On the Definition and Existence of a Minimum Variance Unbiased Estimator for Target Localization

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***Abstract*—The problem of target localization with ideal, binary detectors is considered in one-dimensional space for both the censored and noncensored scenarios. In the censored setting, the problem is equivalent to estimating the center of a uniform distribution, and does not admit a minimum-variance unbiasedestimator (MVUE). However, it is proven that if the detection range is known and the sensor deployment region is large, both censored and noncensored cases will have an MVUE within the class of functions that are invariant to Euclidean motion. In addition, it is shown that when the detection range is unknown, for the censored case one can still form an MVUE whereas in the noncensored case, an MVUE does not exist. Numerical tests support the theoretical findings of this letter.**

***Index Terms*—Censoring, ideal binary detectors, minimum-variance unbiased estimation, target localization.**

I. Introduction

LOCALIZATION of a transmitter using distributed measurements [1], [2] is a fundamental signal processing task originally considered in radar/sonar [3], [4], also enjoying renewed interest due to recent advances in wireless sensors [5]–[7] and applications in 4G/LTE networks [8]–[10]. Binary (or generally quantized [11]) received signal strength (RSS) sensor measurements are often preferred, because of their bandwidth-efficient communication to a fusion center (FC) tasked with forming an estimator of the target's location [12]–[19]. In addition, often a *censoring* scheme is implemented whereby only those detecting binary sensors (which generate a "one" output) will transmit to the FC [12], [15], [19], [20], thus further effecting savings in energy and communication overhead especially for large sensor deployment regions [20].

From a mathematical point of view, the localization problem can be considered as that of point-wise estimation when data samples are available from a population function representing the probability of detection at a specific distance from the target. The ideal noise-free measurement setting is a limiting scenario and its performance yields an important lower bound for all other localizers with uncertain measurements such as those in the presence of noise and fading. In this limiting case, the problem is equivalent to estimating the centroid of a uniform distribution from a limited number of samples. However, for this formulation of the problem a Cramer–Rao bound (CRB) does not exists because the likelihood function is not well behaved [21], [22]. Therefore, a reasonable approach to assess estimation performance in this scenario would be to find a minimum-variance unbiased estimator (MVUE).

This raises the question as to whether an MVUE exists at all. One standard approach to provide an answer is to first examine the existence of a complete sufficient statistic (CSS) [22], since for instance Lehmann and Sheffe show that if a CSS exists, all estimable parametric functions could be uniformly MVU estimated [23]. Unfortunately, in this localization setting a CSS does not exist most of the time as we show in Sections III-A and IV. Bahadur provides a converse to the Lehmann–Sheffe theorem, and shows that if every estimable parametric function of the unknown parameter admits a uniform MVUE, then a CSS exists [24]. However, [25], [26] provide examples to show that even when a CSS does not exist, one may still be able to find an MVUE for *some* (but not every) parametric function. Building on classic results on parameter estimation from uniformly-distributed scalar data, in Section III-A we show an MVUE does not exist for one-dimensional (1-D) target localization with censored observations [27], [28]. However, we introduce a constraint on the function space which is natural for localization estimators, and prove that within the class of functions that are *invariant to Euclidean motion* an MVUE exists. This result offers a novel example along the lines of those in [25], [26]. We also study different variants of the localization problem in 1-D space—namely when sensor observations are not censored as well as when then detection range is unknown—and discuss the existence of MVUEs for each of them (Section IV). Numerical tests in Section V corroborate the analytical findings of this paper, while concluding remarks are given in Section VI.

