Acoustic coupling from a focused transducer to a flat plate and back to the transducer

Xucai Chen and Karl Q. Schwarz
Rochester Center for Biomedical Ultrasound and Department of Medicine, University of Rochester, Rochester, New York 14627

Kevin J. Parker
Rochester Center for Biomedical Ultrasound and Department of Electrical Engineering, University of Rochester, Rochester, New York 14627

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An approximate solution for the acoustic coupling factor (the diffraction correction function) from a focused transducer to a flat plate and back to the transducer is provided. This function is useful for system calibrations where a pulse-echo system or transmit–receive system is used. Numerical solutions are provided for the important case where the flat plate is placed near the focal plane of the transducer. The solution for a flat disk transducer is obtained as a limiting case. Experimental evidence for a focused transducer is provided.

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INTRODUCTION

The acoustic coupling factor (diffraction correction function) from a transducer to a flat reflector and back to the transducer is needed when the reflected signal from a flat plate is used to measure the system transfer function of a transmit–receive system, as in the case of a backscatter measurement system. For a flat disk transducer, this function has been solved by a number of authors, including Lommel,[1] Huntington *et al.*,[2] Williams,[3] and Seki *et al.*[4] Extensive numerical integration of these results has been provided by Khimunin[5,6] and Benson and Kiyohara.[7] The solution for a flat disk transducer with transient excitation has been reported by Rhyne.[8] An exact solution to the steady state problem was obtained by Williams[3] in the integral form as

$$D_L(z;f)=1-(4/\pi)\exp(jkz)\int_0^{\pi/2}\exp[-jk(z^2+4a^2\cos^2\theta)]\sin^2\theta\,d\theta, \quad (1)$$

where $z/2$ is the distance from the transducer to the flat plate, and θ can be regarded as a dummy variable for integration. For $ka \gg 1$, Eq. (1) was reduced by Rogers and Van Buren[9] to

$$D_L(z;f)=1-\exp(-j2\pi/S)[J_0(2\pi/S)+jJ_1(2\pi/S)], \quad (2)$$

where $S=2\pi z/ka^2$ is the normalized distance from the transducer to the flat plate.

In many instances, however, it is more desirable to use a focused transducer and it is the diffraction correction function for such a transducer that will be discussed. An exact solution to this problem in integral form will be shown and an explicit approximate solution will also be shown, along with numerical presentations. The solution for the flat disk transducer is obtained as a limiting case of the general solution. Experimental data from a focused transducer are provided.

I. DERIVATION OF THE DIFFRACTION CORRECTION FUNCTION

A pulse-echo system is shown in Fig. 1. The distance from the transducer (the transmitter) to the flat reflector is denoted by $z/2$. The receiver can be regarded as the mirror image of the transmitter, having the same radius a, and geometrical focal length r_0. The half spread angle of the transducer element, α, is calculated from $\alpha=\arcsin(a/r_0)$. The focusing factor of the transducer is defined as $G_p\equiv ka^2/2r_0=ka\sin\alpha/2$. The medium has density ρ_0 and sound speed c_0.

The effective pressure reflected back to the transmitter by the flat plate, $\bar{p}(z;f)$, is equal to the average of the radiation field of the transmitter on the receiver surface, i.e.,

$$\bar{p}(z;f)=\frac{1}{\pi a^2}\int\int_{S_2} p_1(r,\theta)dS_2, \quad (3)$$

where p_1 is the complex pressure distribution of the transmitter. This expression can be written as a plane traveling wave, modified by a correction term, i.e.,

$$\bar{p}(z;f)=p_0\exp[j(\omega t-kz)]D_F(z;f), \quad (4)$$

where p_0 is the pressure amplitude on the surface of the transmitter, and $D_F(z;f)$ is the desired diffraction correction function. A subscript F, which stands for "focused," is used to distinguish this function from the Lommel diffraction function for a flat transducer.

