Centrality Measures and Link Analysis

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Quantifying vertex importance

- In network analysis many questions relate to vertex importance

Example

- Q1: Which actors in a social network hold the 'reins of power'?
- Q2: How authoritative is a WWW page considered by peers?
- Q3: The 'knock-out' of which genes is likely to be lethal?
- Q4: How critical to the daily commute is a subway station?

- Measures of vertex centrality quantify such notions of importance

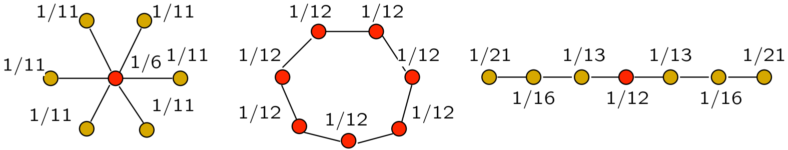
  $\Rightarrow$ Degrees are simplest centrality measures. Let's study others

Closeness centrality

- Rationale: ‘central’ means a vertex is ‘close’ to many other vertices
- **Def:** Distance $d(u, v)$ between vertices u and v is the length of the shortest $u - v$ path. Oftentimes referred to as geodesic distance
- Closeness centrality of vertex v is given by

$$c_{Cl}(v) = \frac{1}{\sum_{u \in V} d(u, v)}$$

- Interpret $v^* = \arg\max_v c_{Cl}(v)$ as the most approachable node in G



Normalization, computation and limitations

- To compare with other centrality measures, often normalize to $[0, 1]$

$$c_{Cl}(v) = \frac{N_v - 1}{\sum_{u \in V} d(u, v)}$$

- Computation: need all pairwise shortest path distances in G
  - $\Rightarrow$ Dijkstra's algorithm in $O(N_v^2 \log N_v + N_v N_e)$ time
- Limitation 1: sensitivity, values tend to span a small dynamic range
  - $\Rightarrow$ Hard to discriminate between central and less central nodes
- Limitation 2: assumes connectivity, if not $c_{Cl}(v) = 0$ for all $v \in V$
  - $\Rightarrow$ Compute centrality indices in different components

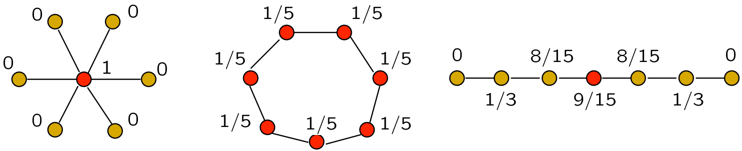
Betweenness centrality

- Rationale: 'central' node is (in the path) 'between' many vertex pairs
- Betweenness centrality of vertex v is given by

$$c_{Be}(v) = \sum_{s \neq t \neq v \in V} \frac{\sigma(s,t|v)}{\sigma(s,t)}$$

  - $\sigma(s,t)$ is the total number of $s-t$ shortest paths
  - $\sigma(s,t|v)$ is the number of $s-t$ shortest paths through $v \in V$

- Interpret $v^* = \arg\max_v c_{Be}(v)$ as the controller of information flow



Computational considerations

- Notice that a $s - t$ shortest path goes through v if and only if

$$d(s,t) = d(s,v) + d(v,t)$$

- Betweenness centralities can be naively computed for all $v \in V$ by:

  Step 1: Use Dijkstra to tabulate $d(s,t)$ and $\sigma(s,t)$ for all s, t
  Step 2: Use the tables to identify $\sigma(s,t|v)$ for all v
  Step 3: Sum the fractions to obtain $c_{Be}(v)$ for all v ($O(N_v^3)$ time)

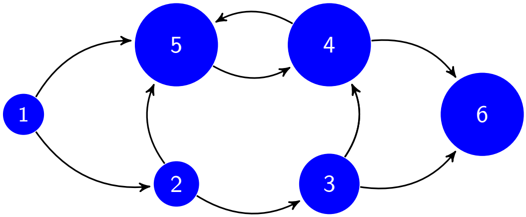
- Cubic complexity can be prohibitive for large networks
- $O(N_v N_e)$-time algorithm for unweighted graphs in:

  U. Brandes, "A faster algorithm for betweenness centrality," *Journal of Mathematical Sociology*, vol. 25, no. 2, pp. 163-177, 2001

Eigenvector centrality

- Rationale: 'central' vertex if 'in-neighbors' are themselves important
  - $\Rightarrow$ Compare with 'importance-agnostic' degree centrality
- Eigenvector centrality of vertex v is implicitly defined as

$$c_{Ei}(v) = \alpha \sum_{(u,v)\in E} c_{Ei}(u)$$



- No one points to 1
- Only 1 points to 2
- Only 2 points to 3, but 2 more important than 1
- 4 as high as 5 with less links
- Links to 5 have lower rank
- Same for 6

Eigenvalue problem

- Recall the adjacency matrix $\mathbf{A}$ and

$$c_{Ei}(v) = \alpha \sum_{(u,v)\in E} c_{Ei}(u)$$

- Vector $\mathbf{c}_{Ei} = [c_{Ei}(1), \ldots, c_{Ei}(N_v)]^\top$ solves the eigenvalue problem

$$\mathbf{A}^T \mathbf{c}_{Ei} = \alpha^{-1} \mathbf{c}_{Ei}$$

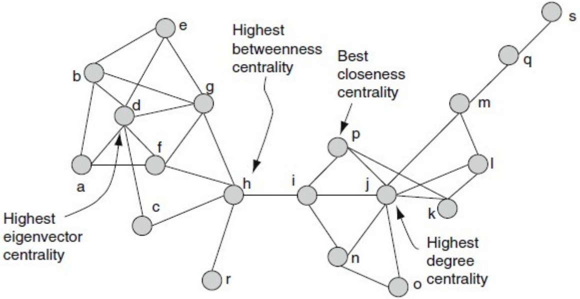
$\Rightarrow$ Typically α^{-1} chosen as largest eigenvalue of $\mathbf{A}^T$ [Bonacich'87]

- If G is strongly connected, by Perron's Theorem then
  - $\Rightarrow$ The largest eigenvalue of $\mathbf{A}^T$ is positive and simple
  - $\Rightarrow$ All the entries in the dominant eigenvector $\mathbf{c}_{Ei}$ are positive
- Can compute $\mathbf{c}_{Ei}$ and α^{-1} via $O(N_v^2)$ complexity power iterations

$$\mathbf{c}_{Ei}(k+1) = \frac{\mathbf{A}^T \mathbf{c}_{Ei}(k)}{\|\mathbf{A}\mathbf{c}_{Ei}(k)\|},\ k = 0, 1, \ldots$$

Example: Comparing centrality measures

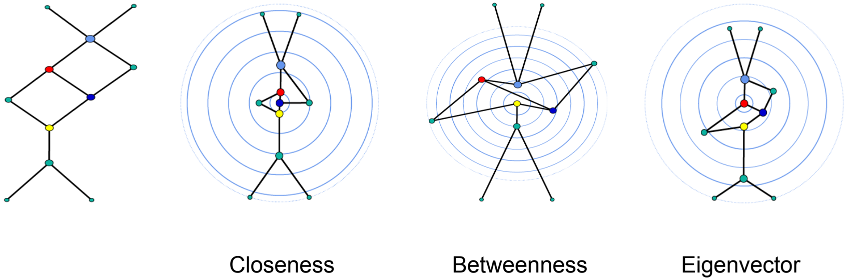
- Q: Which vertices are more central? A: It depends on the context