II. System Model and Problem Formulation

Consider a target with unknown location z_T, which transmits a signal whose power propagates isotropically and is attenuated monotonically as a function of the distance from the target. In this letter, we study the target localization problem in 1-D space, which gives rise to a classic estimation problem with roots in the statistics literature [23], [27]–[29]. Suppose, N sensors are deployed uniformly at random over a region $\mathcal{R} = [A_1, A_2] \subset \mathbb{R}$ of length $L = |A_2 - A_1|$. Let us consider the signal model in [30], which specifies the RSS measured by the ith sensor (located at z_i), say, is given by

$$s_i = \sqrt{P_0/(1 + \alpha|z_i - z_T|^{\beta})} + w_i \quad (1)$$

where $w_i \sim \mathcal{N}(0, 1)$ denotes standard Gaussian noise, $\{\alpha, \beta\}$ are pathloss coefficients, and P_0 is the maximum received power when $z_i = z_T$. Sensor i compares the RSS s_i from the

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target with a prescribed detection threshold τ, to make a binary decision $d_i \in \{0, 1\}$ about the presence ($d_i = 1$) or absence ($d_i = 0$) of the target in its vicinity. An ideal scenario is considered here, where sensors are assumed capable of noise-free target detection in the absence of Rayleigh fading. Under the assumed isotropic pathloss model, the local detection problems boil down to whether sensors are located within a detection range R from the target or not. The value of R naturally depends on the pathloss parameters $\{\alpha, \beta, P_0\}$ and the threshold τ. This ideal scenario can be approximated in practice by averaging s_i over a sufficiently long period of time to mitigate noise, before comparing the result with τ.

Without loss of generality, suppose that the indices of the n detecting sensors (for which $d_i = 1$) are $\mathcal{I}_{\mathrm{D}} = \{1, \ldots, n\}$, and let $\mathcal{Z}_{\mathrm{D}} = \{z_i : i \in \mathcal{I}_{\mathrm{D}}\}$ collect the corresponding locations. Notice that $\forall i \in \mathcal{I}_{\mathrm{D}},\ z_i \in [z_T \pm R]$. Likewise, for the remaining $p = N - n$ nondetecting sensors one can define $\mathcal{I}_{\mathrm{ND}} = \{n+1, \ldots, N\}$ and $\mathcal{Z}_{\mathrm{ND}} = \{z_i : i \in \mathcal{I}_{\mathrm{ND}}\}$. We also assume that at least one sensor is a detecting sensor ($n \geq 1$) and $\mathcal{R}$ is sufficiently large such that $[z_T \pm 3R] \subset [A_1, A_2]$. Hence, the results of this work are also valid when $\mathcal{R}$ is unbounded.

In this context, we will study the target localization problem both with and without sensor observation *censoring*. In the censored scenario of Section III, only those detecting sensors $i \in \mathcal{I}_{\mathrm{D}}$ communicate their own location and decision to the FC, while the remaining ones transmit nothing to save energy. In the noncensored scenario discussed in Section IV, all sensors inform the FC of their locations and decisions. The range R can be considered known or unknown depending on whether or not the propagation model and transmit power are known to the FC – we will investigate estimators for each case separately.

Remark 1 (Scope and value of the analysis): Arguments in Section III extend to target localization in 2-D or 3-D space, but those in Section IV do not [cf., the discussion following (8)]. Notwithstanding, studying the existence of MVUEs for the aforementioned 1-D setting is of independent interest beyond the concrete localization application, and our findings for constrained estimators also contribute to the statistical estimation theory literature. Regarding the assumption on ideal (noise-free) binary detectors, results in this setting can shed valuable insights on the fundamental limits attainable by localization algorithms based on censored observations in the presence of uncertainty, such as noise and Rayleigh fading.

III. Existence of an MVUE Under Censoring

Under censoring, only $\mathcal{Z}_{\mathrm{D}}$ is available to the FC and the localization problem amounts to estimating the center of a uniform distribution from n samples.

