Sampling and Reconstruction of Graph Signals via Weak Submodularity and Semidefinite Relaxation

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BACKGROUND

- Graph Signal Processing
  - Modeling network processes by exploiting the underlying graph structures
  - Applications: sensor and social networks, transportation systems, gene regulatory networks
- Sampling and Reconstruction
  - Selecting a *small representative* subset of graph nodes
  - Applications: resource-constrained sensing in sensor networks, data summarization
- Notation and model

  $\mathbf{x} \in \mathbb{R}^N$: a graph signal with *N* nodes, *non-stationary*

  $\mathbf{A}$: an adjacency matrix of the graph

  $\mathbf{V}$: a basis of the graph signal (here, eigenvectors of the Laplacian matrix **L**)

  $\bar{\mathbf{x}} = \mathbf{V}^\top \mathbf{x}$: graph Fourier transform, *k*-sparse (*bandlimited*), support *K*, $\mathbb{E}[\bar{\mathbf{x}}\bar{\mathbf{x}}^\top] = \mathbf{P}$

  $\mathbf{U} \in \mathbb{R}^{N\times k}$: a submatrix of $\mathbf{V}$ containing columns indexed by *K*

  $\mathbf{y} = \mathbf{x} + \mathbf{n}$: measurement model, **n** Gaussian noise with $\mathbb{E}[\mathbf{n}\mathbf{n}^\top] = \sigma^2 \mathbf{I}_N$
- MSE Formulation of the graph sampling problem

$$\min_S \ \mathrm{Tr}\left(\bar{\mathbf{\Sigma}}_S\right) \quad \text{s.t.} \quad S \subseteq \mathcal{N}, \ |S| \le k \qquad \bar{\mathbf{\Sigma}}_S = \left(\mathbf{P}^{-1} + \sigma^{-2}\mathbf{A}_{S,r}^\top \mathbf{A}_{S,r}\right)^{-1}$$

  - *NP-hard problem!*
- Prior work [1] based on *greedy* heuristics: guarantees *only* in stationary case
- Our approaches: SDP relaxation and randomized greedy

A SDP RELAXATION FORMULATION

- Solve

$$\min_{\mathbf{z},\mathbf{B}} \ \mathrm{Tr}\left(\mathbf{B}\right) \quad \text{s.t.} \quad 0 \le z_i \le 1, \ \sum_{i=1}^n z_i \le k, \ \mathbf{B} \succeq \mathbf{0}.$$

$$\bar{\mathbf{\Sigma}}_z = \left(\mathbf{P}^{-1} + \sigma^{-2}\sum_{i=1}^n z_i \mathbf{u}_i \mathbf{u}_i^\top\right)^{-1} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{C} & \mathbf{I} \\ \mathbf{I} & \bar{\mathbf{\Sigma}}_z^{-1} \end{bmatrix}.$$

- Round **z** to find the selected subset

WEAK SUBMODULARITY OF THE MSE

- [*Submodularity*] Function $f: 2^X \to \mathbb{R}$ is submodular if for $S \subseteq T \subset X, j \in X \backslash T$

$$f_j(S) = f(S \cup \{j\}) - f(S) \ge f(T \cup \{j\}) - f(T) = f_j(T)$$

- [*Monotonicity*] Function $f: 2^X \to \mathbb{R}$ is monotone if $f(S) \le f(T)$ for $S \subseteq T \subset X$
- [*Curvature*] Let $\mathcal{X}_l = \{(S,T,i) | S \subset T \subset X, i \in X\backslash T, |T\backslash S| = l, |X| = n\}$.

  Then, the maximum element-wise curvature is defined as

$$\mathcal{C}_{\max} = \max_{l\in[n-1]} \max_{(S,T,i)\in\mathcal{X}_l} f_i(T)/f_i(S)$$

- An equivalent formulation of graph sampling:

$$\max_S \ f(S) = \mathrm{Tr}(\mathbf{P} - \bar{\mathbf{\Sigma}}_S) \quad \text{s.t.} \quad S \subseteq \mathcal{N}, \quad |S| \le k. \qquad (1)$$