- Each measure identifies a different vertex as most central
  - $\Rightarrow$ None is 'wrong', they target different notions of importance

Example: Comparing centrality measures

- Q: Which vertices are more central? A: It depends on the context



- Small green vertices are arguably more peripheral
  - $\Rightarrow$ Less clear how the yellow, dark blue and red vertices compare

Case study

Centrality measures

Case study: Stability of centrality measures in weighted graphs

Centrality, link analysis and web search

A primer on Markov chains

PageRank as a random walk

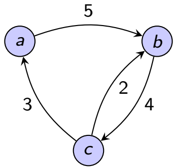
PageRank algorithm leveraging Markov chain structure

Centrality measures robustness

- Robustness to noise in network data is of practical importance
- Approaches have been mostly empirical
  - ⇒ Find average response in random graphs when perturbed
  - ⇒ Not generalizable and does not provide explanations
- Characterize behavior in noisy real graphs
  - ⇒ *Degree and closeness are more reliable than betweenness*
- Q: What is really going on?
  - ⇒ Framework to study formally the stability of centrality measures
- S. Segarra and A. Ribeiro, "Stability and continuity of centrality measures in weighted graphs," *IEEE Trans. Signal Process.*, 2015

Definitions for weighted digraphs

- Weighted and directed graphs $G(V, E, W)$
  - $\Rightarrow$ Set V of N_v vertices
  - $\Rightarrow$ Set $E \subseteq V \times V$ of edges
  - $\Rightarrow$ Map $W : E \to \mathbb{R}_{++}$ of weights in each edge



- Path $P(u, v)$ is an ordered sequence of nodes from u to v
- When weights represent dissimilarities
  - $\Rightarrow$ Path length is the sum of the dissimilarities encountered
- Shortest path length $s_G(u, v)$ from u to v

$$s_G(u, v) := \min_{P(u,v)} \sum_{i=0}^{\ell-1} W(u_i, u_{i+1})$$

Stability of centrality measures

- Space of graphs $\mathcal{G}_{(V,E)}$ with (V,E) as vertex and edge set
- Define the metric $d_{(V,E)}(G,H) : \mathcal{G}_{(V,E)} \times \mathcal{G}_{(V,E)} \to \mathbb{R}_+$

$$d_{(V,E)}(G,H) := \sum_{e \in E} |W_G(e) - W_H(e)|$$

- **Def:** A centrality measure $c(\cdot)$ is *stable* if for any vertex $v \in V$ in any two graphs $G, H \in \mathcal{G}_{(V,E)}$, then

$$\left|c^G(v) - c^H(v)\right| \leq K_G \, d_{(V,E)}(G,H)$$

  - K_G is a constant depending on G only
  - Stability is related to Lipschitz continuity in $\mathcal{G}_{(V,E)}$
  - Independent of the definition of $d_{(V,E)}$ (equivalence of norms)
- Node importance should be robust to small perturbations in the graph

Degree centrality

- Sum of the weights of incoming arcs

$$c_{De}(v) := \sum_{u|(u,v)\in E} W(u,v)$$

  - Applied to graphs where the weights in W represent similarities
  - High $c_{De}(v) \Rightarrow v$ similar to its large number of neighbors

Proposition 1

For any vertex $v \in V$ in any two graphs $G, H \in \mathcal{G}_{(V,E)}$, we have that

$$|c^G_{De}(v) - c^H_{De}(v)| \leq d_{(V,E)}(G,H)$$

i.e., degree centrality c_{De} is a stable measure

- Can show closeness and eigenvector centralities are also stable

Betweenness centrality

- Look at the shortest paths for every two nodes distinct from v

  $\Rightarrow$ Sum the proportion that contains node v

$$c_{Be}(v) := \sum_{s \neq v \neq t \in V} \frac{\sigma(s,t|v)}{\sigma(s,t)}$$

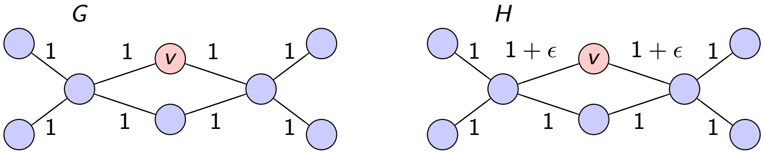
  - $\sigma(s,t)$ is the total number of $s-t$ shortest paths
  - $\sigma(s,t|v)$ is the number of those paths going through v

Proposition 2

The betweenness centrality measure c_{Be} is not stable

Instability of betweenness centrality

- Compare the value of $c_{Be}(v)$ in graphs G and H



$c_{Be}^G(v) = 9$ $\qquad$ $c_{Be}^H(v) = 0$

$\Rightarrow$ Centrality value $c_{Be}^H(v) = 0$ remains unchanged for any $\epsilon > 0$

- For small values of ϵ, graphs G and H become arbitrarily similar

$$9 = |c_{Be}^G(v) - c_{Be}^H(v)| \leq K_G \, d_{(V,E)}(G, H) \to 0$$

$\Rightarrow$ Inequality is not true for any constant K_G

Stable betweenness centrality

- Define $G^v = (V^v, E^v, W^v)$, $V^v = V \setminus \{v\}$, $E^v = E|_{V^v \times V^v}$, $W^v = W|_{E^v}$
  $\Rightarrow$ G^v obtained by deleting from G node v and edges connected to v
- Stable betweenness centrality $c_{SBe}(v)$

$$c_{SBe}(v) := \sum_{s \neq v \neq t \in V} s_{G^v}(s,t) - s_G(s,t)$$

  $\Rightarrow$ Captures impact of deleting v on the shortest paths
- If v is (not) in the $s-t$ shortest path, $s_{G^v}(s,t) - s_G(s,t) > (=) 0$
  $\Rightarrow$ Same notion as (traditional) betweenness centrality c_{Be}

Proposition 3

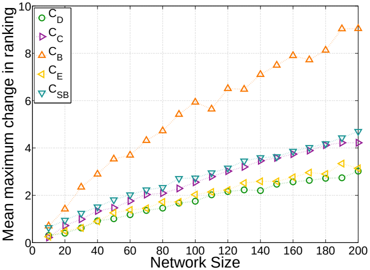
For any vertex $v \in V$ in any two graphs $G, H \in \mathcal{G}_{(V,E)}$, then