A. Known Detection Range

Suppose R is known so the goal is that of estimating z_T from n samples of a uniform distribution over $[z_T \pm R]$. In this setting the pair $\{z_{\mathrm{D}_L}, z_{\mathrm{D}_U}\}$, where $z_{\mathrm{D}_L} := \min_{i \in \mathcal{I}_{\mathrm{D}}} z_i$ and $z_{\mathrm{D}_U} := \max_{i \in \mathcal{I}_{\mathrm{D}}} z_i$, is a minimal sufficient statistic (MSS) for estimating z_T [27], [29]. Intuitively, if the minimum and maximum samples are known the rest of the data will not provide additional information about the target, because any sensor in between must be a detecting sensor. If we consider any estimator of z_T as $\hat{z}_T = v(\mathcal{Z}_{\mathrm{D}})$, the Rao–Blackwell theorem indicates that $h(z_{\mathrm{D}_L}, z_{\mathrm{D}_U}) := \mathbb{E}[v(\mathcal{Z}_{\mathrm{D}}) | \{z_{\mathrm{D}_L}, z_{\mathrm{D}_U}\}]$ will have no greater variance than that of $v(\mathcal{Z}_{\mathrm{D}})$, where $\mathbb{E}\cdot$ denotes the expectation operator [22]. Thus, we can restrict our search for an MVUE to functions of $\{z_{\mathrm{D}_L}, z_{\mathrm{D}_U}\}$.

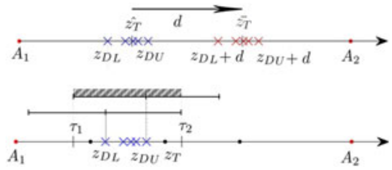

Fig. 1. (Top) noninvariance to Euclidean motion: $\bar{z}_{\mathrm{T}} = \hat{z}_{\mathrm{T}} + d$. (Bottom) *possible target region* illustration under censoring, where $\tau_1 = z_{\mathrm{D}_U} - R$ and $\tau_2 = z_{\mathrm{D}_L} + R$.

Although $\{z_{\mathrm{D}_L}, z_{\mathrm{D}_U}\}$ are MSS for this problem, a CSS does not exist [27], [29]. Furthermore, results in [27], [28] establish that no nonconstant parametric function of z_T (including z_T) can admit a uniform MVUE. However, in localization tasks there are physical characteristics that constraint the set of feasible estimators, and an MVUE may exist among this restricted class. Specifically, in target localization (and perhaps other estimators of physical phenomena) it is reasonable to assume that *estimators are invariant under Euclidean motion*. In other words one expects that if the observations shift, then the estimate should shift accordingly as shown in Fig. 1 (top). Therefore, if we estimate z_{T} by $h(z_{\mathrm{D}_L}, z_{\mathrm{D}_U})$, we should have $h(z_{\mathrm{D}_L} + d, z_{\mathrm{D}_U} + d) = h(z_{\mathrm{D}_L}, z_{\mathrm{D}_U}) + d$.

Let us define the *possible target region* for data $\mathcal{Z}_{\mathrm{D}}$ as $\mathcal{T}(\mathcal{Z}_{\mathrm{D}}) := [z_{\mathrm{D}_U} - R, z_{\mathrm{D}_L} + R]$ as illustrated by the dashed interval in Fig. 1 (bottom). From the definition, it is clear that $\mathcal{T}(\mathcal{Z}_{\mathrm{D}}) = \mathcal{T}(z_{\mathrm{D}_L}, z_{\mathrm{D}_U})$ and $z_T \in \mathcal{T}(z_{\mathrm{D}_L}, z_{\mathrm{D}_U})$. On the other hand, any data for which $z_T \in [z_{\mathrm{D}_U} - R, z_{\mathrm{D}_L} + R]$ are valid observations because this along with $z_{\mathrm{D}_L} \leq z_{\mathrm{D}_U}$ guarantee that $|z_{\mathrm{D}_L} - z_T| \leq R$ and $|z_{\mathrm{D}_U} - z_T| \leq R$. Thus, in the search for an MVUE it suffices to consider functions of $\mathcal{T}(z_{\mathrm{D}_L}, z_{\mathrm{D}_U})$, i.e., $g(\mathcal{T}(z_{\mathrm{D}_L}, z_{\mathrm{D}_U}))$, while invariance under Euclidean motion requires that $g(\mathcal{T}(z_{\mathrm{D}_L} + d, z_{\mathrm{D}_U} + d)) = g(\mathcal{T}(z_{\mathrm{D}_L}, z_{\mathrm{D}_U})) + d$.

