Arbitrages and pricing of stock options

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November 30, 2018

Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing

Arbitrage

- Bet on different events with each outcome paying a random return
- Arbitrage: possibility of devising a betting strategy that
  - $\Rightarrow$ Guarantees a positive return
  - $\Rightarrow$ No matter the combined outcome of the events
- Arbitrages often involve operating in two (or more) different markets

Sports betting example

Ex: Booker 1 $\Rightarrow$ Yankees win pays 1.5:1, Yankees loss pays 3:1

- Bet x on Yankees and y against Yankees. Guaranteed earnings?

$$\text{Yankees win:} \quad 0.5x - y > 0 \ \Rightarrow x > 2y$$
$$\text{Yankees loose:} \quad -x + 2y > 0 \Rightarrow x < 2y$$

$\Rightarrow$ Arbitrage not possible. Notice that $1/(1.5) + 1/3 = 1$

Ex: Booker 2 $\Rightarrow$ Yankees win pays 1.4:1, Yankees loss pays 3.1:1

- Bet x on Yankees and y against Yankees. Guaranteed earnings?

$$\text{Yankees win:} \quad 0.4x - y > 0 \qquad \Rightarrow x > 2.5y$$
$$\text{Yankees loose:} \quad -x + 2.1y > 0 \Rightarrow x < 2.1y$$

$\Rightarrow$ Arbitrage not possible. Notice that $1/(1.4) + 1/(3.1) > 1$

Sports betting example (continued)

- First condition on Booker 1 and second on Booker 2 are compatible

- Bet x on Yankees on Booker 1, y against Yankees on Booker 2
- Guaranteed earnings possible. Make e.g., $x = 2066$, $y = 1000$

$$\begin{aligned} \text{Yankees win:} & \quad 0.5 \times 2066 - 1000 \quad = 33 \\ \text{Yankees loose:} & \quad -2066 + 2.1 \times 1000 = 34 \end{aligned}$$

$\Rightarrow$ Arbitrage possible. Notice that $1/(1.5) + 1/(3.1) < 1$

- Sport bookies coordinate their odds to avoid arbitrage opportunities
  - $\Rightarrow$ Like card counting in casinos, arbitrage betting not illegal
  - $\Rightarrow$ But you will be banned if caught involved in such practices

- If you plan on doing this, do it on, e.g., currency exchange markets

Events, returns, and investment strategy

- Let events on which bets are posted be $k = 1, 2, \ldots, K$
- Let $j = 1, 2, \ldots, J$ index possible joint outcomes
  - Joint realizations, also called "world realization", or "world outcome"
- If world outcome is j, event k yields return r_{jk} per unit invested (bet)
- Invest (bet) x_k in event k $\Rightarrow$ return for world j is $x_k r_{jk}$

  $\Rightarrow$ Bets x_k can be positive ($x_k > 0$) or negative ($x_k < 0$)

  $\Rightarrow$ Positive = regular bet (buy). Negative = short bet (sell)
- Total earnings $\Rightarrow \displaystyle\sum_{k=1}^{K} x_k r_{jk} = \mathbf{x}^T \mathbf{r}_j$
  - Vectors of returns for outcome j $\Rightarrow \mathbf{r}_j := [r_{j1}, \ldots, r_{jK}]^T$ (given)
  - Vector of bets $\Rightarrow \mathbf{x} := [x_1, \ldots, x_K]^T$ (controlled by gambler)

Notation in the sports betting example

Ex: Booker 1 $\Rightarrow$ Yankees win pays 1.5:1, Yankees loose pays 3:1

- There are $K = 2$ events to bet on

  $\Rightarrow$ A Yankees' win ($k = 1$) and a Yankees' loss ($k = 2$)

- Naturally, there are $J = 2$ possible outcomes

  $\Rightarrow$ Yankees won ($j = 1$) and Yankess lost ($j = 2$)

- Q: What are the returns?

$$\begin{aligned} &\text{Yankees win } (j = 1): && r_{11} = 0.5, \quad r_{12} = -1 \\ &\text{Yankees loose } (j = 2): && r_{21} = -1, \quad r_{22} = 2 \end{aligned}$$

  $\Rightarrow$ Return vectors are thus $\mathbf{r}_1 = [0.5, -1]^T$ and $\mathbf{r}_2 = [-1, 2]^T$

- Bet x on Yankees and y against Yankees, vector of bets $\mathbf{x} = [x, y]^T$

Arbitrage (clearly defined now)

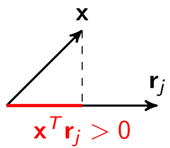
- Arbitrage is possible if there exists investment strategy $\mathbf{x}$ such that

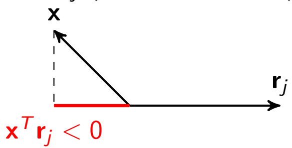
$$\mathbf{x}^T\mathbf{r}_j > 0, \quad \text{for all } j = 1, \ldots, J$$

- Equivalently, arbitrage is possible if

$$\max_{\mathbf{x}} \left(\min_j \left(\mathbf{x}^T\mathbf{r}_j \right) \right) > 0$$

- Earnings $\mathbf{x}^T\mathbf{r}_j$ are the inner product of $\mathbf{x}$ and $\mathbf{r}_j$ (i.e., $\perp$ projection)

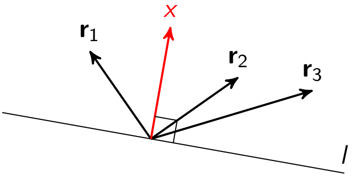




$\Rightarrow$ Positive earnings if angle between $\mathbf{x}$ and $\mathbf{r}_j < \pi/2$ (90°)

When is arbitrage possible?