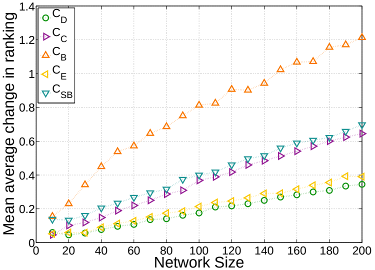
$$|c^G_{SBe}(v) - c^H_{SBe}(v)| \leq 2N_v^2 \; d_{(V,E)}(G,H)$$

i.e., stable betweenness centrality c_{SBe} is a stable measure

Centrality ranking variation in random graphs

- $G_{n,p}$ graphs with $p = 10/n$ and weights $\mathcal{U}(0.5, 1.5)$
  - $\Rightarrow$ Vary n from 10 to 200
  - $\Rightarrow$ Perturb multiplying weights with random numbers $\mathcal{U}(0.99, 1.01)$
- Compare centrality rankings in the original and perturbed graphs

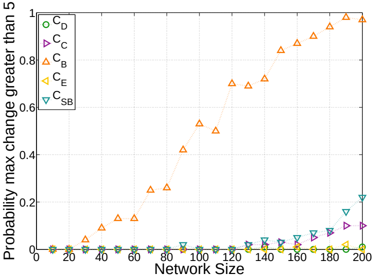


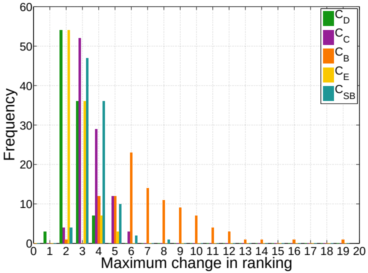


- Betweenness centrality presents larger maximum and average changes

Centrality ranking variation in random graphs

- Compute probability of observing a ranking change ≥ 5
  - $\Rightarrow$ Plot the histogram giving rise to the empirical probabilities

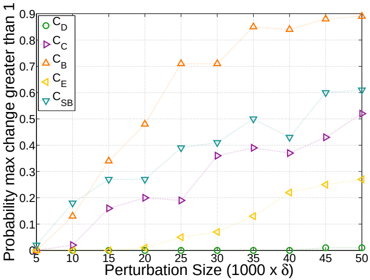


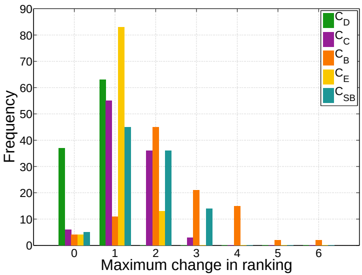


- For c_{Be} some node varies its ranking by 5 positions with high probability
- Long tail in histogram is evidence of instability
  - $\Rightarrow$ Minor perturbation generates change of 19 positions

Centrality ranking variation in an airport graph

- Real-world graph based on the air traffic between popular U.S. airports
  - ⇒ Nodes are $N_v = 25$ popular airports
  - ⇒ Edge weights are the number of yearly passengers between them





- Betweenness centrality still presents the largest variations

Centrality, link analysis and web search

The problem of ranking websites

- Search engines rank pages by looking at the Web itself

  $\Rightarrow$ Enough information intrinsic to the Web and its structure

- Information retrieval is a historically difficult problem

  $\Rightarrow$ Keywords vs complex information needs (synonymy, polysemy)

- Beyond explosion in scale, unique issues arised with the Web
  - Diversity of authoring styles, people issuing queries
  - Dynamic and constantly changing content
  - Paradigm: from scarcity to abundance

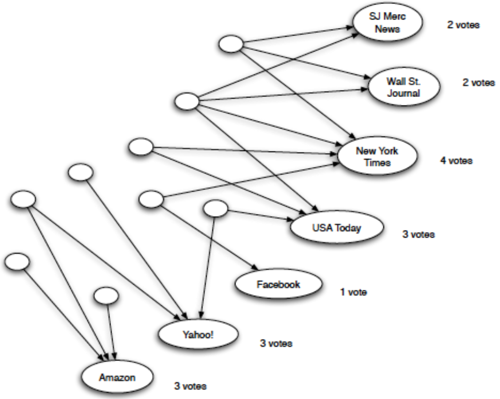
- Finding and indexing documents that are relevant is 'easy'
- Q: Which few of these should the engine recommend?

  $\Rightarrow$ Key is understanding Web structure, i.e., link analysis

Voting by in-links

Ex: Suppose we issue the query 'newspapers'

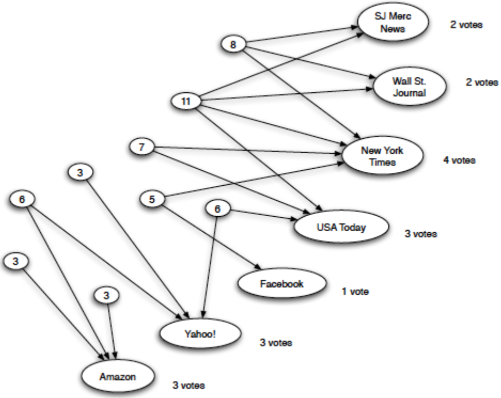
- First, use text-only information retrieval to identify relevant pages



- Idea: Links suggest implicit endorsements of other relevant pages
  - Count in-links to assess the authority of a page on 'newspapers'

A list-finding technique

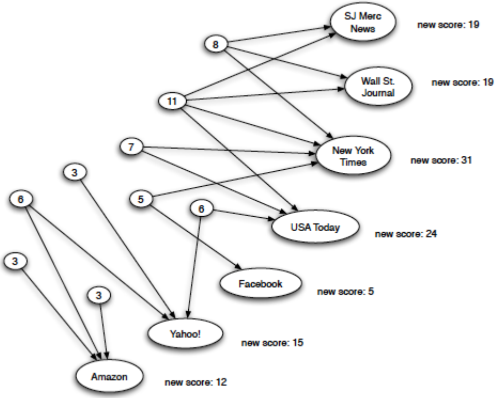
- Query also returns pages that compile lists of relevant resources
  - These hubs voted for many highly endorsed (authoritative) pages



- Idea: Good lists have a better sense of where the good results are
  - Page's hub value is the sum of votes received by its linked pages

Repeated improvement

- Reasonable to weight more the votes of pages scoring well as lists
  - ⇒ Recompute votes summing linking page values as lists



- Q: Why stop here? Use also improved votes to refine the list scores
  - ⇒ Principle of repeated improvement

Hubs and authorities

- Relevant pages fall in two categories: hubs and authorities
- Authorities are pages with useful, relevant content
  - Newspaper home pages
  - Course home pages
  - Auto manufacturer home pages
- Hubs are 'expert' lists pointing to multiple authorities
  - List of newspapers
  - Course bulletin
  - List of US auto manufacturers
- **Rules:** Authorities and hubs have a mutual reinforcement relationship

  $\Rightarrow$ A good hub links to multiple good authorities

  $\Rightarrow$ A good authority is linked from multiple good hubs

Hubs and authorities ranking algorithm

- Hyperlink-Induced Topic Search (HITS) algorithm [Kleinberg'98]
- Each page $v \in V$ has a hub score h_v and authority score a_v

  $\Rightarrow$ Network-wide vectors $\mathbf{h} = [h_1, \ldots, h_{N_v}]^\top$, $\mathbf{a} = [a_1, \ldots, a_{N_v}]^\top$