The pressure distribution of the transmitter can be derived from the velocity potential by

$$p_1(r,\theta)=\rho_0\,\partial\phi(r,\theta)/\partial t=j\rho_0\omega\phi(r,\theta). \quad (5)$$

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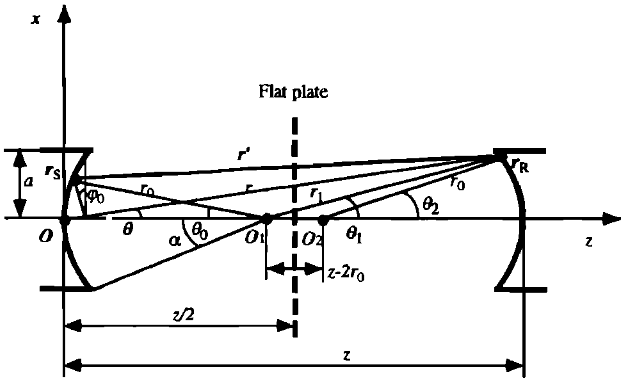

FIG. 1. Coordinate systems used for analysis. The flat plane is positioned z/2 away from the transmitter. The receiver can be considered as the mirror image of the transmitter.

When the size of the transmitter is larger than the wavelength ($ka \gg 1$), the velocity potential of the transmitter can be calculated by the Green's function method (the Rayleigh formulation)

$$\phi(r,\theta)=(u_0/2\pi)\int\int_{S_1}\frac{\exp[j(\omega t-kr')]}{r'}\,dS_1, \quad (6)$$

where u_0 is the normal velocity on the transmitter surface, r' is the distance from a point source on the surface of the transmitter to the point of observation. Defining $p_0 \equiv \rho_0 c u_0$, we have

$$p_1(r,\theta)=j(p_0/u_0)k\phi(r,\theta). \quad (7)$$

Combining Eqs. (3)–(7), we obtain the diffraction correction function

$$D_F(z)=j(k/2\pi^2a^2)\int\int_{S_2}dS_2\int\int_{S_1}\frac{\exp[jk(z-r')]}{r'}\times dS_1. \quad (8)$$

Equation (8) is the exact desired solution in integral form.

The distance from the point source to the point receiver, r', is best described by two spherical coordinate systems with origin $\mathbf{O}_1$ located at the geometrical focus of the transmitter, and origin $\mathbf{O}_2$ located at the geometrical focus of the receiver, as shown in Fig. 1. In system 1, the point source $\mathbf{r}_S$ can be expressed as (r_0,θ_0,φ_0), and the point receiver $\mathbf{r}_R$ can be expressed as $(r_1,\theta_1,0)$. Then

$$r'=\{r_1^2+r_0^2+2r_1r_0(\cos\theta_1\cos\theta_0-\sin\theta_1\sin\theta_0\cos\varphi_0)\}^{1/2}. \quad (9)$$

The point receiver $\mathbf{r}_R$ can also be expressed in system 2 as $(r_0,\theta_2,0)$, where

$$\begin{cases} r_1\cos\theta_1=r_0\cos\theta_2+(z-2r_0),\\ r_1\sin\theta_1=r_0\sin\theta_2. \end{cases} \quad (10)$$

Eliminating r_1 and θ_1 from Eq. (9) by use of Eq. (10), we find

$$r'=\{z^2-2r_0^2[(z/r_0-2)(2-\cos\theta_2-\cos\theta_0)+1-\cos\theta_2\cos\theta_0+\sin\theta_2\sin\theta_0\cos\varphi_0]\}^{1/2}. \quad (11)$$

Letting $U_0=\sin\theta_0/\sin\alpha$, and $U_2=\sin\theta_2/\sin\alpha$, Eq. (11) becomes

$$r'=z-\sin^2\alpha(r^2/z)[(z/r_0-1)(U_0^2+U_2^2)/2+U_0U_2\cos\varphi_0]+O[\sin^4\alpha]+\cdots. \quad (12)$$

Replacing r' in the exponent of Eq. (8) by the first two terms of Eq. (12) and replacing r' in the denominator by the first term of Eq. (12) (similar to the Fresnel approximation), Eq. (8) becomes

$$D_F(z;f)=j\frac{ka^2}{\pi z}\int_{U_2=0}^{1}\int_{U_0=0}^{1}\exp\left[j\frac{ka^2}{2z}\left(\frac{z}{r_0}-1\right)\times(U_2^2+U_0^2)\right]U_2\,U_0\,dU_2\,dU_0\times\int_{\varphi_0=0}^{2\pi}\exp\left(j\frac{ka^2}{z}U_2U_0\cos\varphi_0\right)d\varphi_0. \quad (13)$$