- There is a line that leaves all $\mathbf{r}_j$ vectors to one side

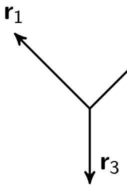


- Arbitrage possible
- Prob. vector $\mathbf{p} = [p_1, \ldots, p_J]^T$ on world outcomes such that

$$\mathbb{E}_{\mathbf{p}}(\mathbf{r}) = \sum_{j=1}^{J} p_j \mathbf{r}_j = \mathbf{0}$$

does not exist

- There is not a line that leaves all $\mathbf{r}_j$ vectors to one side



- Arbitrage not possible
- There is prob. vector $\mathbf{p} = [p_1, \ldots, p_J]^T$ on world outcomes such that

$$\mathbb{E}_{\mathbf{p}}(\mathbf{r}) = \sum_{j=1}^{J} p_j \mathbf{r}_j = \mathbf{0}$$

- Think of p_j as scaling factors

Arbitrage theorem

- Have demonstrated the following result, called arbitrage theorem
  - $\Rightarrow$ Formal proof follows from duality theory in optimization

Theorem

Given vectors of returns $\mathbf{r}_j \in \mathbb{R}^K$ associated with random world outcomes $j = 1, \ldots, J$, an arbitrage is not possible if and only if there exists a probability vector $\mathbf{p} = [p_1, \ldots, p_J]^T$ with $p_j \geq 0$ and $\mathbf{p}^T\mathbf{1} = 1$, such that $\mathbb{E}_{\mathbf{p}}(\mathbf{r}) = \mathbf{0}$. Equivalently,

$$\max_{\mathbf{x}} \left(\min_j \left(\mathbf{x}^T \mathbf{r}_j \right) \right) \leq 0 \quad \Leftrightarrow \quad \sum_{j=1}^{J} p_j \mathbf{r}_j = \mathbf{0}$$

- Prob. vector $\mathbf{p}$ is **NOT** the prob. distribution of events $j = 1, \ldots, J$

Example: Arbitrages in sports betting

Ex: Booker 1 $\Rightarrow$ Yankees win pays 1.5:1, Yankees loose pays 3:1

- There are $K = 2$ events to bet on, $J = 2$ possible outcomes
- Q: What are the returns?

$$\begin{aligned} &\text{Yankees win } (j = 1): && r_{11} = 0.5, \quad r_{12} = -1 \\ &\text{Yankees loose } (j = 2): && r_{21} = -1, \quad r_{22} = 2 \end{aligned}$$

$\Rightarrow$ Return vectors are thus $\mathbf{r}_1 = [0.5, -1]^T$ and $\mathbf{r}_2 = [-1, 2]^T$

- Arbitrage impossible if there is $0 \leq p \leq 1$ such that

$$\mathbb{E}_{\mathbf{p}}(\mathbf{r}) = p \times \begin{bmatrix} 0.5 \\ -1 \end{bmatrix} + (1 - p) \times \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \mathbf{0}$$

$\Rightarrow$ Straightforward to check that $p = 2/3$ satisfies the equation

Example: Arbitrages in geometric random walk

- Consider a stock price $X(nh)$ that follows a geometric random walk

$$X\big((n+1)h\big) = X(nh)e^{\sigma\sqrt{h}Y_n}$$

- Y_n is a binary random variable with probability distribution

$$\mathrm{P}\left(Y_n = 1\right) = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h}\right), \quad \mathrm{P}\left(Y_n = -1\right) = \frac{1}{2}\left(1 - \frac{\mu}{\sigma}\sqrt{h}\right)$$

  $\Rightarrow$ As $h \to 0$, $X(nh)$ becomes geometric Brownian motion

- Q: Are there arbitrage opportunities in trading this stock?

  $\Rightarrow$ Too general, let us consider a narrower problem

Stock flip investment strategy

- Consider the following investment strategy (stock flip):

  Buy: Buy \$1 in stock at time 0 for price $X(0)$ per unit of stock

  Sell: Sell stock at time h for price $X(h)$ per unit of stock

- Cost of transaction is \$1. Units of stock purchased are $1/X(0)$

  $\Rightarrow$ Cash after selling stock is $X(h)/X(0)$

  $\Rightarrow$ Return on investment is $X(h)/X(0) - 1$

- There are two possible outcomes for the price of the stock at time h

  $\Rightarrow$ May have $Y_0 = 1$ or $Y_0 = -1$ respectively yielding

$$X(h) = X(0)e^{\sigma\sqrt{h}}, \qquad X(h) = X(0)e^{-\sigma\sqrt{h}}$$

- Possible returns are therefore

$$r_1 = \frac{X(0)e^{\sigma\sqrt{h}}}{X(0)} - 1 = e^{\sigma\sqrt{h}} - 1, \quad r_2 = \frac{X(0)e^{-\sigma\sqrt{h}}}{X(0)} - 1 = e^{-\sigma\sqrt{h}} - 1$$