  Authority update rule:

  $$a_v(k) = \sum_{(u,v)\in E} h_u(k-1), \text{ for all } v \in V \Leftrightarrow \mathbf{a}(k) = \mathbf{A}^\top \mathbf{h}(k-1)$$

  Hub update rule:

  $$h_v(k) = \sum_{(v,u)\in E} a_u(k), \text{ for all } v \in V \Leftrightarrow \mathbf{h}(k) = \mathbf{A}\mathbf{a}(k)$$

- Initialize $\mathbf{h}(0) = \mathbf{1}/\sqrt{N_v}$, normalize $\mathbf{a}(k)$ and $\mathbf{h}(k)$ each iteration

Limiting values

- Define the hub and authority rankings as

$$\mathbf{a} := \lim_{k\to\infty} \mathbf{a}(k), \quad \mathbf{h} := \lim_{k\to\infty} \mathbf{h}(k)$$

- From the HITS update rules one finds for $k = 0, 1, \ldots$

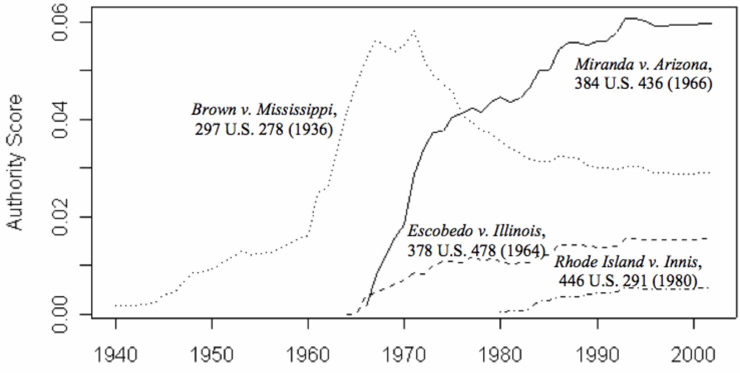
$$\mathbf{a}(k+1) = \frac{\mathbf{A}^\top \mathbf{A}\mathbf{a}(k)}{\|\mathbf{A}^\top \mathbf{A}\mathbf{a}(k)\|}, \quad \mathbf{h}(k+1) = \frac{\mathbf{A}\mathbf{A}^\top \mathbf{h}(k)}{\|\mathbf{A}\mathbf{A}^\top \mathbf{h}(k)\|}$$

- Power iterations converge to dominant eigenvectors of $\mathbf{A}^\top\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\top$

$$\mathbf{A}^\top \mathbf{A}\mathbf{a} = \alpha_a^{-1}\mathbf{a}, \quad \mathbf{A}\mathbf{A}^\top \mathbf{h} = \alpha_h^{-1}\mathbf{h}$$

$\Rightarrow$ Hub and authority ranks are eigenvector centrality measures

Ex: link analysis of citations among US Supreme Court opinions



- Rise and fall of authority of key Fifth Amendment cases [Fowler-Jeon'08]

PageRank

- Node rankings to measure website relevance, social influence
- **Key idea:** in-links as votes, but 'not all links are created equal'
  - $\Rightarrow$ How many links point to a node (outgoing links irrelevant)
  - $\Rightarrow$ How important are the links that point to a node
- PageRank key to Google's original ranking algorithm [Page-Brin'98]
- Inuition 1: fluid that percolates through the network
  - $\Rightarrow$ Eventually accumulates at most relevant Web pages
- Inuition 2: random web surfer (more soon)
  - $\Rightarrow$ In the long-run, relevant Web pages visited more often
- PageRank and HITS success was quite different after 1998

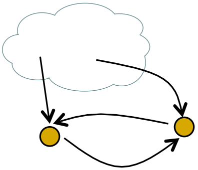
Basic PageRank update rule

- Each page $v \in V$ has PageRank r_v, let $\mathbf{r} = [r_1, \ldots, r_{N_v}]^\top$
  - $\Rightarrow$ Define $\mathbf{P} := (\mathbf{D}^{out})^{-1}\mathbf{A}$, where $\mathbf{D}^{out}$ is the out-degree matrix

PageRank update rule:

$$r_v(k) = \sum_{(u,v)\in E} \frac{r_u(k-1)}{d_u^{out}}, \text{ for all } v \in V \;\Leftrightarrow\; \mathbf{r}(k) = \mathbf{P}^T\mathbf{r}(k-1)$$

- Split current PageRank evenly among outgoing links and pass it on
  - $\Rightarrow$ New PageRank is the total fluid collected in the incoming links
  - $\Rightarrow$ Initialize $\mathbf{r}(0) = \mathbf{1}/N_v$. Flow conserved, no normalization needed
- Problem: 'Spider traps'
  - Accumulate all PageRank
  - Only when not strongly connected



Scaled PageRank update rule

- Apply the basic PageRank rule and scale the result by $s \in (0, 1)$
  Split the leftover $(1 - s)$ evenly among all nodes (evaporation-rain)

  Scaled PageRank update rule:

$$r_v(k) = s \times \sum_{(u,v)\in E} \frac{r_u(k-1)}{d_u^{out}} + \frac{1-s}{N_v}, \text{ for all } v \in V$$

- Can view as basic update $\mathbf{r}(k) = \bar{\mathbf{P}}^T \mathbf{r}(k-1)$ with

$$\bar{\mathbf{P}} := s\mathbf{P} + (1-s)\frac{\mathbf{1}\mathbf{1}^\top}{N_v}$$

$\Rightarrow$ Scaling factor s typically chosen between 0.8 and 0.9

$\Rightarrow$ Power iteration converges to the dominant eigenvector of $\bar{\mathbf{P}}^T$

A primer on Markov chains

Markov chains

- Consider discrete-time index $n = 0, 1, 2, \ldots$
- Time-dependent random state X_n takes values on a countable set
  - In general denote states as $i = 0, 1, 2, \ldots$, i.e., here the state space is $\mathbb{N}$
  - If $X_n = i$ we say "the process is in state i at time n"
- Random process is $X_{\mathbb{N}}$, its history up to n is $\mathbf{X}_n = [X_n, X_{n-1}, \ldots, X_0]^T$
- **Def:** process $X_{\mathbb{N}}$ is a Markov chain (MC) if for all $n \geq 1$, i, j, $\mathbf{x} \in \mathbb{N}^n$

$$\mathrm{P}\left(X_{n+1} = j \,\middle|\, X_n = i, \mathbf{X}_{n-1} = \mathbf{x}\right) = \mathrm{P}\left(X_{n+1} = j \,\middle|\, X_n = i\right) = P_{ij}$$

- Future depends only on current state X_n (memoryless, Markov property)

