Effects of heat conduction and sample size on ultrasonic absorption measurements

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The absorption coefficient of a material can be determined by measuring the heating which occurs as a result of ultrasonic irradiation. When narrow focused beams are used to heat a sample, or when the available volume of a material is restricted to small dimensions, then the effect of heat conduction to surrounding unheated regions becomes significant, complicating the relation between measured temperatures and acoustic parameters. In this paper new analytical expressions, which account for radial and axial heat flow in a medium, are derived for the case of Gaussian-shaped ultrasonic beam patterns in thin or semi-infinite absorbing materials. Solutions are given for temperature histories resulting from an ultrasonic impulse (pulse decay method) or a step input (rate of heating method). The use of these equations in absorption measurements is discussed, and experimental results are given. These expressions provide flexibility in choice of laboratory ultrasonic parameters, and the results are especially useful for many biomedical measurements where the volume of tissue available is restricted.

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LIST OF SYMBOLS

α	amplitude absorption coefficient, Np/cm
β	Gaussian variance, cm^2
c	specific heat, J/g °C
erf(·)	error function, $\mathrm{erf}(z) = (2/\sqrt{\pi})\int_0^z e^{-t^2}\,dt$
I	ultrasonic intensity, W/cm^2
k	thermal diffusivity, cm^2/s
K	thermal conductivity, W/cm °C
ρ	density, g/cm^3
Q_3	instantaneous thermal point source, ° C–cm^3
t	time, seconds
τ	time, seconds
T	temperature, ° C
PD	subscript denoting pulse decay
RH	subscript denoting rate of heating
r,θ,z	axes of cylindrical coordinates
x,y,z	axes of Cartesian coordinates

INTRODUCTION

The ultrasonic absorption coefficient of a material can be determined by measuring the temperature elevation caused by insonation at known frequency and intensity. An embedded thermocouple probe can be used to provide instantaneous temperature readings at a point in the medium. The classical approach utilizes a 1-s duration on time at constant intensity, where the absorption coefficient is determined by measuring the slope of temperature during insonation.[1,2] This rate of heating method presumes that no significant heat conduction to surrounding, cooler regions takes place during measurement, and therefore the relation between temperature slope and absorption is given simply form the energy conservation equation:

$$\rho c \frac{dT}{dt} = 2\alpha I. \tag{1}$$

In practice, thermal conduction from an ultrasonically heated region can occur along two principle axes. In experiments where the thermojunction is located at the center of the mainlobe, heat will flow radially to regions of lower intensity which are cooler. Heat can also flow along the axis of insonation since the surface of the test sample is usually in contact with a coupling medium which is nonabsorbing and therefore unheated. Figure 1 depicts a typical experimental arrangement using a focused beam, coupling medium, and thermocouple embedded in an absorbing material. From this figure one can judge qualitatively that the magnitude of heat loss in the radial r, and axial z, directions will depend on the beamwidth, and thermocouple depth, respectively. Guidelines have been given which relate the radial loss of heat in rate of heating measurements to the focal region size, or beamwidth.[3] No systematic treatment of the axial flow of heat to the sample surface is known by this author. The need to minimize heat conduction effects during rate of heating measurements places restrictions on sample sizes and ultrasonic beamwidths used in practice. The thermal pulse decay method for measuring absorption coefficients was developed[4] to explicitly account for beam shape and radial heat conduction effects, thereby allowing for the use of arbitrarily small, Gaussian shaped focal regions. However, the question of heat conduction in the axial direction towards the unheated coupling medium was not explicitly addressed in the development of the pulse decay equations. The effects of heat transfer to the coupling medium can become prominent, in both pulse decay and rate of heating measurements, when measurements are performed on thin tissue specimens or organs from small animals.

In this paper, the effects of heat conduction to the sur-

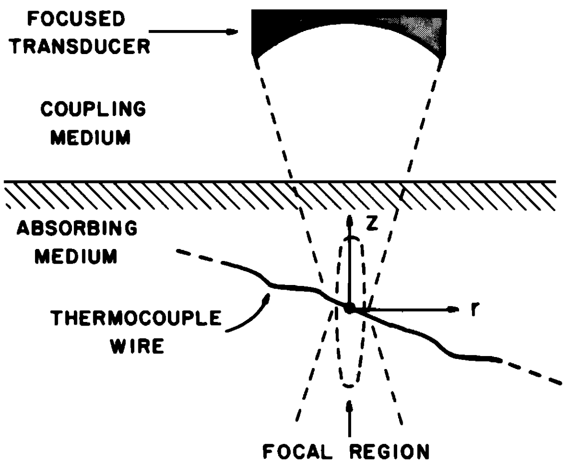

FIG. 1. Typical experimental configuration using focused ultrasonic beam and embedded thermocouple to measure the absorption coefficient.