Let us define two nondimensional parameters Y and Z as

$$Y\equiv(ka^2/z)(z/r_0-1),\qquad Z\equiv ka^2/z, \quad (14)$$

and integrate over φ_0 explicitly, Eq. (13) then becomes

$$D_F(z)=j2Z\int_{U_2=0}^{1}\int_{U_0=0}^{1}\exp\left(j\frac{Y}{2}(U_2^2+U_0^2)\right)\times J_0(ZU_2U_0)U_2\,U_0\,dU_2\,dU_0. \quad (15)$$

The two-parameter problem similar to that given in Eq. (15) has been discussed in detail by Lommel.[1] Let

$$I_0(Y,ZU_2)\equiv\int_{U_0=0}^{1}\exp\left(j\frac{Y}{2}U_0^2\right)J_0(ZU_2U_0)U_0\,dU_0, \quad (16)$$

then

$$D_F(z;f)=j2Z\int_{U_2=0}^{1}\exp\left(j\frac{Y}{2}U_2^2\right)I_0(Y,ZU_2)U_2\,dU_2. \quad (17)$$

Equation (16) can be solved by using Eqs. (35) and (36) in Ref. 10 to obtain

$$I_0(Y,ZU_2)=j\{\exp(-jZ^2U_2^2/2Y)-\exp(jY/2)\times[v_0(Y,ZU_2)-jv_1(Y,ZU_2)]\}/Y, \quad (18)$$

where

$$v_0(Y,ZU_2)=\sum_{n=0}^{\infty}(-1)^n(ZU_2/Y)^{2n}J_{2n}(ZU_2),$$
$$v_1(Y,ZU_2)=\sum_{n=0}^{\infty}(-1)^n(ZU_2/Y)^{2n+1}J_{2n+1}(ZU_2), \quad (19)$$

are the zeroth and first-order Lommel functions of the second kind. Substituting Eqs. (18) and (19) into Eq. (17), we have

$$D_F(z;f)=-\frac{Z}{Y}\left(\frac{\exp[j(Y/2-Z^2/2Y)]-1}{j(Y/2-Z^2/2Y)}-\exp(jY)I_2(Y,Z)\right), \quad (20)$$

where

$$I_2(Y,Z)=2\int_{U_2=0}^{1}\exp\left(-j\frac{Y}{2}(1-U_2^2)\right)[v_0(Y,ZU_2)-jv_1(Y,ZU_2)]U_2\,dU_2. \quad (21)$$

The details for the reduction of Eq. (21) will not be given here. It is sufficient to state that by substituting Eq. (19) into Eq. (21), a summation of complex integrals in the form of $\int_{U_2=0}^{1}U_2^{n+1}J_n(ZU_2)\exp[-j(Y/2)(1-U_2^2)]dU_2$ for $n\geqslant 0$ results and their solutions can be derived from similar expressions given in the text by Gray and Mathews.[10] After some lengthy manipulation, Eq. (20) becomes

$$D_F(z;f)=-\frac{1}{X}\left(\frac{\exp[j(Z/2)(X-1/X)]-1}{j(Z/2)(X-1/X)}-\exp(jZX)[S_0(X,Z)-jS_1(X,Z)]\right), \quad (22)$$

where $X\equiv Y/Z\equiv z/r_0-1$, and

$$S_0(X,Z)=\frac{2}{Z}\sum_{n=0}^{\infty}\left[(-1)^n\left(\sum_{p=-2n,2}^{2n}X^p\right)J_{2n+1}(Z)\right], \quad (23)$$

$$S_1(X,Z)=\frac{2}{Z}\sum_{n=0}^{\infty}\left[(-1)^n\left(\sum_{p=-(2n+1),2}^{(2n+1)}X^p\right)J_{2n+2}(Z)\right].$$

Equation (22) is the solution of the diffraction correction function for a focused transducer when the flat plate is positioned at $z/2$ away from the transmitter. The arguments of $D_F(z;f)$ in Eq. (22) have been changed to express its dependence on the two nondimensional parameters X and Z explicitly.

When the flat plate is positioned at the geometrical focal plane ($z=2r_0$), so that $X=1$ and $Z=ka^2/2r_0\equiv G_p$, functions $S_0(X,Z)$ and $S_1(X,Z)$ reduce to $J_0(Z)$ and $J_1(Z)$, respectively, and Eq. (22) becomes

$$D_F(z=2r_0;f)=-\{1-\exp(jG_p)[J_0(G_p)-jJ_1(G_p)]\}. \quad (24)$$

Notice that Z here is equal to the focusing factor of the transducer, G_p.