  $\Rightarrow$ Future conditionally independent of the past, given the present

Matrix representation

- Group the P_{ij} in a transition probability "matrix" $\mathbf{P}$

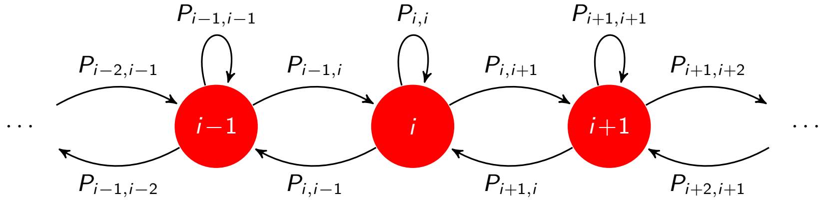
$$\mathbf{P} = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \dots & P_{0j} & \dots \\ P_{10} & P_{11} & P_{12} & \dots & P_{1j} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{i0} & P_{i1} & P_{i2} & \dots & P_{ij} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$\Rightarrow$ Not really a matrix if number of states is infinite

- Row-wise sums should be equal to one, i.e., $\sum_{j=0}^{\infty} P_{ij} = 1$ for all i

Graph representation

- A graph representation or state transition diagram is also used



- Useful when number of states is infinite, skip arrows if $P_{ij} = 0$
- Again, sum of per-state outgoing arrow weights should be one

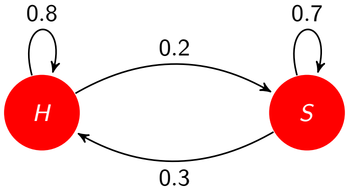
Example: Bipolar mood

- I can be happy ($X_n = 0$) or sad ($X_n = 1$)

  $\Rightarrow$ My mood tomorrow is only affected by my mood today

- Model as Markov chain with transition probabilities

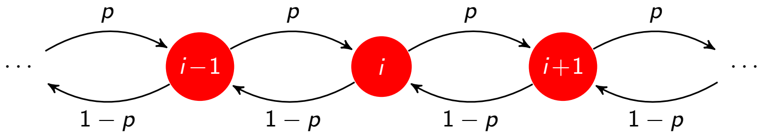
$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$



- Inertia $\Rightarrow$ happy or sad today, likely to stay happy or sad tomorrow
- But when sad, a little less likely so ($P_{00} > P_{11}$)

Example: Random (drunkard's) walk

- Step to the right w.p. p, to the left w.p. $1-p$
  - $\Rightarrow$ Not that drunk to stay on the same place



- States are $0, \pm 1, \pm 2, \dots$ (state space is $\mathbb{Z}$), infinite number of states
- Transition probabilities are

$$P_{i,i+1} = p, \qquad P_{i,i-1} = 1-p$$

- $P_{ij} = 0$ for all other transitions

Multiple-step transition probabilities

- Q: What can be said about multiple transitions?
- Probabilities of X_{m+n} given X_m $\Rightarrow$ n-step transition probabilities

$$P_{ij}^{n} = \mathrm{P}\left(X_{m+n} = j \,\middle|\, X_m = i\right)$$

$\Rightarrow$ Define the matrix $\mathbf{P}^{(n)}$ with elements P_{ij}^{n}

Theorem

The matrix of n-step transition probabilities $\mathbf{P}^{(n)}$ is given by the n-th power of the transition probability matrix $\mathbf{P}$, i.e.,

$$\mathbf{P}^{(n)} = \mathbf{P}^{n}$$

Henceforth we write $\mathbf{P}^{n}$

Unconditional probabilities

- All probabilities so far are conditional, i.e., $P_{ij}^n = \mathrm{P}\left(X_n = j \,\middle|\, X_0 = i\right)$

  $\Rightarrow$ May want unconditional probabilities $p_j(n) = \mathrm{P}\left(X_n = j\right)$
- Requires specification of initial conditions $p_i(0) = \mathrm{P}\left(X_0 = i\right)$
- Using law of total probability and definitions of P_{ij}^n and $p_j(n)$

$$p_j(n) = \mathrm{P}\left(X_n = j\right) = \sum_{i=0}^{\infty} \mathrm{P}\left(X_n = j \,\middle|\, X_0 = i\right) \mathrm{P}\left(X_0 = i\right)$$

$$= \sum_{i=0}^{\infty} P_{ij}^n p_i(0)$$

- In matrix form (define vector $\mathbf{p}(n) = [p_1(n), p_2(n), \ldots]^T$)

$$\mathbf{p}(n) = \left(\mathbf{P}^n\right)^T \mathbf{p}(0)$$

Limiting distributions

- MCs have one-step memory. Eventually they forget initial state
- Q: What can we say about probabilities for large n?

$$\pi_j := \lim_{n\to\infty} \mathrm{P}\left(X_n = j \,\middle|\, X_0 = i\right) = \lim_{n\to\infty} P_{ij}^n$$

$\Rightarrow$ Assumed that limit is independent of initial state $X_0 = i$

- We've seen that this problem is related to the matrix power $\mathbf{P}^n$

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}, \qquad \mathbf{P}^7 = \begin{pmatrix} 0.6031 & 0.3969 \\ 0.5953 & 0.4047 \end{pmatrix}$$

$$\mathbf{P}^2 = \begin{pmatrix} 0.7 & 0.3 \\ 0.45 & 0.55 \end{pmatrix}, \qquad \mathbf{P}^{30} = \begin{pmatrix} 0.6000 & 0.4000 \\ 0.6000 & 0.4000 \end{pmatrix}$$

- Matrix product converges $\Rightarrow$ probs. independent of time (large n)
- All rows are equal $\Rightarrow$ probs. independent of initial condition

Limit distribution of ergodic Markov chains

Theorem

For an ergodic (i.e. irreducible, aperiodic, and positive recurrent) MC, $\lim_{n\to\infty} P_{ij}^n$ *exists and is independent of the initial state* i*, i.e.,*

$$\pi_j = \lim_{n\to\infty} P_{ij}^n$$

Furthermore, steady-state probabilities $\pi_j \geq 0$ *are the unique nonnegative solution of the system of linear equations*

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \qquad \sum_{j=0}^{\infty} \pi_j = 1$$

- Limit probs. independent of initial condition exist for ergodic MC
  $\Rightarrow$ Simple algebraic equations can be solved to find π_j

Markov chains meet eigenvalue problems

- Define vector steady-state distribution $\boldsymbol{\pi} := [\pi_0, \pi_1, \ldots, \pi_J]^T$
- Limit distribution is unique solution of

$$\boldsymbol{\pi} = \mathbf{P}^T \boldsymbol{\pi}, \qquad \boldsymbol{\pi}^T \mathbf{1} = 1$$

- Eigenvector $\boldsymbol{\pi}$ associated with eigenvalue 1 of $\mathbf{P}^T$
  - Eigenvectors are defined up to a scaling factor
  - Normalize to sum 1
- All other eigenvalues of $\mathbf{P}^T$ have modulus smaller than 1
- Computing $\boldsymbol{\pi}$ as eigenvector is computationally efficient

Ergodicity

- **Def:** Fraction of time $T_i^{(n)}$ spent in i-th state by time n is

$$T_i^{(n)} := \frac{1}{n}\sum_{m=1}^{n} \mathbb{I}\left\{X_m = i\right\}$$

- Compute expected value of $T_i^{(n)}$

$$\mathbb{E}\left[T_i^{(n)}\right] = \frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[\mathbb{I}\left\{X_m = i\right\}\right] = \frac{1}{n}\sum_{m=1}^{n} \mathrm{P}\left(X_m = i\right)$$