For this estimator class, the localization mean-squared error (MSE) given sensor observations with $|\mathcal{Z}_{\mathrm{D}}| = n$ is as follows:

$$\mathbb{E}(\hat{z}_T - z_T)^2 = 2\binom{n}{2} \int_{z_T - R}^{z_T + R} \int_{z_T - R}^{z_T + R} \left(\frac{y - x}{L}\right)^{n-2} \times \left(\frac{1}{L}\right)^2 \left(g(\mathcal{T}(x, y)) - z_T\right)^2 \mathbb{I}\{x \leq y\} dy dx \tag{2}$$

where $\mathbb{I}\{X\}$ is the indicator function of event X. Notice that in (2) the binomial coefficient compensates for the fact that any two of the n detecting sensors could end up as $\{z_{\mathrm{D}_L}, z_{\mathrm{D}_U}\}$; and $(\frac{z_{\mathrm{D}_U} - z_{\mathrm{D}_L}}{L})^{n-2}$ is the probability that all $n - 2$ other detecting sensors are located in $[z_{\mathrm{D}_L}, z_{\mathrm{D}_U}]$. Now, introduce a change of variable $t := y - x \in [0, 2R]$ to eliminate y, namely

$$\mathbb{E}(\hat{z}_T - z_T)^2 = n(n-1) \int_{z_T - R}^{z_T + R} \int_0^{2R} \left(\frac{t}{L}\right)^{n-2} \left(\frac{1}{L}\right)^2 \times \left(g(\mathcal{T}(x, x + t)) - z_T\right)^2 \mathbb{I}\{z_{\mathrm{T}} \in [t + x - R, R + x]\} dt dx.$$

The assumption regarding invariance of g to Euclidean motion implies $g(\mathcal{T}(x, t + x)) = g(\mathcal{T}(0, t)) + x$. In addition, $\mathbb{I}\{z_T \in$

$[t+x-R, R+x]\} = \mathbb{I}\{z_T - x \in [t-R, R]\}$, so yet another change of variable $t' = z_T - x$ yields

$$\mathbb{E}(\hat{z}_T - z_T)^2 = n(n-1)\int_{-R}^{+R}\int_0^{2R}\left(\frac{t}{L}\right)^{n-2}\left(\frac{1}{L}\right)^2 \times (g(\mathcal{T}(0,t)) - t')^2\, \mathbb{I}\{t' \in [t-R, R]\} dt dt'. \quad (3)$$

Changing the order of integration, noting that $\mathcal{T}(0,t) = [t-R, R]$ and that $t \geq 0$ implies $[t-R,\ R] \subset [-R,\ R]$, then

$$\mathbb{E}(\hat{z}_T - z_T)^2 = n(n-1)\int_0^{2R}\left(\frac{t}{L}\right)^{n-2}\left(\frac{1}{L}\right)^2 \times \left[\int_{t-R}^{R} (g([t-R,\ R]) - t')^2\, dt'\right] dt. \quad (4)$$

Clearly, $g([t-R,\ R])$ does not depend on t', and if selected as the *center of gravity* of the inner integration interval, i.e.,

$$g([t-R,\ R]) = \text{CG}([t-R, R]) := \frac{(t-R)+R}{2} \quad (5)$$

then the inner integral is minimized [31]–[33]. Since the integrand of the outer integral is positive, this choice also minimizes the MSE in (4).