rounding, unheated coupling medium are considered for both the pulse decay and rate of heating measurements. Exact analytical expressions are developed which describe the temperature history in both pulse decay and rate of heating experiments, for any Gaussian shaped beamwidth size, and arbitrary sample thickness. Procedurally, exact expressions are first developed for the pulse decay curves. These results are interpreted as the output of a linear system to an ultrasonic impulse. Then, expressions for rate of heating temperature histories are easily obtained by considering this experiment as eliciting the response of the same linear system to a step input. The results are of practical importance in biomedical ultrasound since sample volumes are unavoidably small in many experiments using animal tissues. In work on human tissue, biopsy samples of millimeter dimensions are sometimes available, and the expressions derived in this paper could permit meaningful absorption measurements to be obtained from these restricted volumes, enhancing the ultrasonic characterization of normal and diseased tissue.

I. THEORY

A. Radial heat conduction only

In the pulse decay method, a short burst of ultrasound is used to generate an initial temperature elevation in the absorbing medium. The intensity distribution is assumed to have the Gaussian form:

$$I(r) = I_0 e^{-r^2/\beta} \tag{2}$$

and is assumed to have no variation in the axial direction. This, of course, precludes any heat transfer in the z direction. Within this approximation, the initial excess temperature distribution in the material following a short pulse of Δt-s duration is

$$T(r) = T_0 e^{-r^2/\beta}, \tag{3}$$

where the proportionality between temperature and intensity is

$$T_0 = 2\alpha I_0 \Delta t/\rho c. \tag{4}$$

The resulting temperature history at any distance r from the center of the focal region is given by[4]:

$$T_{\rm PD}(r,t) = \frac{2\alpha I_0 \Delta t}{\rho c[1+(4kt/\beta)]} \exp\left(\frac{-r^2}{(4kt+\beta)}\right). \tag{5}$$

In practice, the first second of thermocouple reading is influenced by the localized "viscous heating" effect. Data obtained between 1 and 10 s are compared with Eq. (5) with α as the sole unknown coefficient to be determined by least squares error curve fitting.[4]

The pulse decay equations can also be used to generate analytical expressions for rate of heating curves. Beginning with the simplest case, we assume a pulse decay experiment is performed where the thermocouple is centered with respect to the focal region. In that case, $r=0$ and the decay history is obtained from Eq. (5) as

$$T_{\rm PD}(r=0,t) = 2\alpha I_0 \Delta t/\{\rho c[1+(4kt/\beta)]\}. \tag{6}$$

If a rate of heating experiment is next performed, the intensity is maintained at a constant value of I_0 for long durations, as opposed to an impulse interval Δt. The observed temperature can therefore be expressed as the convolution of the impulse response of the system, Eq. (6), with a step input[5,6]:

$$T_{\rm RH}(t) = \int_0^t \frac{2\alpha I_0}{\rho c[1+(4k\tau/\beta)]}\, d\tau. \tag{7}$$

Evaluating the convolution integral yields:

$$T_{\rm RH}(t) = \frac{2\alpha I_0}{\rho c}\left(\frac{\beta}{4k}\right)\ln\left(1+\frac{4kt}{\beta}\right). \tag{8}$$

The above expression indicates that temperature does not, strictly speaking, increase linearly with time during rate of heating experiments. However, Eq. (8) provides a quantitative guide to when conduction can be neglected. Using the approximation

$$\ln(1+x) \cong x, \quad \text{for } x \ll 1, \tag{9}$$

then for short durations such that

$$4kt/\beta \ll 1, \tag{10}$$

Eq. (8) can be rewritten as

$$T_{\rm RH}(t) = \frac{2\alpha I_0}{\rho c}\left(\frac{\beta}{4k}\right)\frac{4kt}{\beta} = \frac{2\alpha I_0 t}{\rho c}. \tag{11}$$

The last expression would also be obtained directly from energy conservation, Eq. (1), neglecting heat conduction. However, by deriving Eq. (8), the effects of conduction are explicitly included as a function of beam shape and time.

