarrow$ Individuals removed at exponential times with mean $1/\mu_i$
  $\Rightarrow$ Death rate = downward transition rate = $q_{i,i-1} = \mu_i$
- Transition time rates $\Rightarrow \nu_i = \lambda_i + \mu_i,\ i > 0$ and $\nu_0 = \lambda_0$



- Limit distribution/balance equations: Rate out of j = Rate into j

$$(\lambda_i + \mu_i)P_i = \lambda_{i-1}P_{i-1} + \mu_{i+1}P_{i+1}$$
$$\lambda_0 P_0 = \mu_1 P_1$$

Finding solution of balance equations

- Start expressing all probabilities in terms of P_0
- Equation for P_0

$$\lambda_0 P_0 = \mu_1 P_1$$

- Sum eqs. for P_1 and P_0

$$\begin{aligned} \lambda_0 P_0 &= \mu_1 P_1 \\ (\lambda_1 + \mu_1) P_1 &= \lambda_0 P_0 + \mu_2 P_2 \end{aligned}$$

$$\lambda_1 P_1 = \mu_2 P_2$$

- Sum result and eq. for P_2

$$\begin{aligned} \lambda_1 P_1 &= \mu_2 P_2 \\ (\lambda_2 + \mu_2) P_2 &= \lambda_1 P_1 + \mu_3 P_3 \end{aligned}$$

$$\lambda_2 P_2 = \mu_3 P_3$$

$\vdots$

- Sum result and eq. for P_i

$$\begin{aligned} \lambda_{i-1} P_{i-1} &= \mu_i P_i \\ (\lambda_i + \mu_i) P_i &= \lambda_{i-1} P_{i-1} + \mu_{i+1} P_{i+1} \end{aligned}$$

$$\lambda_i P_i = \mu_{i+1} P_{i+1}$$

Finding solution of balance equations (continued)

- Recursive substitutions on red equations on the right

$$P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$P_2 = \frac{\lambda_1}{\mu_2} P_1 \quad = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0$$

$$\vdots$$

$$P_{i+1} = \frac{\lambda_i}{\mu_{i+1}} P_i = \frac{\lambda_i \lambda_{i-1} \ldots \lambda_0}{\mu_{i+1} \mu_i \ldots \mu_1} P_0$$

- To find P_0 use $\sum_{i=0}^{\infty} P_i = 1 \;\Rightarrow 1 = P_0 + \sum_{i=1}^{\infty} \frac{\lambda_i \lambda_{i-1} \ldots \lambda_0}{\mu_{i+1} \mu_i \ldots \mu_1} P_0$

$$\Rightarrow P_0 = \left[1 + \sum_{i=1}^{\infty} \frac{\lambda_i \lambda_{i-1} \ldots \lambda_0}{\mu_{i+1} \mu_i \ldots \mu_1} \right]^{-1}$$

Glossary

- Continuous-time Markov chain
- Markov property
- Time-homogeneous CTMC
- Transition probability function
- Exponential transition time
- Transition probabilities
- Embedded discrete-time MC
- Transition rates
- Birth and death process
- Poisson process
- M/M/1 queue
- Chapman-Kolmogorov equations
- Kolmogorov's forward equations
- Kolmogorov's backward equations
- Limiting probabilities
- Matrix exponential
- Unconditional probabilities
- Recurrent and transient states
- Ergodicity
- Balance equations