Following the steps leading to (4), the bias of g is as follows:

$$\mathbb{E}(\hat{z}_T - z_T) = n(n-1)\int_0^{2R}\left(\frac{t}{L}\right)^{n-2}\left(\frac{1}{L}\right)^2 \times \left[\int_{t-R}^{R} (g([t-R,\ R]) - t')\, dt'\right] dt \quad (6)$$

which vanishes for the choice in (5). Noting that $\text{CG}([x+t-R, x+R]) = \text{CG}([t-R, R]) + x$, then it follows that $g(\mathcal{T}(x,y)) = \text{CG}(\mathcal{T}(x,y))$ will be the MVUE among all estimators that are invariant to Euclidean motion.

B. Unknown Detection Range

In this section we study the problem when R is unknown, implying that $\{z_t, R\}$ are now both unknown parameters. In this case, $\{z_{\text{D}_L}, z_{\text{D}_U}\}$ not only is an MSS but is also *complete* [27], [29]. Consider the sample mean estimator $v(\mathcal{Z}_\text{D}) = \frac{1}{n}\sum_{i\in\mathcal{I}_\text{D}} z_i$. According to the Rao–Blackwell theorem, $\hat{z}_T = \mathbb{E}[v(\mathcal{Z}_\text{D})|\{z_{\text{D}_L}, z_{\text{D}_U}\}] = \frac{z_{\text{D}_L} + z_{\text{D}_U}}{2}$ is the MVUE [22]. Interestingly, this is the center of gravity MVUE in Section III-A, but derived without constraining the class of admissible estimators and for a different reason (i.e., CSS).

IV. Existence of an MVUE Without Censoring

In the absence of censoring, all sensors report their locations and decisions to the FC. Therefore, the problem is that of estimating the center of a uniform distribution over $[z_T \pm R]$ given n samples, and p additional informative samples from another uniform distribution over the complementary set $\mathcal{R} \setminus [z_T \pm R]$. Consider defining

$$z_{\text{D}_L} := \min_{i\in\mathcal{I}_\text{D}} z_i, \quad z_{\text{D}_U} := \max_{i\in\mathcal{I}_\text{D}} z_i$$

$$z_{\text{ND}_L} := \max_{\substack{i\in\mathcal{I}_\text{ND}\\ z_i < z_{\text{D}_L}}} z_i, \quad z_{\text{ND}_U} := \min_{\substack{i\in\mathcal{I}_\text{ND}\\ z_i > z_{\text{D}_U}}} z_i.$$

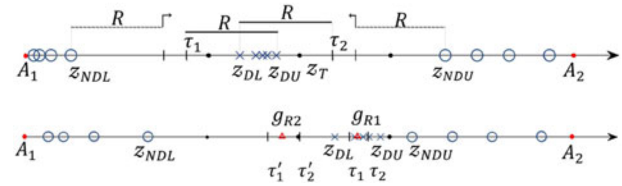

Fig. 2. (top) Relative locations of $z_{\text{D}L}, z_{\text{D}U}, z_{\text{ND}L}, z_{\text{ND}U}$ in noncensored scenario and the effect in τ_1 and τ_2 determination. (bottom) Value of g_R for $R_1 = 0.7$ and $R_2 = 2.5$ when $(z_{\text{D}L}, z_{\text{D}U}, z_{\text{ND}L}, z_{\text{ND}U}) = (2.8, 3.8, -1.3, 4.5)$.

One can show that z_{ND_L} summarizes the information of all nondetecting sensors located at $z < z_{\text{D}_L}$; and likewise $z_{\text{ND}U}$ summarizes the information of all nondetecting sensors located at $z > z_{\text{D}_U}$. As in the censored case, the pair $\{z_{\text{D}_L}, z_{\text{D}_U}\}$ summarizes the information of all detecting sensors.