The "viscous heating" effect will add an unknown magnitude to the thermocouple reading, in practice. The excess heating will, however, approach a quasiequilibrium value within about 0.3 s in soft materials where small ($<76\ \mu$m diam) thermocouple wires are used.[2,3] Therefore, the slope of temperature rise measured after 0.3 s can be directly related to true absorption and conduction effects using Eqs. (5)–(8) and the relations between the impulse response and the step response of a linear system. Specifically, the slope of temperature measured subsequent to 0.3 s is

$$\frac{\partial T_{\rm RH}}{\partial t} = \frac{2\alpha I_0}{\rho c[1+(4kt/\beta)]}. \tag{12}$$

As expected, the simple relation given by Eq. (1) is obtained for beamwidths and measurements times such that $(4kt/\beta)\ll 1$.

More generally, when the thermocouple is not centered in the focal region, the rate of heating method would give

$$\frac{\partial T_{\mathrm{RH}}}{\partial t}=\frac{2\alpha I_0}{\rho c[1+(4kt/\beta)]}e^{-r^2/(4kt+\beta)}. \tag{13}$$

Equations (12) and (13) remove previous restrictions on the use of rate of heating absorption measurements. Beamwidths smaller than 3 mm in diameter can be used, and placement of the thermocouple at the center of the focal region is no longer required. The extension of pulse decay and rate of heating measurements to very thin samples requires additional considerations which are developed below.

B. Axial heat flow

In modeling heat flow to the water coupling medium, we assume that the effects of convection in the fluid, and thermal resistance at the absorber–water interface, are negligible. Furthermore, we assume that in soft biomaterials with large water content, the values of K and k are identical to water. Under these conditions, the problem of absorption and conduction in a layer of material can be handled in a straightforward manner by use of Green's functions. The derivation begins with the solution for an instantaneous point heat source of strength Q_3 liberated at time $t=0$ in a homogeneous medium, as measured at a distance r_s in spherical coordinates,[6]

$$T(r_s,t)=[Q_3/8(\pi kt)^{3/2}]e^{-r_s^2/4kt}. \tag{14}$$

To build up the solution for an ultrasonic impulse, we next consider a semi-infinite line source located along the z axis of Fig. 2 from $z'=0$ to ∞. With Q_2 as the source strength per unit length, the temperature at observation point (x_0,y_0,z_0) can be written in terms of Eq. (14) using superposition:

$$T(x_0,y_0,z_0,t)=\int_{z'=0}^{\infty}\frac{Q_2}{8(\pi kt)^{3/2}}\exp\left(\frac{-[x_0^2+y_0^2+(z'-z_0)^2]}{4kt}\right)dz'. \tag{15}$$

The integration over z' can be expressed in terms of the error function[7]

$$T(x_0,y_0,z_0,t)=\frac{Q_2}{8(\pi kt)}\exp\left(\frac{-(x_0^2+y_0^2)}{4kt}\right)\left(1+\operatorname{erf}\frac{z_0}{\sqrt{4kt}}\right). \tag{16}$$

The error function rises monotonically from 0 at zero argument, to 0.84 at unity argument, to 1.0 at infinite argument.[7] Thus, for small times such that

$$t\ll z_0^2/4k \tag{17}$$

the error function term is nearly unity; and Eq. (16) would be identical to the infinite line case. In other words, Eq. (17) tells how long it takes an observer at depth z_0 to "see" the surface effects on heat transfer. The derivation proceeds with a superposition of semi-infinite line sources to form a single cylindrical shell around the observation point. Thus, line sources of strength $Q_2=Q_1r'\,d\theta'$ are placed at distance r' around observation point at z_0. (Here, a cylindrical coordinate system is used with r' being the distance in the x',y' plane from the observation point.)

$$\begin{aligned}T(r'=0,z_0,t)&=\int_{\theta'=0}^{2\pi}\frac{Q_1r'}{8(\pi kt)}e^{(r'^2/4kt)}\left(1+\operatorname{erf}\frac{z_0}{\sqrt{4kt}}\right)d\theta'\\&=\frac{Q_1r'e^{-r'^2/4kt}}{4kt}\left(1+\operatorname{erf}\frac{z_0}{\sqrt{4kt}}\right).\end{aligned} \tag{18}$$

Finally, the Gaussian initial temperature distributions caused by an impulse of ultrasound centered on the observation point is modeled as a superposition of appropriately weighted cylindrical sources. Hence

$$Q_1(r')=T_0e^{-r'^2/\beta}\,dr' \tag{19}$$

and

$$T(r'=0,z_0,t)=\int_{r'=0}^{\infty}(T_0e^{-r'^2/\beta})\frac{r'e^{-r'^2/4kt}}{4kt}\left(1+\operatorname{erf}\frac{z_0}{\sqrt{4kt}}\right)dr'. \tag{20}$$

Evaluating the integral and designating the result as the pulse decay solution with Eq. (4) substituted for T_0, we have

$$T_{\mathrm{PD}}(r=0,z,t)=\frac{2\alpha I_0\Delta t}{2\rho c[1+(4kt/\beta)]}\left(1+\operatorname{erf}\frac{z}{4kt}\right), \tag{21}$$

where the prime notation on r and subscript on z have been dropped to generalize the result. The above equation gives the results for a centered pulse decay experiment including separable terms which account for heat transfer in the radial and axial directions. Note that for small times (or large depths) such that the condition on Eq. (17) is satisfied, the error function has a value of nearly unity, and Eq. (21) reduces to the case of Eq. (6) where axial heat flow is ignored.