WEAK SUBMODULARITY OF THE MSE

Theorem 1:
The objective function $f(S)$ is a monotonically increasing set function , $f(\emptyset) = 0$ and

$$f_j(S) = \frac{\mathbf{u}_j^\top \bar{\mathbf{\Sigma}}_S^2 \mathbf{u}_j}{\sigma^2 + \mathbf{u}_j^\top \bar{\mathbf{\Sigma}}_S \mathbf{u}_j} \qquad \bar{\mathbf{\Sigma}}_{S\cup\{j\}} = \bar{\mathbf{\Sigma}}_S - \frac{\bar{\mathbf{\Sigma}}_S \mathbf{u}_j \mathbf{u}_j^\top \bar{\mathbf{\Sigma}}_S}{\sigma^2 + \mathbf{u}_j^\top \bar{\mathbf{\Sigma}}_S \mathbf{u}_j}.$$

Further,

$$\mathcal{C}_{\max} \le \frac{\lambda_{\max}^2(\mathbf{P})}{\lambda_{\min}^2(\mathbf{P})}\left(1 + \frac{\lambda_{\max}(\mathbf{P})}{\sigma^2}\right)^3$$

- Intuition: A *well-conditioned* **P** ensures *weak submodularity*

A RANDOMIZED GREEDY ALGORITHM

- An accelerated graph sampling scheme, inspired by the algorithm in [2] (only for submodular objectives and hence not MSE)
- Randomized step:
  - select a *random subset* R in each iteration with $|R| = \frac{N}{k}\log(1/\epsilon)$
  - essentially reducing the number of oracle calls
  - for $\epsilon = e^{-k}$ we obtain the greedy algorithm in [1]
  - speed gain of $k/\log(1/\epsilon)$ compared to the state-of-the-art scheme in [1]

Algorithm 1 Randomized Greedy Algorithm for Graph Sampling

1: Initialize $S = \emptyset$, $\bar{\mathbf{\Sigma}}_S = \mathbf{P}$.
2: **while** $|S| < k$
3: Choose R by sampling $s = \frac{N}{k}\log(1/\epsilon)$ indices uniformly from $\mathcal{N}\backslash S$
4: $j_s = \mathrm{argmax}_{j\in R} \frac{\mathbf{u}_j^\top \bar{\mathbf{\Sigma}}_S^2 \mathbf{u}_j}{\sigma^2 + \mathbf{u}_j^\top \bar{\mathbf{\Sigma}}_S \mathbf{u}_j}$
5: $\bar{\mathbf{\Sigma}}_{S\cup\{j_s\}} = \bar{\mathbf{\Sigma}}_S - \frac{\bar{\mathbf{\Sigma}}_S \mathbf{u}_j \mathbf{u}_j^\top \bar{\mathbf{\Sigma}}_S}{\sigma^2 + \mathbf{u}_j^\top \bar{\mathbf{\Sigma}}_S \mathbf{u}_j}$
6: Set $S \leftarrow S \cup \{j_s\}$
7: **end while**

PERFORMANCE GUARANTEES

- Guarantee on expected MSE of selected nodes:

Theorem 2:
Let $\alpha = (1 - e^{-\frac{1}{c}} - \frac{\epsilon^\beta}{c})$ where $e^{-k} \le \epsilon < 1$, $c = \max\{1, \mathcal{C}\}$, and $\beta = 1 + \max\{0, \frac{s}{2n} - \frac{1}{2(n-s)}\}$. Let S be the set returned by the randomized greedy algorithm and let O denote the optimal set of nodes. Then,

$$\mathbb{E}\left[\mathrm{Tr}(\bar{\mathbf{\Sigma}}_S)\right] \le \alpha \mathrm{Tr}(\bar{\mathbf{\Sigma}}_O) + (1-\alpha)\mathrm{Tr}(\mathbf{P}_x).$$

- Intuition: average MSE over ensemble of sampling problem is near optimal

- Proof idea: in each iteration, *R with high probability* contains a node indexed by *O* if $|R| = \frac{N}{k}\log(1/\epsilon)$.
- Next, a *probably approximately correct (PAC)* view:
  - Effect of randomization: in i^{th} iteration $f_{j_{rg}}(S_{rg}) = \eta_i f_{j_g}(S_g)$ where $0 < \eta_i \le 1$ are random variables.