A. Known Detection Range

With known R, the *possible target region* can be defined as follows:

$$\mathcal{T}(\mathcal{Z}_\text{D}, \mathcal{Z}_\text{ND}) := \mathcal{T}(\mathcal{Z}_\text{D}) \setminus ([z_{\text{ND}_L} \pm R] \cup [z_{\text{ND}_U} \pm R]) \quad (7)$$

where recall $\mathcal{T}(\mathcal{Z}_\text{D}) := [z_{\text{D}_U} - R, z_{\text{D}_L} + R]$ is the possible target region under censoring. The following conditions categorize the relative positions of $\{z_{\text{D}_L}, z_{\text{D}_U}, z_{\text{ND}_L}, z_{\text{ND}_U}\}$

i) $(z_{\text{D}U} - R) \geq (z_{\text{ND}L} + R) \& (z_{\text{D}L} + R) \leq (z_{\text{ND}U} - R)$,
ii) $(z_{\text{D}U} - R) < (z_{\text{ND}L} + R) \& (z_{\text{D}L} + R) > (z_{\text{ND}U} - R)$,
iii) $(z_{\text{D}U} - R) < (z_{\text{ND}L} + R) \& (z_{\text{D}L} + R) \leq (z_{\text{ND}U} - R)$,
iv) $(z_{\text{D}U} - R) \geq (z_{\text{ND}L} + R) \& (z_{\text{D}L} + R) > (z_{\text{ND}U} - R)$.

For instance, Fig. 2 (top) illustrates a condition (i) setting. Depending on which of these conditions is true, $\mathcal{T}(\mathcal{Z}_\text{D}, \mathcal{Z}_\text{ND})$ can be calculated based on only two of the elements in $\{z_{\text{D}_L}, z_{\text{D}_U}, z_{\text{ND}_L}, z_{\text{ND}_U}\}$ as follows:

$$\mathcal{T}(\mathcal{Z}_\text{D}, \mathcal{Z}_\text{ND}) = \begin{cases} [z_{\text{D}U} - R, z_{\text{D}L} + R], & (i) \text{ holds} \\ [z_{\text{ND}L} + R, z_{\text{ND}U} - R], & (ii) \text{ holds} \\ [z_{\text{ND}L} + R, z_{\text{D}L} + R], & (iii) \text{ holds} \\ [z_{\text{D}U} - R, z_{\text{ND}U} - R], & (iv) \text{ holds.} \end{cases} \quad (8)$$

The interval $\mathcal{T}(\mathcal{Z}_\text{D}, \mathcal{Z}_\text{ND})$ incorporates all information regarding the target location. Moreover, a set of observations is valid if and only if $z_T \in \mathcal{T}(\mathcal{Z}_\text{D}, \mathcal{Z}_\text{ND})$. It can be shown that $\mathcal{T}(\mathcal{Z}_\text{D}, \mathcal{Z}_\text{ND})$ is a MSS for the estimation of z_T, although it is not complete.[1] Notice also that defining a possible target region as well as enumerating all possible sensor location configurations becomes markedly more challenging for localization in 2-D or 3-D space [cf. Remark 1].

Because the four conditions in (8) cannot be simultaneously satisfied for any data sample, we can partition the observation space based upon which condition is met. Proceeding like in Section III-A, it is possible to compute the bias and estimation MSE for an estimator $g(\mathcal{T}(\mathcal{Z}_\text{D}, \mathcal{Z}_\text{ND}))$, over each of these

[1] Let $\mathcal{T}(\mathcal{Z}_\text{D}, \mathcal{Z}_\text{ND}) = [\tau_1, \tau_2]$. To show incompleteness we can equivalently consider the MSS as $\{\tau_1, (\tau_2 - \tau_1)\}$. This way, $\tau_2 - \tau_1$ is an ancillary statistic whose probability density does not depend on z_T, but when combined with τ_1 provides information about z_T. Therefore, the MSS cannot be decoupled from the ancillary part of the data, an indication of incompleteness [27].

four disjoint regions in observation space. Interestingly, one can still show that CG $(\mathcal{T}(\mathcal{Z}_{\rm D}, \mathcal{Z}_{\rm ND}))$ is unbiased and minimizes the MSE on each region among all estimators that are invariant to Euclidean motion, provided the deployment region $\mathcal{R}$ is sufficiently large. It thus follows CG $(\mathcal{T}(\mathcal{Z}_{\rm D}, \mathcal{Z}_{\rm ND}))$ is an MVUE over the entire observation space. Details are omitted due to limited space, but can be found in an online appendix [34].