If the thermocouple were positioned at the center of a sample of finite thickness $2Z$ (as opposed to the semi-infinite case), then the upper limit of integration of Eq. (15) would change to $z'=2Z$. This yields a slightly modified term which propagates through the derivation of the pulse decay

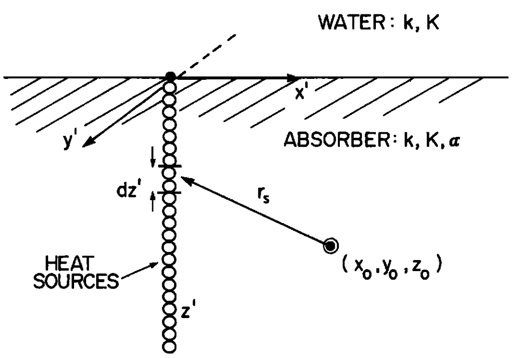

FIG. 2. Geometry of derivations for heating in a semi-infinite medium.

history. Hence, for finite sample thickness we obtain:

$$T_{\mathrm{PD}}(r=0,Z,t)=\frac{2\alpha I_0\Delta t}{\rho c[1+(4kt/\beta)]}\left(\operatorname{erf}\frac{Z}{4kt}\right). \tag{22}$$

Here, the erf(·) term can be interpreted as a separable correction function which accounts for the finite sample thickness. Generalizing the results to the off-axis case ($r\neq 0$), gives

$$T_{\mathrm{PD}}(r,Z,t)=\frac{2\alpha I_0\Delta t}{\rho c[1+(4kt/\beta)]}\times\exp[-r^2/(4kt+\beta)]\,[\operatorname{erf}(Z/\sqrt{4kt})]. \tag{23}$$

Expressions valid for use with rate of heating experiments may also be obtained directly from the pulse decay equations. As mentioned previously, the slope of temperature rise is used to eliminate effects of viscous heating during cw insonation, so using the relations between step response and impulse response of a linear system one derives the relation:

$$\frac{\partial T_{\mathrm{RH}}}{\partial t}=\frac{2\alpha I_0}{\rho c[1+(4kt/\beta)]}\times\exp[-r^2/(4kt+\beta)]\,[\operatorname{erf}(Z/\sqrt{4kt})], \tag{24}$$

which is written directly from Eq.(23) assuming the thermocouple is positioned at depth Z within a sample of thickness $2Z$. For simplified experimental conditions where $r=0$, and t is small enough to satisfy Eqs. (10) and (17), this rate of heating expression reduces to the simple case of no conduction given by Eq. (1).

II. RESULTS

To demonstrate the effect of heat conduction to the coupling medium, rate of heating experiments were performed on samples of soft polyethylene plastic of varying thickness. The results are shown in Fig. 3. In these measurements, a 0.9-MHz beam was used with a half-intensity focal beamwidth of 4.3 mm. This focal region diameter is sufficiently large to prevent appreciable radial heat flow at 0.5 s following commencement of ultrasonic heating.[3] In the case of the 1.8-mm sample, a thermocouple, located at the sample center, records a slope at 0.5 s which is very close to the value of dT/dt obtained by using Eq. (1) with known values[8] of ρ, c, α, and I. Radial and axial heat conduction are not accounted for in Eq. (1), and are not significant effects at 0.5 s in this experimental configuration as demonstrated by the close match between calculated and measured slopes.