- As $n \to \infty$, probabilities $\mathrm{P}\left(X_m = i\right) \to \pi_i$ (ergodic MC). Then

$$\lim_{n\to\infty} \mathbb{E}\left[T_i^{(n)}\right] = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \mathrm{P}\left(X_m = i\right) = \pi_i$$

- For ergodic MCs same is true without expected value $\Rightarrow$ Ergodicity

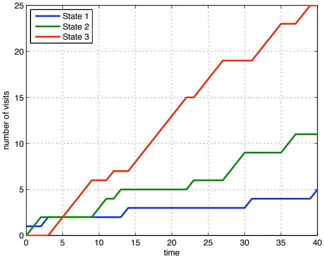
$$\lim_{n\to\infty} T_i^{(n)} = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \mathbb{I}\left\{X_m = i\right\} = \pi_i, \quad \text{a.s.}$$

Example: Ergodic Markov chain

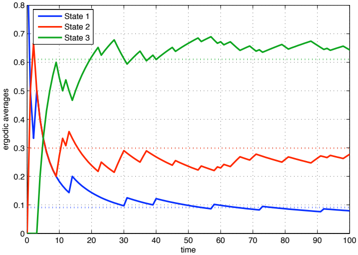
- Consider an ergodic Markov chain with transition probability matrix

$$\mathbf{P} := \begin{pmatrix} 0 & 0.3 & 0.7 \\ 0.1 & 0.5 & 0.4 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}$$

Visits to states, $nT_i^{(n)}$



Ergodic averages, $T_i^{(n)}$

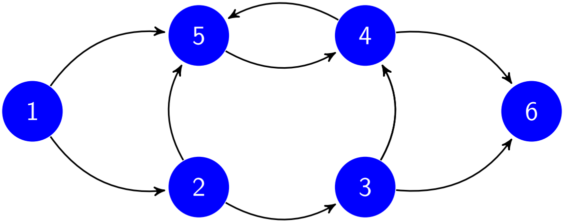


- Ergodic averages slowly converge to $\boldsymbol{\pi} = [0.09, 0.29, 0.61]^T$

PageRank: Random walk formulation

Preliminary definitions

- Graph $G = (V, E)$ $\Rightarrow$ vertices $V = \{1, 2, \ldots, J\}$ and edges E



- Outgoing neighborhood of i is the set of nodes j to which i points

$$n(i) := \{j : (i, j) \in E\}$$

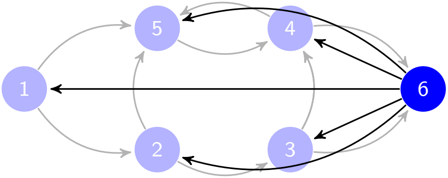
- Incoming neighborhood of i is the set of nodes that point to i:

$$n^{-1}(i) := \{j : (j, i) \in E\}$$

- Strongly connected G $\Rightarrow$ directed path joining any pair of nodes

Definition of rank

- Agent A chooses node i, e.g., web page, at random for initial visit
- Next visit randomly chosen between links in the neighborhood $n(i)$
  - $\Rightarrow$ All neighbors chosen with equal probability
- If reach a dead end because node i has no neighbors
  - $\Rightarrow$ Chose next visit at random equiprobably among all nodes
- Redefine graph $\mathcal{G} = (V, E)$ adding edges from dead ends to all nodes
  - $\Rightarrow$ Restrict attention to connected (modified) graphs



- Rank of node i is the average number of visits of agent A to i

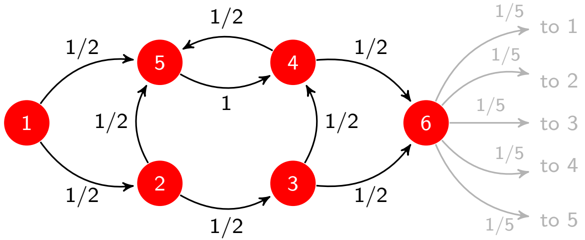
Equiprobable random walk

- Formally, let A_n be the node visited at time n
- Define transition probability P_{ij} from node i into node j

$$P_{ij} := \mathrm{P}\left(A_{n+1} = j \,\middle|\, A_n = i\right)$$

- Next visit equiprobable among i's $N_i := |n(i)|$ neighbors

$$P_{ij} = \frac{1}{|n(i)|} = \frac{1}{N_i}, \qquad \text{for all } j \in n(i)$$



- Still have a graph
- But also a MC
- Red (not blue) circles

Formal definition of rank

- **Def:** Rank r_i of i-th node is the time average of number of visits

$$r_i := \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \mathbb{I}\left\{A_m = i\right\}$$

$\Rightarrow$ Define vector of ranks $\mathbf{r} := [r_1, r_2, \ldots, r_J]^T$

- Rank r_i can be approximated by average r_{ni} at time n

$$r_{ni} := \frac{1}{n}\sum_{m=1}^{n} \mathbb{I}\left\{A_m = i\right\}$$

$\Rightarrow$ Since $\lim_{n\to\infty} r_{ni} = r_i$, it holds $r_{ni} \approx r_i$ for n sufficiently large

$\Rightarrow$ Define vector of approximate ranks $\mathbf{r}_n := [r_{n1}, r_{n2}, \ldots, r_{nJ}]^T$

- If modified graph is connected, rank independent of initial visit

Ranking algorithm

Output : Vector $\mathbf{r}(i)$ with ranking of node i
Input : Scalar n indicating maximum number of iterations
Input : Vector $N(i)$ containing number of neighbors of i
Input : Matrix $\mathbf{N}(i, j)$ containing indices j of neighbors of i

$m = 1$; $\mathbf{r}$=zeros(J,1); % Initialize time and ranks
$A_0 =$ random('unid',J); % Draw first visit uniformly at random
while $m < n$ **do**
    jump = random('unid',$N(A_{m-1})$); % Neighbor uniformly at random
    $A_m = \mathbf{N}(A_{m-1}, \text{jump})$; % Jump to selected neighbor
    $\mathbf{r}(A_m) = \mathbf{r}(A_m) + 1$; % Update ranking for A_m
    $m = m + 1$;
end
$\mathbf{r} = \mathbf{r}/n$; % Normalize by number of iterations n

Social graph example

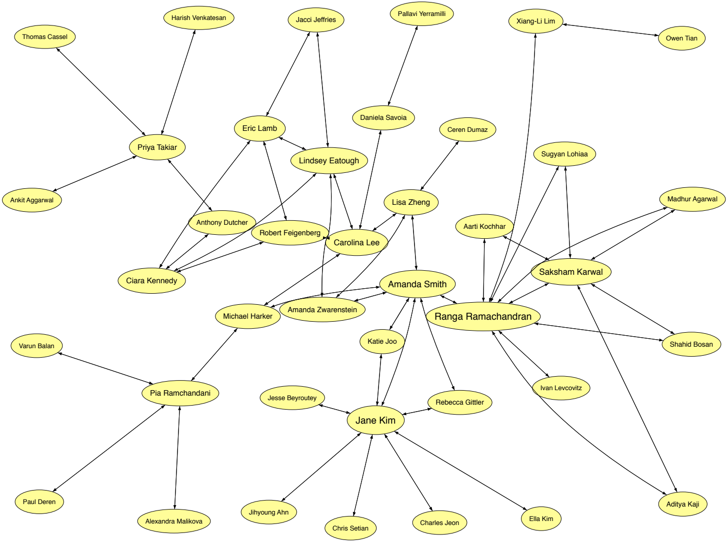
- Asked probability students about homework collaboration
- Created (crude) graph of the social network of students in the class