PERFORMANCE GUARANTEES

Theorem 3:
Instate the notation and hypotheses of Theorem 2. Assume $\{\eta_i\}_{i=1}^k$ are independent such that $\mathbb{E}[\eta_i] \ge \mu$.Then, for all *0<q<1* and for some *C>0* with probability at least $1 - e^{-Ck}$ it holds that

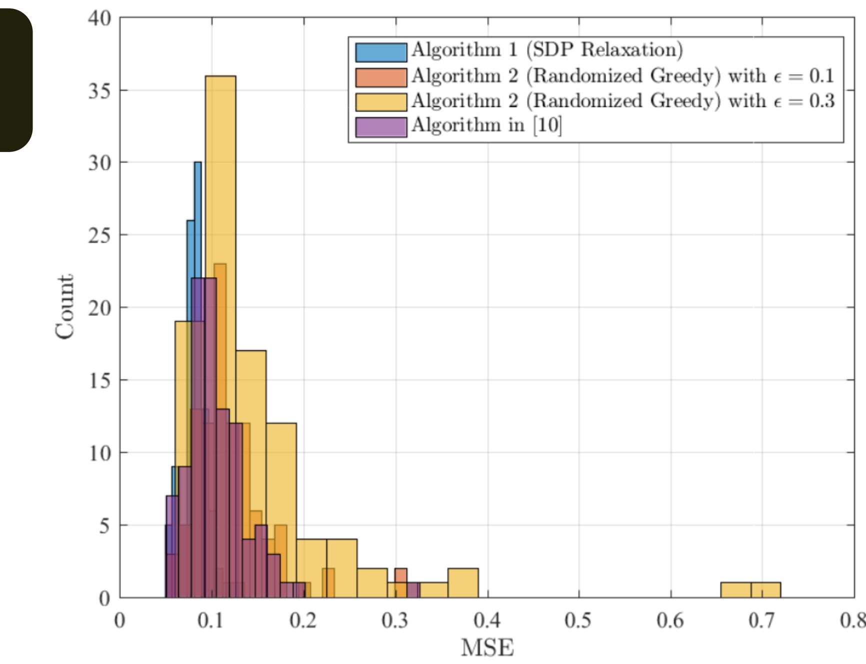
$$\mathrm{Tr}(\bar{\mathbf{\Sigma}}_{S_{rg}}) \le (1 - e^{-\frac{(1-q)\mu}{c}})\mathrm{Tr}(\bar{\mathbf{\Sigma}}_O) + e^{-\frac{(1-q)\mu}{c}}\mathrm{Tr}(\mathbf{P}).$$

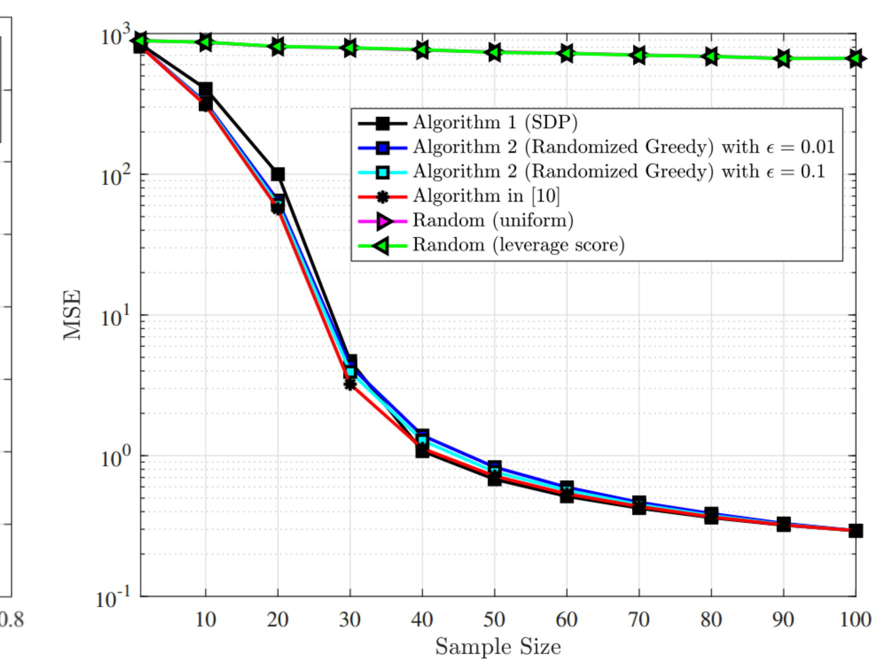
Intuition: MSE for a single sampling problem is also near optimal