Present value of returns

- One dollar at time h is not the same as 1 dollar at time 0
  - $\Rightarrow$ Must take into account the time value of money
- Interest rate of a risk-free investment is α continuously compounded
  - $\Rightarrow$ In practice, α is the money-market rate (time value of money)
- Prices have to be compared at their present value
- The present value (at time 0) of $X(h)$ is $X(h)e^{-\alpha h}$
  - $\Rightarrow$ Return on investment is $e^{-\alpha h}X(h)/X(0) - 1$
- Present value of possible returns (whether $Y_0 = 1$ or $Y_0 = -1$) are

$$r_1 = \frac{e^{-\alpha h}X(0)e^{\sigma\sqrt{h}}}{X(0)} - 1 \quad = e^{-\alpha h}e^{\sigma\sqrt{h}} - 1,$$

$$r_2 = \frac{e^{-\alpha h}X(0)e^{-\sigma\sqrt{h}}}{X(0)} - 1 = e^{-\alpha h}e^{-\sigma\sqrt{h}} - 1$$

No arbitrage condition

- Arbitrage not possible if and only if there exists $0 \leq q \leq 1$ such that

$$qr_1 + (1-q)r_2 = 0$$

$\Rightarrow$ Arbitrage theorem in one dimension (only one bet, stock flip)

- Substituting r_1 and r_2 for their respective values

$$q\left(e^{-\alpha h}e^{\sigma\sqrt{h}} - 1\right) + (1-q)\left(e^{-\alpha h}e^{-\sigma\sqrt{h}} - 1\right) = 0$$

- Can be easily solved for q. Expanding product and reordering terms

$$qe^{-\alpha h}e^{\sigma\sqrt{h}} + (1-q)e^{-\alpha h}e^{-\sigma\sqrt{h}} = 1$$

- Multiplying by $e^{\alpha h}$ and grouping terms with a q factor

$$q\left(e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}\right) = e^{\alpha h} - e^{-\sigma\sqrt{h}}$$

No arbitrage condition (continued)

- Solving for q finally yields $\Rightarrow q = \dfrac{e^{\alpha h} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$
- For small h we have $e^{\alpha h} \approx 1 + \alpha h$ and $e^{\pm\sigma\sqrt{h}} \approx 1 \pm \sigma\sqrt{h} + \sigma^2 h/2$
- Thus, the value of q as $h \to 0$ may be approximated as

$$q \approx \frac{1 + \alpha h - \left(1 - \sigma\sqrt{h} + \sigma^2 h/2\right)}{1 + \sigma\sqrt{h} - \left(1 - \sigma\sqrt{h}\right)} = \frac{\sigma\sqrt{h} + \left(\alpha - \sigma^2/2\right) h}{2\sigma\sqrt{h}}$$
$$= \frac{1}{2}\left(1 + \frac{\alpha - \sigma^2/2}{\sigma}\sqrt{h}\right)$$

- Approximation proves that at least for small h, then $0 < q < 1$

  $\Rightarrow$ Arbitrage not possible
- Also, suspiciously similar to probabilities of geometric random walk

  $\Rightarrow$ Key observation as we'll see next

Risk neutral measure

Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing

No arbitrage condition on geometric random walk

- Stock prices $X(nh)$ follow geometric random walk (drift μ, variance σ^2)
  $\Rightarrow$ Risk-free investment has return α (time value of money)
- Arbitrage is not possible in stock flips if there is $0 \leq q \leq 1$ such that

$$q = \frac{e^{\alpha h} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$$

- Notice that q satisfies the equation (which we'll use later on)

$$qe^{\sigma\sqrt{h}} + (1-q)e^{-\sigma\sqrt{h}} = e^{\alpha h}$$

- Q: Can we have arbitrage using a more complex set of possible bets?

General investment strategy

- Consider the following general investment strategy:

  Observe: Observe the stock price at times $h, 2h, \ldots, nh$

  Compare: Is $X(h) = x_1, X(2h) = x_2, \ldots, X(nh) = x_n$?

  Buy: If above answer is yes, buy stock at price $X(nh)$

  Sell: Sell stock at time mh $(m > n)$ for price $X(mh)$

- Possible bets are the observed values of the stock $x_1, x_2, \ldots, x_n$

  $\Rightarrow$ There are 2^n possible bets

- Possible outcomes are value at time mh and observed values

  $\Rightarrow$ There are 2^m possible outcomes

Explanation of general investment strategy

- There are 2^n possible bets:
  - Bet $1 = n$ price increases in $1, \dots, n$
  - Bet $2 =$ price increases in $1, \dots, n-1$ and price decrease in n
  - ...
- For each bet we have 2^{m-n} possible outcomes:
  - $m - n$ price increases in $n+1, \dots, m$
  - Price increases in $n+1, \dots, m-1$ and price decrease in m
  - ...