B. Unknown Detection Range

Here, we revisit the previous analysis when R is itself an unknown parameter along z_T. Similar to Section IV-A, the most informative sensor locations are $\{z_{{\rm D}_L}, z_{{\rm D}_U}, z_{{\rm ND}_L}, z_{{\rm ND}_U}\}$. However since R is unknown, defining a possible target region is not as straightforward as before [cf. (8)]. In fact, R should be constrained to the following range of possible values:

$$\frac{z_{{\rm D}_U} - z_{{\rm D}_L}}{2} < R < \frac{z_{{\rm ND}_U} - z_{{\rm ND}_L}}{2}$$

to guarantee that the interval in (7) is not empty.

Given statistics $\{z_{{\rm D}_L}, z_{{\rm D}_U}, z_{{\rm ND}_L}, z_{{\rm ND}_U}\}$, depending on the actual value of R only one of the conditions (i)–(iv) [cf. Section IV-A] will be satisfied. Accordingly, the values of $\mathcal{T}$ and g would be different in each case. Define $g_R\,(\mathcal{Z}_{\rm D}, \mathcal{Z}_{\rm ND})$ as follows:

$$g_R := \begin{cases} \frac{z_{{\rm D}_U} + z_{{\rm D}_L}}{2}, & R \leq \min\left\{\frac{z_{{\rm D}_U} - z_{{\rm ND}_L}}{2}, \frac{z_{{\rm ND}_U} - z_{{\rm D}_L}}{2}\right\} \\ \frac{z_{{\rm ND}_L} + z_{{\rm ND}_U}}{2}, & R > \max\left\{\frac{z_{{\rm D}_U} - z_{{\rm ND}_L}}{2}, \frac{z_{{\rm ND}_U} - z_{{\rm D}_L}}{2}\right\} \\ \frac{z_{{\rm ND}_L} + z_{{\rm D}_L}}{2} + R, & \frac{z_{{\rm D}_U} - z_{{\rm ND}_L}}{2} < R \leq \frac{z_{{\rm ND}_U} - z_{{\rm D}_L}}{2} \\ \frac{z_{{\rm D}_U} + z_{{\rm ND}_U}}{2} - R, & \frac{z_{{\rm ND}_U} - z_{{\rm D}_L}}{2} < R \leq \frac{z_{{\rm D}_U} - z_{{\rm ND}_L}}{2} \end{cases} \quad (9)$$

where each of the four branches corresponds to (i)–(iv), and the estimator values are defined as CG $(\mathcal{T}(\mathcal{Z}_{\rm D}, \mathcal{Z}_{\rm ND}))$; see (8).

Let us consider two different values for R, namely R_1 and R_2 such that each of them enables a different branch in (9). According to the discussion in Section IV-A, $g_{R_1}(\mathcal{Z}_{\rm D}, \mathcal{Z}_{\rm ND})$ and $g_{R_2}(\mathcal{Z}_{\rm D}, \mathcal{Z}_{\rm ND})$ are MVUEs when $R = R_1$ and $R = R_2$, respectively; but they might take different values as exemplified in Fig. 2 (bottom). This establishes an MVUE does not exist for the uncensored case when R is unknown, even if we constrain the search space to those functions invariant to Euclidean motion.