In contrast, when the material thickness is reduced to 0.55 mm, the center thermocouple records a temperature history which is clearly affected by heat conduction to the top and bottom surfaces. Attempts to obtain the temperature slope before conduction effects become significant are thwarted by the presence of a "viscous heating" phenomenon around the embedded thermocouple which is prominent in the first 0.3 s of insonation.[1–4] From the theory developed in the previous section; the slope of temperature in the 0.55-mm sample can be predicted from the slope of the 1.8-mm piece by accounting for the presence of significant axial heat flow in the thinner sample. Thus, the value of the slope A (°C/s) was multiplied by the term $\operatorname{erf}(Z/\sqrt{4kt})$, where $Z=0.055/2$ cm; $k=1.6\times10^{-3}\,\mathrm{cm}^2/\mathrm{s}$, and $t=0.5$ s, to obtain a predicted value of the slope of the 0.55-mm sample. The result is denoted as slope B, and is superimposed on the measured rate of heating curve in Fig. 3. Slope B appears to be a better match to the data between 0.4 and 0.45 s, but the fit at 0.5 s is reasonable, considering the small mismatch between thermal diffusivities of plastic and water, a condition which is not accounted for in the theory.

A demonstration of axial heat conduction in pulse decay measurements was obtained using thermocouples embedded at 0.75- and 2.5-mm depth in an absorbing, castable rubber material of total thickness 20 mm (essentially semi-infinite). Figure 4 shows measured pulse decay curves for both depths, where a 0.1 s on time, 1.25-MHz focused beam was employed. A comparison of the temperature histories must account for the attenuation which occurs in the sample between the two depths, and also the greater effect of heat transfer to the sample surface in the case of the shallow (0.75-mm depth) thermojunction. Accordingly, the raw data from the 2.5-mm thermocouple were scaled by a factor of 1.25 to compensate for attenuation losses between 0.75 and 2.5 mm. This figure was based on a radiation force measurement of

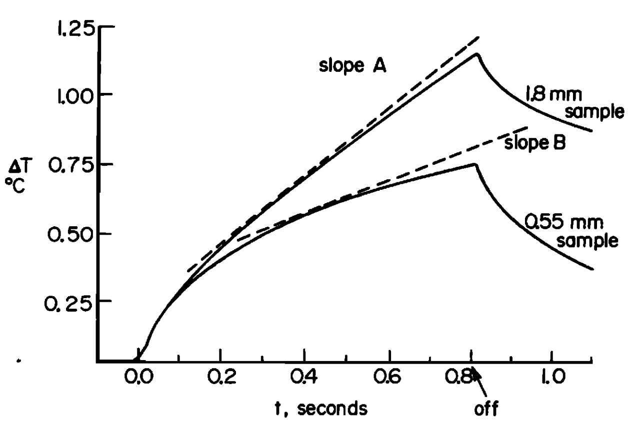

FIG. 3. Rate of heating curves obtained in soft polyethylene strips of different thicknesses, using 0.9-MHz ultrasound. Solid lines—measured temperature rise. Dashed lines—theoretical values for the slope of temperature at 0.5 s.

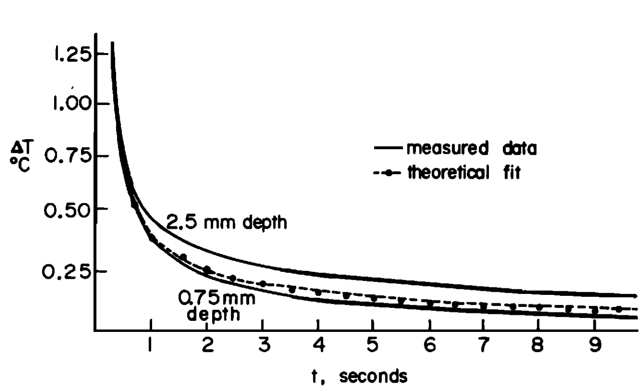

FIG. 4. Pulse decay results from 1.25-MHz insonation of thermocouples at different depths in an absorbing rubber compound. Solid lines—experimental data. Dashed lines—calculated values using theory to "convert" the deep thermocouple data into readings from the shallower depth. The theory accounts for the greater effect of heat flow to the surface in the case of the shallow thermocouple.

attenuation. The scaled version is plotted along with the raw data from 0.75-mm depth in Fig. 4. From the theoretical result presented in Eq. (21). the remaining difference between the 0.75-and 2.5-mm heating curves should be attributable to error function terms. Accordingly, the scaled 2.5-mm data were multiplied by a compensating factor of $[1+\mathrm{erf}(0.075/\sqrt{4kt})/1+\mathrm{erf}(0.25/\sqrt{4kt})]$, with $k=1.6\times10^{-3}$ cm^2/s. This correction function should reduce the magnitude of the 2.5-mm curve, at each point in time, to precisely overlap the data from 0.75 mm. The result is shown as a dotted line in Fig. 4, and although the match of the compensated 2.5-mm data to the 0.75-mm data is not precise over the entire curve, the results are within the error imposed by the approximately 10% uncertainty in the location of each thermojunction, the values of α and k, and by the small mismatch between thermal properties of the absorbing material and water.