	$X(h)$	$X(2h)$	$X(3h)$	$X(nh)$	$X((n+1)h)$	$X((n+2)h)$	$X(mh)$
bet 1	$e^{\sigma\sqrt{h}}$	$e^{2\sigma\sqrt{h}}$	$e^{3\sigma\sqrt{h}}$	$e^{n\sigma\sqrt{h}}$	$X(nh)e^{\sigma\sqrt{h}}$	$X(nh)e^{2\sigma\sqrt{h}}$	$X(nh)e^{m\sigma\sqrt{h}}$
bet 2	$e^{\sigma\sqrt{h}}$	$e^{2\sigma\sqrt{h}}$	$e^{3\sigma\sqrt{h}}$	$e^{(n-2)\sigma\sqrt{h}}$	$X(nh)e^{\sigma\sqrt{h}}$	$X(nh)e^{2\sigma\sqrt{h}}$	$X(nh)e^{(m-2)\sigma\sqrt{h}}$
bet 2^n	$e^{-\sigma\sqrt{h}}$	$e^{-2\sigma\sqrt{h}}$	$e^{-3\sigma\sqrt{h}}$	$e^{-n\sigma\sqrt{h}}$	$X(nh)e^{-\sigma\sqrt{h}}$	$X(nh)e^{-2\sigma\sqrt{h}}$	$X(nh)e^{-m\sigma\sqrt{h}}$

outcomes per each bet

- Table assumes $X(0) = 1$ for simplicity

Candidate no arbitrage probability measure

- Define the prob. distribution $\mathbf{q}$ over possible outcomes as follows
- Start with a sequence of i.i.d. binary RVs Y_n, probabilities

$$\mathrm{P}\left(Y_n = 1\right) = q, \quad \mathrm{P}\left(Y_n = -1\right) = 1 - q$$

$\Rightarrow$ With $q = \left(e^{\alpha h} - e^{-\sigma\sqrt{h}}\right) / \left(e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}\right)$ as in slide 18

- Joint prob. distribution $\mathbf{q}$ on $X(h), X(2h), \ldots, X\left(mh\right)$ from

$$X\left((n+1)h\right) = X(nh) e^{\sigma\sqrt{h} Y_n}$$

$\Rightarrow$ Recall this is **NOT** the prob. distribution of $X(nh)$

- Will show that expected value of earnings with respect to $\mathbf{q}$ is null
  $\Rightarrow$ By arbitrage theorem, arbitrages are not possible

Return for given outcome

- Consider a time 0 unit investment in given arbitrary outcome
- Stock units purchased depend on the price $X(nh)$ at buying time

$$\text{Units bought } = \frac{1}{X(nh)e^{-\alpha nh}}$$

$\Rightarrow$ Have corrected $X(nh)$ to express it in time 0 values

- Cash after selling stock given by price $X(mh)$ at sell time m

$$\text{Cash after sell } = \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}}$$

- Return is then $\Rightarrow r\big(X(h), \ldots, X(mh)\big) = \dfrac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1$

$\Rightarrow$ Depends on $X(mh)$ and $X(nh)$ only

Expected return with respect to measure $\mathbf{q}$

- Expected value of all possible returns with respect to $\mathbf{q}$ is

$$\mathbb{E}_{\mathbf{q}}\left[r\big(X(h),\ldots,X(mh)\big)\right] = \mathbb{E}_{\mathbf{q}}\left[\frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1\right]$$

- Condition on observed values $X(h),\ldots,X(nh)$

$$\mathbb{E}_{\mathbf{q}}\left[r\big(X(h),\ldots,X(mh)\big)\right] = \mathbb{E}_{\mathbf{q}(1:n)}\left[\mathbb{E}_{\mathbf{q}(n+1:m)}\left[\frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \,\Big|\, X(h),\ldots,X(nh)\right]\right]$$

- In innermost expectation $X(nh)$ is given. Furthermore, process X is Markov, so conditioning on $X(h),\ldots,X((n-1)h)$ is irrelevant. Thus

$$\mathbb{E}_{\mathbf{q}}\left[r\big(X(h),\ldots,X(mh)\big)\right] = \mathbb{E}_{\mathbf{q}(1:n)}\left[\frac{\mathbb{E}_{\mathbf{q}(n+1:m)}\left[X(mh)\,\big|\,X(nh)\right]e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1\right]$$

Expected value of future values (measure q)

- Need to find expectation of future value $\mathbb{E}_{\mathbf{q}(n+1:m)}\left[X(mh)\,\middle|\,X(nh)\right]$

- From recursive relation for $X(nh)$ in terms of Y_n sequence

$$\begin{aligned}
X(mh) &= X\big((m-1)h\big)e^{\sigma\sqrt{h}Y_{m-1}} \\
&= X\big((m-2)h\big)e^{\sigma\sqrt{h}Y_{m-1}}e^{\sigma\sqrt{h}Y_{m-2}} \\
&\vdots \\
&= X\big(nh\big)e^{\sigma\sqrt{h}Y_{m-1}}e^{\sigma\sqrt{h}Y_{m-2}}\ldots e^{\sigma\sqrt{h}Y_{n}}
\end{aligned}$$