  $\Rightarrow$ Used ranking algorithm to understand connectedness

  Ex: I want to know how well students are coping with the class

  $\Rightarrow$ Best to ask people with higher connectivity ranking
- 2009 data from "UPenn's ECE440"

Ranked class graph



Convergence metrics

- Recall $\mathbf{r}$ is vector of ranks and $\mathbf{r}_n$ of rank iterates
- By definition $\lim_{n\to\infty} \mathbf{r}_n = \mathbf{r}$. How fast $\mathbf{r}_n$ converges to $\mathbf{r}$ ($\mathbf{r}$ given)?
- Can measure by ℓ_2 distance between $\mathbf{r}$ and $\mathbf{r}_n$

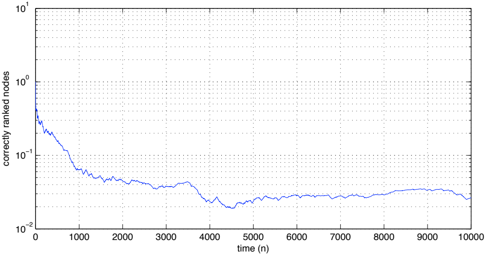
$$\zeta_n := \|\mathbf{r} - \mathbf{r}_n\|_2 = \left(\sum_{i=1}^{J} (r_{ni} - r_i)^2 \right)^{1/2}$$

- If interest is only on highest ranked nodes, e.g., a web search
  - $\Rightarrow$ Denote $r^{(i)}$ as the index of the i-th highest ranked node
  - $\Rightarrow$ Let $r_n^{(i)}$ be the index of the i-th highest ranked node at time n
- First element wrongly ranked at time n

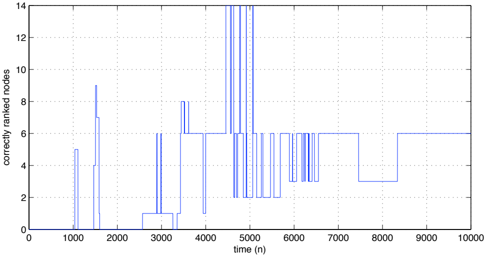
$$\xi_n := \arg\min_i \{ r^{(i)} \neq r_n^{(i)} \}$$

Evaluation of convergence metrics

Distance



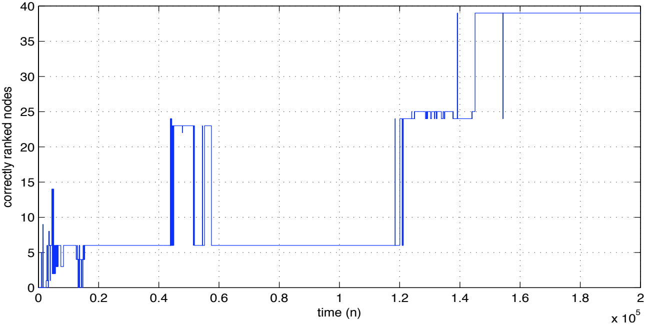
First element wrongly ranked



- Distance close to 10^{-2} in $\approx 5 \times 10^3$ iterations
- Bad: Two highest ranks in $\approx 4 \times 10^3$ iterations
- Awful: Six best ranks in $\approx 8 \times 10^3$ iterations
- (Very) slow convergence

When does this algorithm converge?

- Cannot confidently claim convergence until 10^5 iterations
  - $\Rightarrow$ Beyond particular case, slow convergence inherent to algorithm



- Example has 40 nodes, want to use in network with 10^9 nodes!
  - $\Rightarrow$ Leverage properties of MCs to obtain a faster algorithm

PageRank: Fast algorithms

Limit probabilities

- Recall definition of rank $\Rightarrow r_i := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{I}\{A_m = i\}$
- Rank is time average of number of state visits in a MC

  $\Rightarrow$ Can be as well obtained from limiting probabilities
- Recall transition probabilities $\Rightarrow P_{ij} = \frac{1}{N_i}$, for all $j \in n(i)$
- Stationary distribution $\boldsymbol{\pi} = [\pi_1, \pi_1, \ldots, \pi_J]^T$ solution of

$$\pi_i = \sum_{j \in n^{-1}(i)} P_{ji}\pi_j = \sum_{j \in n^{-1}(i)} \frac{\pi_j}{N_j} \qquad \text{for all } i$$

  $\Rightarrow$ Plus normalization equation $\sum_{i=1}^{J} \pi_i = 1$
- As per ergodicity of MC (strongly connected G) $\Rightarrow \mathbf{r} = \boldsymbol{\pi}$

Matrix notation, eigenvalue problem

- As always, can define matrix $\mathbf{P}$ with elements P_{ij}

$$\pi_i = \sum_{j \in n^{-1}(i)} P_{ji}\pi_j = \sum_{j=1}^{J} P_{ji}\pi_j \qquad \text{for all } i$$

- Right hand side is just definition of a matrix product leading to

$$\boldsymbol{\pi} = \mathbf{P}^T\boldsymbol{\pi}, \qquad \boldsymbol{\pi}^T\mathbf{1} = 1$$

$\Rightarrow$ Also added normalization equation

- Idea: solve system of linear equations or eigenvalue problem on $\mathbf{P}^T$

$\Rightarrow$ Requires matrix $\mathbf{P}$ available at a central location

$\Rightarrow$ Computationally costly (sparse matrix $\mathbf{P}$ with 10^{18} entries)

What are limit probabilities?

- Let $p_i(n)$ denote probability of agent A visiting node i at time n

$$p_i(n) := \mathrm{P}\left(A_n = i\right)$$

- Probabilities at time $n+1$ and n can be related

$$\mathrm{P}\left(A_{n+1} = i\right) = \sum_{j \in n^{-1}(i)} \mathrm{P}\left(A_{n+1} = i \,\middle|\, A_n = j\right) \mathrm{P}\left(A_n = j\right)$$

- Which is, of course, probability propagation in a MC

$$p_i(n+1) = \sum_{j \in n^{-1}(i)} P_{ji} p_j(n)$$

- By definition limit probabilities are (let $\mathbf{p}(n) = [p_1(n), \ldots, p_J(n)]^T$)

$$\lim_{n \to \infty} \mathbf{p}(n) = \boldsymbol{\pi} = \mathbf{r}$$

$\Rightarrow$ Compute ranks from limit of propagated probabilities

Probability propagation

- Can also write probability propagation in matrix form

$$p_i(n+1) = \sum_{j \in n^{-1}(i)} P_{ji} p_j(n) = \sum_{j=1}^{J} P_{ji} p_j(n) \qquad \text{for all } i$$