When r_0 is allowed to approach infinity, $X=-1$ holds for all values of z. The functions $S_0(X,Z)$ and $S_1(X,Z)$ reduce to $J_0(Z)$ and $-J_1(Z)$, respectively, and Eq. (22) reduces to the solution for a flat disk transducer given in Eq. (3), where $2\pi/S$ is used in place of Z.

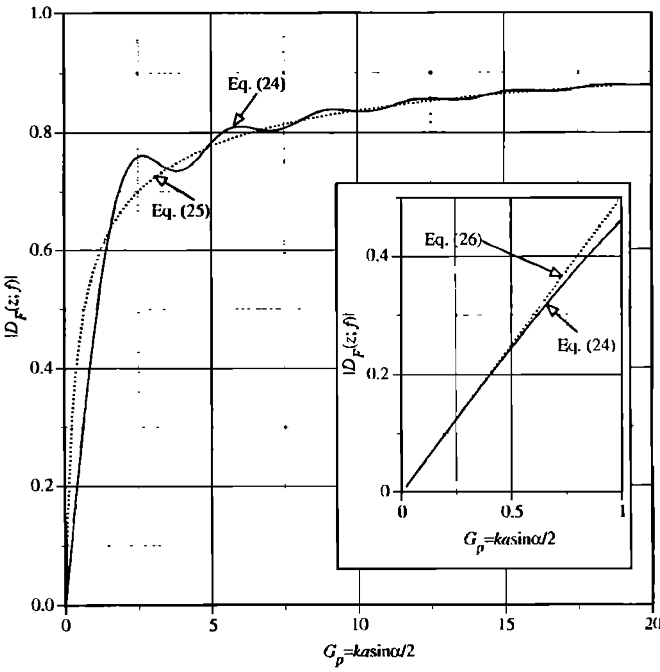

FIG. 2. Amplitude of the acoustic coupling factor $|D_F(z;f)|$ as a function of focusing factor G_p when the flat plate is positioned in the focal plane ($z'=1$). Dashed lines represent the approximate solutions given in Eqs. (25) and (26).

II. NUMERICAL PRESENTATION

The amplitude of $D_F(z;f)$ can be calculated from Eq. (22) readily as a function of X and Z. For comparison purposes, however, it is more advantageous to treat $D_F(z;f)$ as a function of $z'\equiv z/2r_0$ and G_p, where z' is the distance from the transducer to the flat plate ($z/2$) normalized by the geometrical focal length r_0, and G_p is the focusing factor of the transducer defined earlier. Then $X\equiv 2z'-1$, and $Z\equiv G_p/z'$. When the flat plate is at the focal plane ($z'=1$), $X=1$, and $Z=G_p$.

The numerical value of $|D_F(z;f)|$ when $z'=1$ is shown in Fig. 2. It is not a monotonically increasing function of G_p. For $G_p>\pi$, $|D_F(z;f)|$ can be replaced by the following empirical approximation

$$|D_F(z=2r_0;f)|\approx\exp(-\sqrt{1/\pi G_p}),\quad G_p>\pi. \quad (25)$$

In the range specified, the relative error of the approximation in Eq. (25) is less than 1%. For $G_p<1$, $|D_F(z;f)|$ can be replaced by the following empirical approximation

$$|D_F(z=2r_0;f)|\approx G_p/2,\quad G_p<1. \quad (26)$$

The condition $G_p<1$ indicates that the geometrical focal length of the transducer is larger than the Rayleigh distance of a flat transducer with the same frequency and size. Equation (26) states that the pressure distribution of the transducer has reached spherical spreading at the geometrical focal plane for such a combination of frequency and size.

Figure 3 shows the variation of $|D_F(z;f)|$ as a function of z' for several values of G_p. For high focusing factors, $|D_F(z;f)|$ diverges rapidly from its value in the focal plane. This indicates that the positioning of the reference