An integral-differential relationship between rate of heating and pulse decay methods has been assumed in the theory. To demonstrate this experimentally, both pulse decay and rate of heating curves were obtained at 0.75-mm depth in a castable rubber absorber, using focused 1.25-MHz insonation. The result of the 0.1 s on time pulse decay experiment is shown in Fig. 5. The integral of the decay curve, when scaled by a factor of $1/\Delta t$ and plotted as a continuous function of time, should yield precisely the data obtained from a step input of ultrasound at the same intensity, according to the general relations which exist between the impulse response of a linear, time invariant system. Figure 6 shows the rate-of-heating data obtained at an intensity level 0.33 times lower than in the pulse decay experiment. Also shown in Fig. 6 is the results of numerical integration of the pulse decay experiment of Fig. 5, scaled by a factor of (0.33)/(0.1) which accounts for the intensity change between experimetns and the on time used in the pulse decay case. The overlap of these curves is good overall even though both curves include the effects of "viscous heating" which is localized around the thermojunction. The small deviation may be attributable to the use of a finite duration input of acoustic energy which is short but not truly impulsive in the pulse decay experiment.

In summary, these experiments verify the impulse response–step response relationship between pulse decay and rate of heating approaches, and also validate the use of error functions to account for axial heat flow.

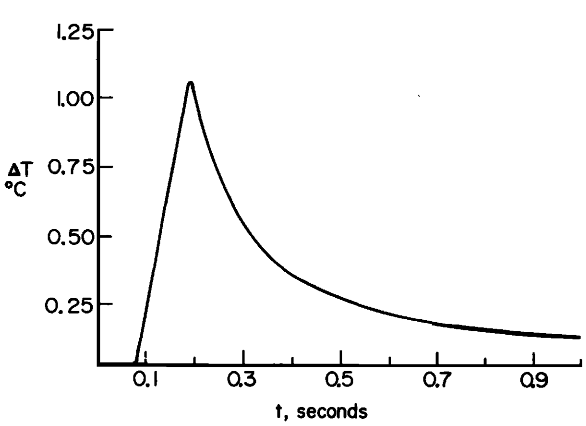

FIG. 5. Pulse decay results using a 1.25-MHz focused beam and a thermocouple located at 0.75 mm in an absorbing material.

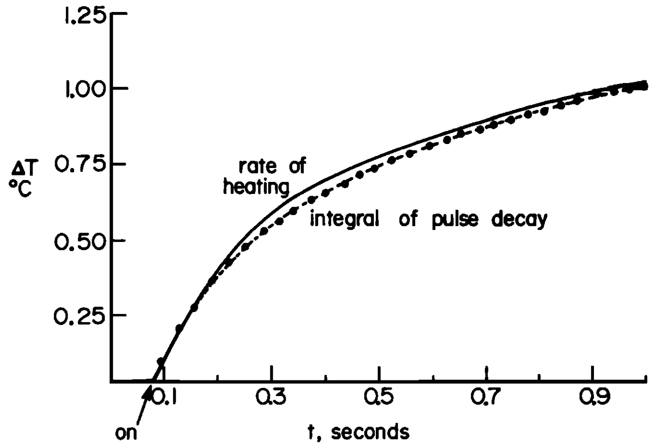

FIG. 6. Comparison of rate of heating and the integral of the pulse decay experiment shown in Fig. 5. These data demonstrate the impulse response–step response relationship between pulse decay and rate of heating experiments.

III. DISCUSSION

The practicability of any measurement technique depends in part on the number of parameters which must be known *a priori*, and the complexity of mathematics used in analysis. Therefore, it is useful to consider the degree of difficulty encountered in the approaches described herein. The simplest situation is the rate of heating measurement where heat conduction effects can be ignored. Here the absorption coefficient is given by Eq. (1), and accurate values for ρ, c, and I must be known. Measurement of the slope, dT/dt can be done graphically since temperature is presumably increasing at a constant rate during a substantial interval.

In cases where radial heat transfer cannot be ignored, then use of the central pulse decay [Eq. (6)], or rate of heating [Eq. (12)] methods require the additional parameters k and β. These are not usually difficult to determine since the thermal diffusivity of many soft tissues is close to that of water,[9] and the ultrasonic beamwidth can be measured in the medium using the embedded thermocouple.[4] The data are no longer a straight line, however, and must be curve fit to Eq. (6) or (12) to determine the unknown value of α. While estimations of α can still be obtained graphically, it is straightforward to perform a least squares error curve fit using a minicomputer where the temperature histories have been digitized and stored.