Proof idea: Applying *Bernstein inequality* on sum of marginal gains

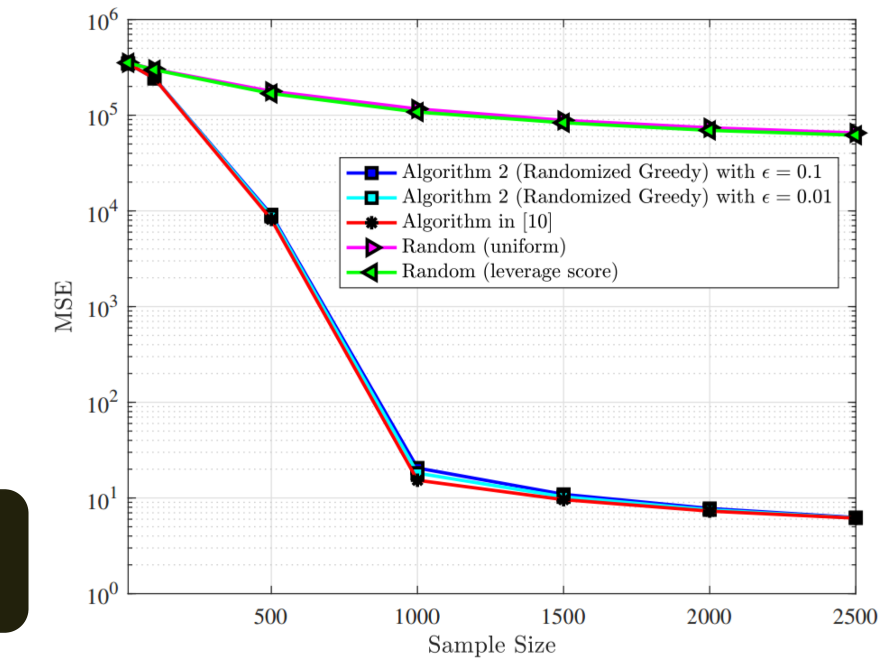
RESULTS

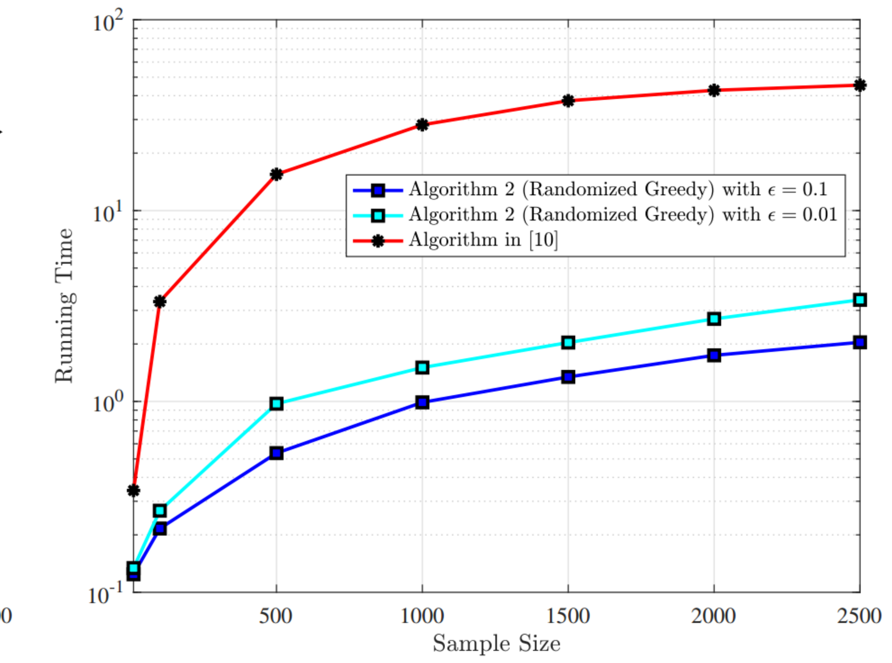
- Simulated Erdos-Renyi graph





- Real-world graph: Minnesota road network





CONCLUSION

- Our contributions:
  - proved *weak submodularity* of (1) for non-stationary graph signals
  - proposed an SDP relaxation framework for sampling and reconstruction
  - proposed a *randomized greedy* algorithm with performance guarantees
  - demonstrated superiority of the proposed methods using simulated and real-world graphs
- Future work:
  - Handling unknown support, extension to nonlinear models

[1] L. FO Chamon and A. Ribeiro, "Greedy sampling of graph signals," IEEE TSP, 2018.

[2] B. Mirzasoleiman, A. Badanidiyuru, A. Karbasi, J. Vondrak, and A. Krause, "Lazier than lazy greedy," in AAAI, 2015.