- All the Y_n are independent. Then, upon taking expectations

$$\mathbb{E}_{\mathbf{q}(n+1:m)}\left[X(mh)\,\middle|\,X(nh)\right] = X\big(nh\big)\mathbb{E}\left[e^{\sigma\sqrt{h}Y_{m-1}}\right]\mathbb{E}\left[e^{\sigma\sqrt{h}Y_{m-2}}\right]\ldots\mathbb{E}\left[e^{\sigma\sqrt{h}Y_{n}}\right]$$

- Need to determine expectation of relative price change $\mathbb{E}\left[e^{\sigma\sqrt{h}Y_{n}}\right]$

Expectation of relative price change (measure **q**)

- The expected value of the relative price change $\mathbb{E}\left[e^{\sigma\sqrt{h}Y_n}\right]$ is

$$\mathbb{E}\left[e^{\sigma\sqrt{h}Y_n}\right] = e^{\sigma\sqrt{h}}\,\text{Pr}\left[Y_n = 1\right] + e^{-\sigma\sqrt{h}}\,\text{Pr}\left[Y_n = -1\right]$$

- According to definition of measure **q**, it holds

$$\text{Pr}\left[Y_n = 1\right] = q, \qquad \text{Pr}\left[Y_n = -1\right] = 1 - q$$

- Substituting in expression for $\mathbb{E}\left[e^{\sigma\sqrt{h}Y_n}\right]$

$$\mathbb{E}\left[e^{\sigma\sqrt{h}Y_n}\right] = e^{\sigma\sqrt{h}}\,q + e^{-\sigma\sqrt{h}}\,(1-q) = e^{\alpha h}$$

$\Rightarrow$ Follows from definition of probability q [cf. slide 18]

- Reweave the quilt:
  - (i) Use expected relative price change to compute expected future value
  - (ii) Use expected future value to obtain desired expected return

Reweave the quilt

- Plug $\mathbb{E}\left[e^{\sigma\sqrt{h}Y_n}\right] = e^{\alpha h}$ into expression for expected future value

$$\mathbb{E}_{\mathbf{q}(n+1:m)}\left[X(mh)\,\middle|\,X(nh)\right] = X(nh)\,e^{\alpha h}e^{\alpha h}\dots e^{\alpha h} = X(nh)\,e^{\alpha(m-n)h}$$

- Substitute result into expression for expected return

$$\mathbb{E}_{\mathbf{q}}\left[r\big(X(h),\dots,X(mh)\big)\right] = \mathbb{E}_{\mathbf{q}(1:n)}\left[\frac{X(nh)\,e^{\alpha(m-n)h}e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1\right]$$

- Exponentials cancel out, finally yielding

$$\mathbb{E}_{\mathbf{q}}\left[r\big(X(h),\dots,X(mh)\big)\right] = \mathbb{E}_{\mathbf{q}(1:n)}\left[1-1\right] = 0$$

$\Rightarrow$ Arbitrage not possible if $0 \leq q \leq 1$ exists (true for small h)

What if prices follow a geometric Brownian motion?

- Suppose stock prices follow a geometric Brownian motion, i.e.,

$$X(t) = X(0)e^{Y(t)}$$

  $\Rightarrow$ $Y(t)$ Brownian motion with drift μ and variance σ^2

- Q: What is the no arbitrage condition?

- Approximate geometric Brownian motion by geometric random walk

  $\Rightarrow$ Approximation arbitrarily accurate by letting $h \to 0$

- No arbitrage measure $\mathbf{q}$ exists for geometric random walk
  - This requires h sufficiently small
  - Notice that prob. distribution $\mathbf{q} = \mathbf{q}(h)$ is a function of h

- Existence of the prob. distribution $\mathbf{q} := \lim_{h \to 0} \mathbf{q}(h)$ proves that

  $\Rightarrow$ Arbitrages are not possible in stock trading

No arbitrage probability distribution

- Recall that as $h \to 0 \ \Rightarrow q \approx \dfrac{1}{2}\left(1 + \dfrac{\alpha - \sigma^2/2}{\sigma}\sqrt{h}\right)$

  $$\Rightarrow 1 - q \approx \frac{1}{2}\left(1 - \frac{\alpha - \sigma^2/2}{\sigma}\sqrt{h}\right)$$

- Thus, measure $\mathbf{q} := \lim_{h \to 0} \mathbf{q}(h)$ is a geometric Brownian motion

  $\Rightarrow$ Variance σ^2 (same as stock price)

  $\Rightarrow$ Drift $\alpha - \sigma^2/2$

- Measure showing arbitrage impossible a geometric Brownian motion

  $\Rightarrow$ Which is also the way stock prices evolve as $h \to 0$

- Furthermore, the variance is the same as that of stock prices

  $\Rightarrow$ Different drifts $\ \Rightarrow \mu$ for stocks and $\alpha - \sigma^2/2$ for no arbitrage

Expected investment growth

- Compute expected return on an investment on stock $X(t)$
  - $\Rightarrow$ Buy 1 share of stock at time 0. Cash invested is $X(0)$
  - $\Rightarrow$ Sell stock at time t. Cash after sell is $X(t)$
- Expected value of cash after sell given $X(0)$ is

$$\mathbb{E}\left[X(t)\,\middle|\,X(0)\right] = X(0)e^{(\mu+\sigma^2/2)t}$$

- Alternatively, invest $X(0)$ risk free in the money market
  - $\Rightarrow$ Guaranteed cash at time t is $X(0)e^{\alpha t}$
- Invest in stock only if $\mu + \sigma^2/2 > \alpha$ $\Rightarrow$ "Risk premium" exists

Proof of expected return formula

- Stock prices follow a geometric Brownian motion $X(t) = X(0)e^{Y(t)}$

  $\Rightarrow$ $Y(t)$ Brownian motion with drift μ and variance σ^2

- Q: What is the expected return $\mathbb{E}\left[X(t) \,\middle|\, X(0)\right]$?