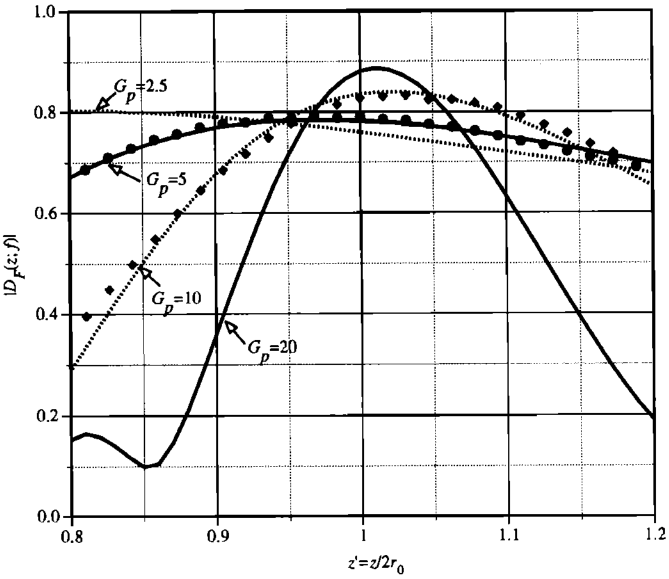

FIG. 3. Amplitude of the acoustic coupling factor $|D_F(z;f)|$ as a function of normalized distance z' for focusing factors G_p=2.5, 5, 10, and 20. Experimental data are shown for $G_p=5$ (f=3.73 MHz) (●) and $G_p=10$ (f=7.47 MHz) (◆). The transducer parameters: a=6.35 mm and r_0=63.5 mm.

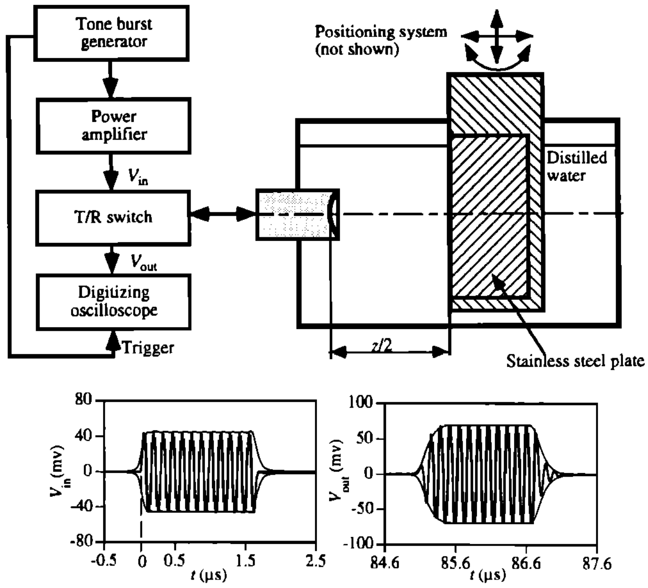

FIG. 5. Schematic diagram of the experimental apparatus. The transducer was driven by a gated tone burst; the reflected signal from the stainless steel plate was acquired by the oscilloscope. Alignment and positioning systems are not shown. The input waveform before the power amplifier and the output waveform for $G_p=10$ (f=7.47 MHz) at $z'=1$ are also shown.

plate is crucial at high frequencies. This fact is emphasized in Fig. 4 where the numerical values of $|D_F(z;f)|$ are compared for z'=0.5, 0.9, 1.0, and 1.1. At $G_p=10$ and $z'=0.9$, $|D_F(z;f)|$ differs from its value in the focal plane by more than 20%. This difference increases as the plate is moved further away from the geometrical focal plane. At $z'=0.5$, for example, $X=0$, and $Z=2G_p$. Equation (22) fails to converge for this special case. For these parameters, Eq. (8) becomes

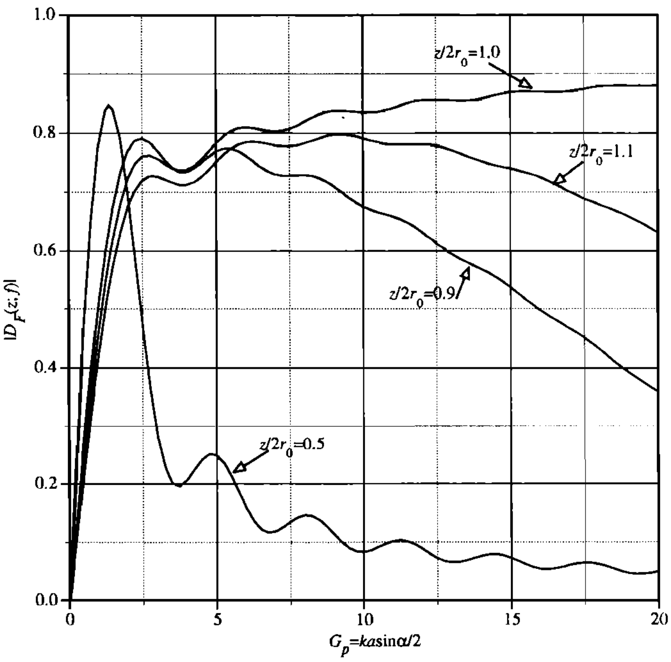

FIG. 4. Amplitude of the acoustic coupling factor $|D_F(z;f)|$ as a function of focusing factor G_p for normalized distances z'=0.5, 0.9, 1.0, and 1.1.