V. Numerical Tests

Here we carry out a numerical test to support the theoretical findings of this letter in the censoring case (Section III). Recalling the CRB does not exist because the likelihood function is discontinuous, we compare the MSE of the center of gravity estimator (MVUE) with the median estimator. Let us denote the order statistics of the elements of $\mathcal{Z}_{\rm D}$ by $\{z_{(1)}, \ldots, z_{(n)}\}$. Then the median estimator is given by

$$\text{med}(\mathcal{Z}_{\rm D}) = \begin{cases} z_{\left(\frac{n+1}{2}\right)}, & n \text{ is odd} \\ \frac{1}{2}\left(z_{\left(\frac{n}{2}\right)} + z_{\left(\frac{n}{2}+1\right)}\right), & n \text{ is even} \end{cases}.$$

By symmetry of the observations around z_T, med $(\mathcal{Z}_{\rm D})$ is unbiased. Moreover, it is invariant to Euclidean motion.

For each trial $N = \lfloor \rho \times L \rfloor$ sensors are scattered uniformly in $\mathcal{R} = [-100, 100]$, where $L = |\mathcal{R}|$ and ρ is the density of sensor deployment. We assume $z_T = 0$, which implies that detecting sensors are the ones located in $[-R, R]$. Trials with no detecting sensors are ignored to calculate the MSE (averaged over 10 000 trials) for both CG $(\mathcal{T}(\mathcal{Z}_{\rm D}))$ and med $(\mathcal{Z}_{\rm D})$. Fig. 3 depicts the log MSE of the aforementioned estimators versus ρ when R is fixed and equal to 1 and 5. Apparently, CG$(\mathcal{T}(\mathcal{Z}_{\rm D}))$ markedly outperforms med $(\mathcal{Z}_{\rm D})$ for both values of R. When ρ is small, trials resulting in only one or two detecting sensors are increasingly likely, explaining with both estimators yield comparable performance.

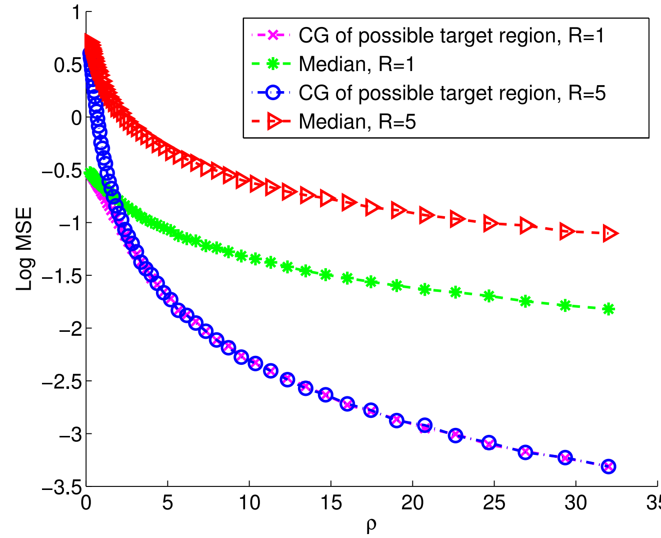

Fig. 3. Log MSE versus the density of sensor deployment (ρ) of the center of gravity (CG) and the median estimators for $R = 1$ and $R = 5$.

VI. Conclusion

We considered the target localization problem in 1-D space using ideal (noise-free) binary RSS measurements. In lieu of a CRB, finding an MVUE for this admittedly simplistic scenario is still important to establish a fundamental performance limit for all other estimators in the presence of uncertainty. Even though in statistical folklore a version of the problem has been categorized as one that does not admit an MVUE, we show the result might be different when one incorporates constraints to the family of candidate estimators. Specifically, we assume the location estimators are *invariant under Euclidean motion* and establish that when the detection range is known, the MVUE does exist for both the censored and noncensored scenarios. However, even a constrained MVUE does not exist in the absence of censoring when the detection range is unknown. We believe that this constraint could be relevant to facilitate novel estimators of other physical phenomena beyond target localization, with desirable properties such as minimum variance.

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