- Note first that $\mathbb{E}\left[X(t) \,\middle|\, X(0)\right] = X(0)\mathbb{E}\left[e^{Y(t)} \,\middle|\, X(0)\right]$

- Using that $Y(t)$ has independent increments

$$\mathbb{E}\left[e^{Y(t)} \,\middle|\, X(0)\right] = \mathbb{E}\left[e^{Y(t)}\right]$$

  $\Rightarrow$ Next we focus on computing $\mathbb{E}\left[e^{Y(t)}\right]$

Proof of expected return formula (cont.)

- Since $Y(t) \sim \mathcal{N}(\mu t, \sigma^2 t)$

$$\mathbb{E}\left[e^{Y(t)}\right] = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} e^{y} e^{-\frac{(y-\mu t)^2}{2\sigma^2 t}} dy$$

- Completing the squares in the argument of the exponential we have

$$y - \frac{(y-\mu t)^2}{2\sigma^2 t} = \frac{-y^2 + 2(\mu+\sigma^2)ty - \mu^2 t^2}{2\sigma^2 t}$$
$$= -\frac{\left(y-(\mu+\sigma^2)t\right)^2}{2\sigma^2 t} + \frac{2\mu\sigma^2 t^2 + \sigma^4 t^2}{2\sigma^2 t}$$

- The blue term does not depend on y, red integral equals 1

$$\mathbb{E}\left[e^{Y(t)}\right] = e^{\left(\mu+\frac{\sigma^2}{2}\right)t} \times \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} e^{-\frac{\left(y-(\mu+\sigma^2)t\right)^2}{2\sigma^2 t}} dy = e^{\left(\mu+\frac{\sigma^2}{2}\right)t}$$

- Putting the pieces together, we obtain

$$\mathbb{E}\left[X(t) \,\middle|\, X(0)\right] = X(0)\mathbb{E}\left[e^{Y(t)}\right] = X(0)e^{(\mu+\sigma^2/2)t}$$

Risk neutral measure

- Compute expected return as if **q** were the actual distribution
  - $\Rightarrow$ Recall that **q** is NOT the actual distribution
  - $\Rightarrow$ As before, cash invested is $X(0)$ and cash after sale is $X(t)$
- Expected cash value is different because prob. distribution is different

$$\mathbb{E}_{\mathbf{q}}\left[X(t)\,\middle|\,X(0)\right] = X(0)e^{(\alpha-\sigma^2/2+\sigma^2/2)t} = X(0)e^{\alpha t}$$

  - $\Rightarrow$ Same return as risk-free investment regardless of parameters
- Measure **q** is called risk neutral measure
  - $\Rightarrow$ Risky stock investments yield same return as risk-free one
  - $\Rightarrow$ "Alternate universe", investors do not demand risk premiums
- Pricing of derivatives, e.g., options, is always based on expected returns with respect to risk neutral valuation (pricing in alternate universe)
  - $\Rightarrow$ Basis for Black-Scholes formula for option pricing

Martingale as basis for fair pricing

- A continuous-time process $X(t)$ is a martingale if for $t, s \geq 0$

$$\mathbb{E}\left[X(t+s)\,\middle|\,X(u), 0 \leq u \leq t\right] = X(t)$$

$\Rightarrow$ Expected future value = present value, even given process history

- Model of a fair, e.g., gambling game. Excludes winning strategies

$\Rightarrow$ Even with prior info. of outcomes (cards drawn from the deck)

- For risk-neutral measure $\mathbf{q}$, time 0 prices $e^{-\alpha t}X(t)$ form a martingale

$$\mathbb{E}_{\mathbf{q}}\left[e^{-\alpha(t+s)}X(t+s)\,\middle|\,e^{-\alpha u}X(u), 0 \leq u \leq t\right] = e^{-\alpha t}X(t)$$

- **Key principle:** stock price = expected discounted payoff

$$X(0) = \mathbb{E}_{\mathbf{q}}\left[e^{-\alpha t}X(t)\,\middle|\,X(0)\right]$$

$\Rightarrow$ Fair pricing, cannot devise a winning strategy (arbitrage)

Stock prices form a martingale under **q** (proof)

- Recall measure **q** is a geometric Brownian motion $X(t) = e^{Y(t)}$
  - $\Rightarrow$ Variance σ^2 (same as stock price)
  - $\Rightarrow$ Drift $\alpha - \sigma^2/2$

Proof.