- Right hand side is just definition of a matrix product leading to

$$\mathbf{p}(n+1) = \mathbf{P}^T \mathbf{p}(n)$$

- Idea: can approximate rank by large n probability distribution

  $\Rightarrow \mathbf{r} = \lim_{n \to \infty} \mathbf{p}(n) \approx \mathbf{p}(n)$ for n sufficiently large

Ranking algorithm

- Algorithm is just a recursive matrix product, a power iteration

Output : Vector $\mathbf{r}(i)$ with ranking of node i
Input : Scalar n indicating maximum number of iterations
Input : Matrix $\mathbf{P}$ containing transition probabilities

$m = 1$; % Initialize time
$\mathbf{r}$=(1/J)ones(J,1); % Initial distribution uniform across all nodes
while $m < n$ **do**
    $\mathbf{r} = \mathbf{P}^T\mathbf{r}$; % Probability propagation
    $m = m + 1$;
end

Interpretation of probability propagation

- Q: Why does the random walk converge so slow?
- A: Need to register a large number of agent visits to every state

  Ex: 40 nodes, say 100 visits to each $\Rightarrow 4 \times 10^3$ iters.
- Smart idea: Unleash a large number of agents K

$$r_i = \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}\left\{A_{km} = i\right\}$$

- Visits are now spread over time and space

  $\Rightarrow$ Converges "K times faster"

  $\Rightarrow$ But haven't changed computational cost

Interpretation of prob. propagation (continued)

- Q: What happens if we unleash infinite number of agents K?

$$r_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}\left\{A_{km} = i\right\}$$

- Using law of large numbers and expected value of indicator function

$$r_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{E}\left[\mathbb{I}\left\{A_m = i\right\}\right] = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathrm{P}\left(A_m = i\right)$$

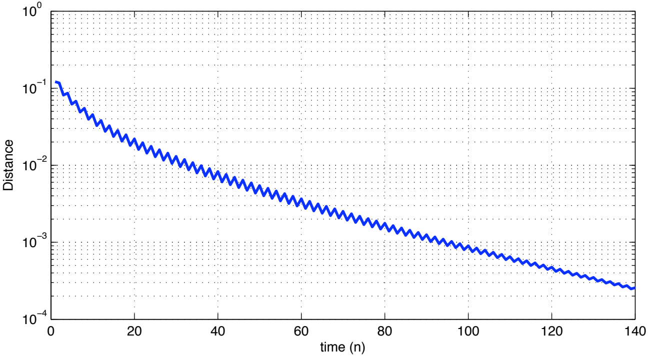
- Graph walk is an ergodic MC, then $\lim_{m\to\infty} \mathrm{P}\left(A_m = i\right)$ exists, and

$$r_i = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} p_i(m) = \lim_{n\to\infty} p_i(n)$$

$\Rightarrow$ Probability propagation $\approx$ Unleashing infinitely many agents

Distance to rank

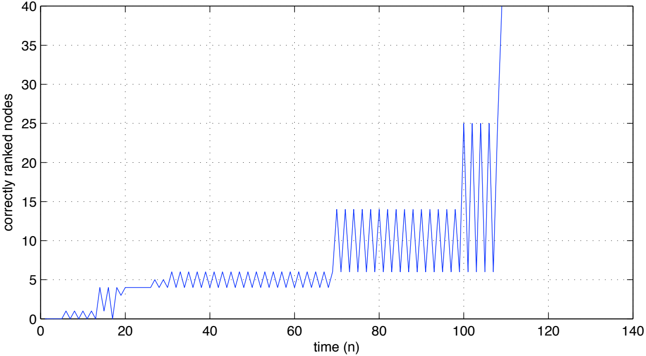
- Initialize with uniform probability distribution $\Rightarrow \mathbf{p}(0) = (1/J)\mathbf{1}$
  - $\Rightarrow$ Plot distance between $\mathbf{p}(n)$ and $\mathbf{r}$



- Distance is 10^{-2} in ≈ 30 iters., 10^{-4} in ≈ 140 iters.
  - $\Rightarrow$ Convergence two orders of magnitude faster than random walk

Number of nodes correctly ranked

- Rank of highest ranked node that is wrongly ranked by time n



- Not bad: All nodes correctly ranked in 120 iterations
- Good: Ten best ranks in 70 iterations
- Great: Four best ranks in 20 iterations

Distributed algorithm to compute ranks

- Nodes want to compute their rank r_i
  - $\Rightarrow$ Can communicate with neighbors only (incoming + outgoing)
  - $\Rightarrow$ Access to neighborhood information only
- Recall probability update

$$p_i(n+1) = \sum_{j \in n^{-1}(i)} P_{ji} p_j(n) = \sum_{j \in n^{-1}(i)} \frac{1}{N_j} p_j(n)$$

  - $\Rightarrow$ Uses local information only
- Distributed algorithm. Nodes keep local rank estimates $r_i(n)$
  - Receive rank (probability) estimates $r_j(n)$ from neighbors $j \in n^{-1}(i)$
  - Update local rank estimate $r_i(n+1) = \sum_{j \in n^{-1}(i)} r_j(n)/N_j$
  - Communicate rank estimate $r_i(n+1)$ to outgoing neighbors $j \in n(i)$
- Only need to know the number of neighbors of my neighbors

Distributed implementation of random walk

- Can communicate with neighbors only (incoming + outgoing)

  $\Rightarrow$ But cannot access neighborhood information

  $\Rightarrow$ Pass agent ('hot potato') around

- Local rank estimates $r_i(n)$ and counter with number of visits V_i
- Algorithm run by node i at time n

```
if Agent received from neighbor then
    V_i = V_i + 1
    Choose random neighbor
    Send agent to chosen neighbor
end
n = n + 1; r_i(n) = V_i/n;
```

- Speed up convergence by generating many agents to pass around

Comparison of different algorithms

- Random walk (RW) implementation
  - $\Rightarrow$ Most secure. No information shared with other nodes
  - $\Rightarrow$ Implementation can be distributed
  - $\Rightarrow$ Convergence exceedingly slow
- System of linear equations
  - $\Rightarrow$ Least security. Graph in central server
  - $\Rightarrow$ Distributed implementation not clear
  - $\Rightarrow$ Convergence not an issue
  - $\Rightarrow$ But computationally costly to obtain approximate solutions
- Probability propagation
  - $\Rightarrow$ Somewhat secure. Information shared with neighbors only
  - $\Rightarrow$ Implementation can be distributed
  - $\Rightarrow$ Convergence rate acceptable (orders of magnitude faster than RW)

Glossary

- Centrality measure
- Closeness centrality
- Dijkstra's algorithm
- Betweenness centrality
- Information controller
- Eigenvector centrality
- Perron's Theorem
- Power method
- Information retrieval
- Link analysis
- Repeated improvement
- Hubs and authorities
- HITS algorithm
- PageRank
- Spider traps
- Scaled PageRank updates
- Ergodic Markov chain
- Limiting probabilities
- Random walk on a graph
- Long-run fraction of state visits
- Probability propagation
- Distributed algorithm