$$
\begin{aligned}
D_F(z=r_0;f) &= j2Z\int_{U_2=0}^{1}\int_{U_0=0}^{1} J_0(ZU_2U_0)\,U_2U_0\,dU_2\,dU_0 \\
&= -2j[1-J_0(Z)]/Z \\
&= -j[1-J_0(2G_p)]/G_p. \qquad (27)
\end{aligned}
$$

The amplitude of this function, also plotted in Fig. 4, shows the loss of reflected acoustic energy through interference at high frequencies.

III. EXPERIMENTAL EVIDENCE

An experiment was performed to test the theory presented above. A schematic diagram of the experimental apparatus is shown in Fig. 5. A gated tone burst, generated by a function generator (model HP 8116A, Hewlett–Packard GmbH, Germany) and amplified 55 dB (model A150, ENI, Inc., Rochester, NY), was used to excite the focused transducer. The transducer was a focused, broadband transducer (model V309, Panametrics, Waltham, MA), with an active element a=6.35 mm in radius, and a geometrical focus r_0=63.5 mm. The flat plate was made of type 304 stainless steel (model TB7545-5, Panametrics, Waltham, MA), with a thickness of 23 mm. A goniometer was used to adjust the angle between the transducer and the plate so that total reflected power was maximum when the plate was positioned in the focal plane. A digitally controlled positioning system (UniSlide® series, Velmex Inc., East Bloomfield, NY) with a precision of 0.01 mm was used to move the plate along the axis of the transducer. Via a transmit/receive (T/R) switch (model RDX-6, RITEK Inc., Warwick, RI), the reflected signal was connected directly to a digitizing oscilloscope (model 9430, LeCroy Corporation, Chestnut Ridge, NY). The radio-frequency (rf) signal was digitized at a 100-MHz sampling rate with 10-bit precision. An antialiasing filter with a cut-off frequency of 45 MHz was used before digitization. Time-domain averaging of 100 traces of digitized rf signal was used for each measurement to reduce broadband noise. The digitized rf signal was transferred via an IEEE 488 interface to a personal computer for storage and processing. The envelope of the rf signal was calculated from the amplitude of the complex analytic function associated with the rf signal.[11]

The received signal at the oscilloscope when the flat plate is positioned at a distance $z/2$ away from the transducer can be written as

$$V_{\text{out}}(z;t) = V_{\text{in,amp}} X(f) F_R(f) X(f) \times \exp[\,j(\omega t - kz)\,] D_F(z;f), \tag{28}$$

where $V_{\text{in,amp}}$ was the amplitude of electrical signal driving the transducer, $X(f)$ was the electroacoustical coupling factor of the transducer, and $F_R(f)$ was the filter function of the receiving circuitry. The filter function $F_R(f)$, which was used to remove the requirement that the transmit-receive circuitry be symmetric, could have included the gain function if any amplification circuitry was used between the T/R switch and the oscilloscope. The combination $|X^2(f)F_R(f)|$, which is a function of frequency only, is the system transfer function which needs to be determined for the calibration of pulse-echo systems. It is clear from Eq. (28) that system calibration is accomplished if the amplitude of $V_{\text{out}}(z;t)$ is measured and $|D_F(z;f)|$ is calculated from Eq. (8) or Eq. (22).

To completely verify the theory presented in Secs. I and II, it is necessary to determine the system transfer function accurately and independently. This is often difficult, if not impossible. For a particular frequency, however, the system transfer function is a constant, and the variation in the amplitude of $V_{\text{out}}(z;t)$ as a function of z can be used to verify the theoretical results.

The input waveform before amplification and the output waveform for $G_p=10$ when the flat plate was positioned in the focal plane are shown in Fig. 5. A total of 12 acoustic cycles were used. Steady state was reached after roughly $r_0 \sin^2\alpha/c_0 = 0.42\ \mu s$, which corresponds to the difference in propagation delays from a point on the transmitter to a point on the "image" receiver.