$$\mathbb{E}_{\mathbf{q}}\left[e^{-\alpha(t+s)}e^{Y(t+s)} \,\middle|\, e^{-\alpha u}e^{Y(u)}, 0 \leq u \leq t\right]$$

$$= \mathbb{E}_{\mathbf{q}}\left[e^{-\alpha(t+s)}e^{Y(t+s)} \,\middle|\, e^{-\alpha t}e^{Y(t)}\right] \qquad Y(t) \text{ is Markov}$$

$$= \mathbb{E}_{\mathbf{q}}\left[e^{-\alpha(t+s)}e^{[Y(t+s)-Y(t)]+Y(t)} \,\middle|\, e^{-\alpha t}e^{Y(t)}\right] \qquad \text{Add and subtract } Y(t)$$

$$= e^{-\alpha t}e^{Y(t)}\mathbb{E}_{\mathbf{q}}\left[e^{-\alpha s}e^{[Y(t+s)-Y(t)]}\right] \qquad \text{Independent increments}$$

$$= e^{-\alpha t}X(t)\mathbb{E}_{\mathbf{q}}\left[e^{-\alpha s}e^{Y(s)}\right] \qquad \text{Stationary increments}$$

$$= e^{-\alpha t}X(t) \qquad \mathbb{E}_{\mathbf{q}}\left[e^{Y(s)}\right] = e^{(\mu+\sigma^2/2)s} = e^{\alpha s}$$

$\square$

Black-Scholes formula for option pricing

Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing

Options

- An option is a contract to buy shares of a stock at a future time
  - Strike time t = Convened time for stock purchase
  - Strike price K = Price at which stock is purchased at strike time
- At time t, option holder may decide to

  $\Rightarrow$ Buy a stock at strike price K = exercise the option

  $\Rightarrow$ Do not exercise the option
- May buy option at time 0 for price c
- Q: How do we determine the option's worth, i.e., price c at time 0?
- A: Given by the Black-Scholes formula for option pricing

Stock price model

- Let $e^{\alpha t}$ be the compounding of a risk-free investment
- Let $X(t)$ be the stock's price at time t

  $\Rightarrow$ Modeled as geometric Brownian motion, drift μ, variance σ^2
- Risk neutral measure **q** is also a geometric Brownian motion

  $\Rightarrow$ Drift $\alpha - \sigma^2/2$ and variance σ^2

Return of option investment

- At time t, the option's worth depends on the stock's price $X(t)$
- If stock's price smaller or equal than strike price $\Rightarrow X(t) \leq K$
  - $\Rightarrow$ Option is worthless (better to buy stock at current price)
- Since had paid c for the option at time 0, lost c on this investment
  - $\Rightarrow$ Return on investment is $r = -c$
- If stock's price larger than strike price $\Rightarrow X(t) > K$
  - $\Rightarrow$ Exercise option and realize a gain of $X(t) - K$
- To obtain return express as time 0 values and subtract c

$$r = e^{-\alpha t}\big(X(t) - K\big) - c$$

- May combine both in single equation $\Rightarrow r = e^{-\alpha t}\big(X(t) - K\big)_+ - c$
  - $\Rightarrow (\cdot)_+ := \max(\cdot, 0)$ denotes projection onto positive reals $\mathbb{R}_+$

Option pricing

- Select option price c to prevent arbitrage opportunities

$$\mathbb{E}_{\mathbf{q}}\left[e^{-\alpha t}\big(X(t)-K\big)_+ - c\right] = 0$$

$\Rightarrow$ Expectation is with respect to risk neutral measure $\mathbf{q}$

- From above condition, the no-arbitrage price of the option is

$$c = e^{-\alpha t}\mathbb{E}_{\mathbf{q}}\left[\big(X(t)-K\big)_+\right]$$

$\Rightarrow$ Source of Black-Scholes formula for option valuation

$\Rightarrow$ Rest of derivation is just evaluating $\mathbb{E}_{\mathbf{q}}\left[\big(X(t)-K\big)_+\right]$

- Same argument used to price any derivative of the stock's price

Use fact that **q** is a geometric Brownian motion

- Let us evaluate $\mathbb{E}_{\mathbf{q}}\left[\left(X(t)-K\right)_+\right]$ to compute option's price c
- Recall **q** is a geometric Brownian motion $\Rightarrow X(t) = X_0 e^{Y(t)}$

  $\Rightarrow X_0 =$ price at time 0

  $\Rightarrow Y(t)$ BMD, $\mu \; (= \alpha - \sigma^2/2)$ and variance σ^2
- Can rewrite no arbitrage condition as

$$c = e^{-\alpha t}\mathbb{E}_{\mathbf{q}}\left[\left(X_0 e^{Y(t)} - K\right)_+\right]$$

- $Y(t)$ is a Brownian motion with drift. Thus, $Y(t) \sim \mathcal{N}(\mu t, \sigma^2 t)$

$$c = e^{-\alpha t}\frac{1}{\sqrt{2\pi\sigma^2 t}}\int_{-\infty}^{\infty}\left(X_0 e^{y} - K\right)_+ e^{-(y-\mu t)^2/(2\sigma^2 t)}\, dy$$