If axial heat flow is also significant, then the error function term must be incorporated into pulse decay [Eq. (22)] or rate of heating [Eq. (24)] analyses. This additional complexity requires specification of the thermocouple depth Z, which can be determined by visual or microscopic inspection in transparent or incised samples; or by high resolution pulse echo interrogation where echoes can be obtained from the sample surface and the embedded thermocouple. Computationally, erf(·) can be programmed using a polynomial approximation valid for all positive arguments.[7] Thus, a curve

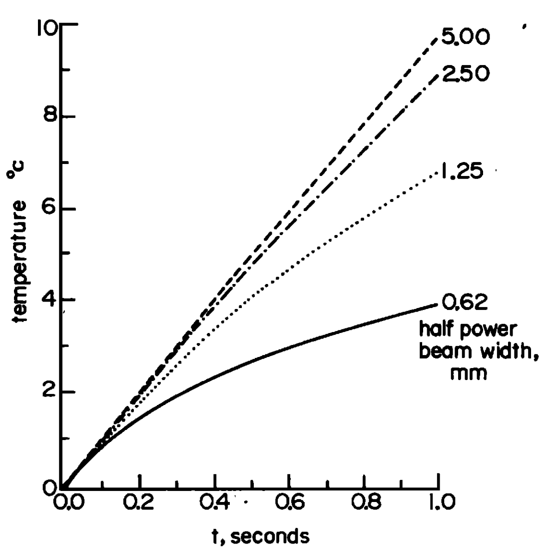

FIG. 7. Calculated rate of heating curves for various beamwidths. The effects of viscous heating and axial heat flow are not incorporated into these calculations. These data indicate that radial heat flow may be ignored when an ultrasonic beamwidth of approximately 3 mm or greater is used and observation times are limited to approximately 0.5 s.

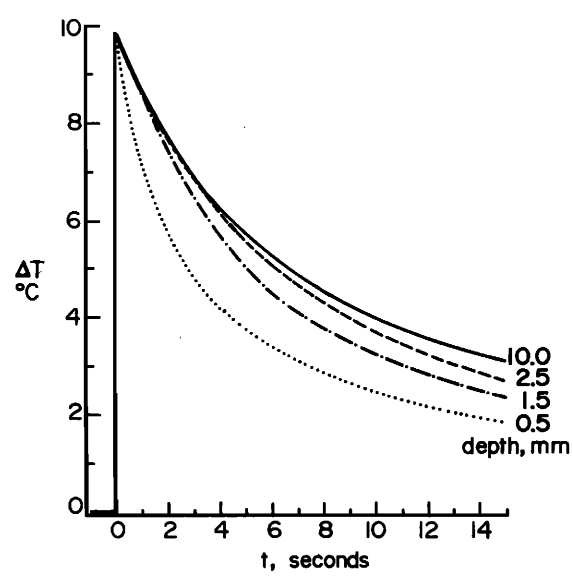

FIG. 8. Calculated pulse decay temperatures, including axial and radial heat flow, for thermojunctions at various depths in a semi-infinite medium. The results indicate that axial heat conduction to the coupling medium may be ignored when the thermojunction is at least 3 mm deep and observation times are limited to no more than 10 s.

fit to determine the value of α can still be accomplished rapidly by minicomputer. One additional parameter is required if off-axis pulse decay or rate of heating experiments are performed. The radial distance, r, must be known for use of Eqs. (5), (13), (23), and (24). Since precision positioners are frequently used in laboratory alignment, an accurate lateral displacement of a focused beam with respect to a thermojunction can be readily accomplished.

Another procedural question concerns the choice of rate of heating versus pulse decay approaches. Theoretically, these are linked by the general relations between the step response and impulse response of a linear system, and therefore the same values are obtained from either taking the derivative of rate of heating curves, or taking the absolute temperature elevation of a pulse decay curve. There are important practical considerations, however. One disadvantage of the rate of heating approach is that a time-varying derivative must be calculated from a signal which generally contains noise, introducing some computational issues and uncertainties. However, an advantage of the rate of heating method is that the absolute temperature rise obtained using cw insonation is much greater than the absolute temperatures obtainable under identical conditions but utilizing a short pulse of ultrasound. This becomes important when low absorption and/or restricted output intensity (from avoidance of cavitation or shock thresholds, for example) result in a low thermocouple signal to noise ratio.

Bounds on when radial and axial heat conduction can be neglected can be obtained from Eqs. (5)–(24).