Shown in Fig. 3 are the amplitudes of the reflected signal as functions of normalized position of the flat plate for two focusing factors, $G_p=5$ and 10, which correspond to $f=3.73$ and 7.47 MHz, respectively. Arbitrary multiplicative constants were used to correct for the unknown system transfer function. Good agreement between theory and the experimental data was observed near the focal plane.

It was also observed theoretically and experimentally that, for a fixed frequency, the maximum in $|D_F(z;f)|$ occurs farther away from the transducer surface than the on-axis pressure maximum, which is always before the geometrical focus. For the transducer used in this experiment, the pressure maximum was found at 47.7 mm away from the transducer surface at 3.73 MHz, corresponding to a normalized distance of 0.75; the pressure maximum was found at 57.4 mm away from the transducer surface at 7.47 MHz, corresponding to a normalized distance of 0.90.

IV. DISCUSSIONS AND CONCLUSION

Some approximations were used in the derivation. The leading term of error in the phase angle $k(z-r')$ is

$$\begin{aligned}\Delta[k(z-r')] = (ka^4/4r_0^3)\{&[(z'-1/2)(U_0^4+U_2^4) \\ &- U_0^2U_2^2]/(2z') + [(z'-1/2)(U_0^2+U_2^2) \\ &+ U_0U_2\cos\varphi_0]^2/(4z'^3)\},\end{aligned} \tag{29}$$

and the necessary condition for the Fresnel-type approximation to be valid is $|\Delta[k(z-r')]| \ll \pi$. If we limit our discussion to $z'>0.5$ (since most of the measurements are made near or at the focal plane), we find $|\Delta[k(z-r')]| < ka^4/4r_0^3$. The necessary condition for the approximation to be valid is therefore $ka^4/4r_0^3 \ll \pi$, or

$$G_p \sin^2\alpha/2\pi \ll 1. \tag{30}$$

This condition is satisfied by a variety of frequency-size combinations for transducers of practical interest.

It was also assumed that the flat plate is perfectly reflective, i.e., its reflection coefficient is 1. For a flat interface, the pressure reflection coefficient is given by Snell's law as $R = 1-2[1+(\rho c/\rho_0 c_0)\cos\theta_{\text{In}}]^{-1}$, where $\rho_0 c_0$ is the acoustic impedance of the medium, ρc is the acoustic impedance of the flat plate, and θ_{In} is the angle of incidence. Strictly speaking, this reflection coefficient should be inserted in the integrand in Eq. (8) as a multiplicative factor. The numerical value of $\cos\theta_{\text{In}}$, however, varies only slightly when the flat plate is sufficiently away from the transmitter. For example, for $z'>0.5$, we have $1 \geqslant \cos\theta_{\text{In}} \geqslant 1-2\sin^2\alpha$. For weakly focused transducers, $\sin^2\alpha \ll 1$. This condition has already been implied by Eq. (30). Allowing $\cos\theta_{\text{In}}=1$, the reflection coefficient can be treated as a constant, equal to the reflection coefficient at the medium–plate interface with normal incidence. This coefficient can be directly inserted in the front of Eq. (8) as well as in the front of the final solution given by Eq. (22). It must be stressed here that the nonperfect reflector must be of sufficient thickness so that the back wall echo is separated from the front wall echo.

Thus we have provided the acoustic coupling factor (the diffraction correction function) from a focused transducer to a flat plate and back to the focused transducer as a function of two nondimensional parameters X and Z, or alternatively, z' and G_p. This function is required for the system calibration of a transmit-receive system. Approximate forms of the function have been provided for $z'=1$, when the flat plate is positioned in the focal plane. Numerical evaluation of the function indicates that positioning of the flat plate for high focusing factors is crucial. The theory includes the solution for a flat disk transducer as a limiting case. Experimental results have been provided to support the theoretical solution.

In Secs. II and III, only the amplitudes of the diffraction correction function were presented. The phase of the diffraction correction function can be computed from Eq. (8) or Eq. (22) and can be used for the accurate measurement of phase velocity in the medium with a focused transducer. It is worthwhile to point out that the theoretical solution provided is also valid when intrinsic absorption of the medium is included. In that case, the wave number k should be replaced by $k-j\alpha$, where α is the absorption coefficient of the medium. The amplitude of the diffraction correction function can then be used for the accurate measurement of the absorption coefficient of the medium with a focused transducer. The phase of the diffraction function can be used for the accurate measurement of the phase velocity in the medium with a focused transducer.

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