Evaluation of the integral

- Note that $\left(X_0 e^{Y(t)} - K\right)_+ = 0$ for all values $Y(t) \leq \log(K/X_0)$
- Because integrand is null for $Y(t) \leq \log(K/X_0)$ can write

$$c = e^{-\alpha t} \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{\log(K/X_0)}^{\infty} (X_0 e^{y} - K)\, e^{-(y-\mu t)^2/(2\sigma^2 t)}\, dy$$

- Change of variables $z = (y - \mu t)/\sqrt{\sigma^2 t}$. Associated replacements

Variable:	y	$\Rightarrow \sqrt{\sigma^2 t} z + \mu t$
Differential:	dy	$\Rightarrow \sqrt{\sigma^2 t}\, dz$
Integration limit:	$\log(K/X_0)$	$\Rightarrow a := \dfrac{\log(K/X_0) - \mu t}{\sqrt{\sigma^2 t}}$

- Option price then given by

$$c = e^{-\alpha t} \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} \left(X_0 e^{\sqrt{\sigma^2 t} z + \mu t} - K\right) e^{-z^2/2}\, dz$$

Split in two integrals

- Separate in two integrals $c = e^{-\alpha t}(I_1 - I_2)$ where

$$I_1 := \frac{1}{\sqrt{2\pi}} \int_a^\infty X_0 e^{\sqrt{\sigma^2 t}z + \mu t} e^{-z^2/2}\, dz$$
$$I_2 := \frac{K}{\sqrt{2\pi}} \int_a^\infty e^{-z^2/2}\, dz$$

- Gaussian Φ function (ccdf of standard normal RV)

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2}\, dz$$

$\Rightarrow$ Comparing last two equations we have $I_2 = K\Phi(a)$

- Integral I_1 requires some more work

Evaluation of the first integral

- Reorder terms in integral I_1

$$I_1 := \frac{1}{\sqrt{2\pi}} \int_a^\infty X_0 e^{\sqrt{\sigma^2 t} z + \mu t} e^{-z^2/2}\, dz = \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_a^\infty e^{\sqrt{\sigma^2 t} z - z^2/2}\, dz$$

- The exponent can be written as a square minus a "constant" (no z)

$$-\left(z - \sqrt{\sigma^2 t}\right)^2 /2 + \sigma^2 t/2 = -z^2/2 + \sqrt{\sigma^2 t} z - \sigma^2 t/2 + \sigma^2 t/2$$

- Substituting the latter into I_1 yields

$$\begin{aligned} I_1 &= \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_a^\infty e^{-\left(z - \sqrt{\sigma^2 t}\right)^2/2 + \sigma^2 t/2}\, dz \\ &= \frac{X_0 e^{\mu t + \sigma^2 t/2}}{\sqrt{2\pi}} \int_a^\infty e^{-\left(z - \sqrt{\sigma^2 t}\right)^2/2}\, dz \end{aligned}$$

Evaluation of the first integral (continued)

- Change of variables $u = z - \sqrt{\sigma^2 t} \Rightarrow du = dz$ and integration limit

$$a \Rightarrow b := a - \sqrt{\sigma^2 t} = \frac{\log(K/X_0) - \mu t}{\sqrt{\sigma^2 t}} - \sqrt{\sigma^2 t}$$

- Implementing change of variables in I_1

$$I_1 = \frac{X_0 e^{\mu t + \sigma^2 t/2}}{\sqrt{2\pi}} \int_b^\infty e^{-u^2/2}\, du = X_0 e^{\mu t + \sigma^2 t/2} \Phi(b)$$

- Putting together results for I_1 and I_2

$$c = e^{-\alpha t}(I_1 - I_2) = e^{-\alpha t} X_0 e^{\mu t + \sigma^2 t/2} \Phi(b) - e^{-\alpha t} K \Phi(a)$$

- For non-arbitrage stock prices (measure $\mathbf{q}$) $\Rightarrow \alpha = \mu + \sigma^2/2$

  $\Rightarrow$ Substitute to obtain Black-Scholes formula

Black-Scholes

- Black-Scholes formula for option pricing. Option cost at time 0 is

$$c = X_0\Phi(b) - e^{-\alpha t}K\Phi(a)$$

$$\Rightarrow a := \frac{\log(K/X_0) - \mu t}{\sqrt{\sigma^2 t}} \text{ and } b := a - \sqrt{\sigma^2 t}$$

- Note further that $\mu = \alpha - \sigma^2/2$. Can then write a as

$$a = \frac{\log(K/X_0) - \left(\alpha - \sigma^2/2\right) t}{\sqrt{\sigma^2 t}}$$

$\Rightarrow X_0 =$ stock price at time 0, $\sigma^2 =$ volatility of stock

$\Rightarrow K =$ option's strike price, $t =$ option's strike time

$\Rightarrow \alpha =$ benchmark risk-free rate of return (cost of money)

- Black-Scholes formula independent of stock's mean tendency μ

Glossary

- Arbitrage
- Investment strategy
- Bets, events, outcomes
- Returns and earnings
- Arbitrage theorem
- Geometric Brownian motion
- Stock flip
- Time value of money
- Continuously-compounded interest
- Present value
- Risk-free investment
- Expected return
- Risk premium
- Risk neutral measure
- Pricing of derivatives
- Stock option
- Strike time and price
- Option price
- Stock volatility
- Black-Scholes formula