The effects of radial heat flow on rate of heating curves are demonstrated in Fig. 7, where Eq. (8) (no axial flow included) was used to calculate temperatures of centered, rate of heating experiments given different values of β (beamwidths). It was assumed that

$$2\alpha I_0/\rho c = 10.0\,^\circ\mathrm{C/s} \text{ and } k = 1.5\times 10^{-3}\ \mathrm{cm^2/s},$$

while β was varied from a high value of $9\times 10^{-2}\mathrm{cm}^2$ (half-intensity beamwidth of 5 mm) to $1.4\times 10^{-2}\mathrm{cm}^2$ (half-intensity beamwidth of 0.62 mm). The results show that at 0.5 s, the slopes of the top two curves are within approximately 10% of the value which would be obtained if no heat transfer, were permitted [Eq. (1)]. The smaller beam width cases have slopes at 0.5 s which can only be related to absorption by Eq. (12) which includes radial heat flow. This is in agreement with the guidelines given by Goss *et al.*[3] who used a numerical model. We generalize from Eqs. (7)–(11) and state that, in centered rate of heating experiments, radial heat flow can be neglected (within 10% accuracy) and therefore Eq. (1) may be used, when

$$4kt/\beta \leqslant 0.1. \tag{25}$$

Thus, for experiments on soft tissue where measurement times are $t \leqslant 0.6$ s and $k = 1.5\times 10^{-3}\ \mathrm{cm^2/s}$, then the inequality is satisfied for $\beta \geqslant 0.036\ \mathrm{cm}^2$, corresponding to the use of a half-intensity beamwidth of 3 mm or greater.

The effect of axial heat flow is more easily seen in pulse decay experiments where observation times may stretch to 10 s or longer. Figure 8 shows calculated pulse decay experiments using Eq. (21) where it was assumed that

$$2\alpha I_0 \Delta t/\rho c = 10.0\,^\circ\mathrm{C}, \quad \beta = 0.04\ \mathrm{cm}^2$$

and

$$k = 1.5\times 10^{-3}\ \mathrm{cm^2/s},$$

while the thermocouple location z was varied from 0.5 mm to essentially infinite depth from the sample surface. These curves show that in order to neglect axial heat flow at 10.0 s, the thermocouple depth must be greater than 2.5 mm. We generalize this result to all rate of heating or pulse decay experiments by referring to Eqs. (21)–(24) and concluding that axial heat flow to the sample surfaces can be neglected (within 10% accuracy) when

$$\mathrm{erf}(Z/\sqrt{4kt}) \geqslant 0.9, \tag{26}$$

thus,

$$Z/\sqrt{4kt}>1.2. \tag{27}$$

For rate of heating experiments where the inequality must hold for $t = 0.6$ s, we have $Z>0.7$ mm. For pulse decay experiments where we wish to maintain the inequality to $t = 10.0$ s, we obtain $Z>3.0$ mm. Thus, to within approximately 10% accuracy the heat flow to the coupling medium may be neglected, permitting the use of simpler expressions, when the embedded thermocouple is located around 1-mm depth for rate of heating experiments, and 3-mm depth for the pulse decay measurements. Since the radial and axial heat flow terms are separable, the above guidelines regarding thermocouple depth hold true independent of the beamwidth or β value, used.

IV. CONCLUSION

Analytical expressions have been developed which model the flow of heat in radial and axial directions during ultrasonic heating, assuming the use of a Gausssian shaped intensity distribution. These equations can be used to measure absorption coefficients under a wide range of experimental conditions, using on or off axis, rate of heating or pulse decay methods, with arbitrary beamwidths and sample thickness. Key assumptions are that the coupling medium has no absorption or convection, but has a thermal diffusivity equal to that of the absorbing sample. Also, the intensity loss with depth due to absorption is considered to negligibly contribute to axial heat flow, so the equations presented herein would not be valid for deep thermocouple measurements in a highly absorbing medium.

The derivations provide guidelines as to when axial or radial flow may be neglected, permitting use of simpler expressions relating temperature histories and absorption. Specifically, the axial flow of heat may be neglected in typical pulse decay experiments when the thermocouple is located at least 3mm in depth and observation times are less than 10 s. For rate of heating experiments with observation times between 0.5 and 0.6 s, the thermojunction must be approximately 1 mm deep. Radial heat flow in centered rate of heating experiments can also be neglected providing observation times are limited to 0.6 s, and the half-intensity beamwidth is at least 3 mm.

These results should lead to more accurate measurements of absorption over an extended range of frequencies and sample sizes. In addition, the derivations may be of value in modeling ultrasonic applications such as hyperthermia or focal ablation of tissues.

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