the multiplication permits choosing between truncation and roundoff by setting an appropriate parameter. The multiplication Fortran procedure is given in the Appendix.

III. Conclusions

A simulation of two's-complement binary addition and multiplication with positive integer numbers is presented for simulating finite precision fixed-point operations of digital filters with high-level programming languages.

Addition with or without saturation, truncation, or roundoff quantization, multiplication by coefficients with magnitude greater than one, and variable word length are facilities provided by the simulation.

This appears to be a flexible tool for software simulation of hardware digital filter designs in operating conditions. Eventually, this method could be extended to other signal processing algorithms that are to be implemented in fixed-point arithmetic.

Appendix

```
      FUNCTION ADDITN ( TERM1, TERM2, OVCNT, RN )
C   Two's complement fixed point addition simulation.
C Global Variables: ADDITN  :  addition.
C                   TERM1   :  addend 1.
C                   TERM2   :  addend 2.
C                   OVCNT   :  overflow counter.
C                   RN      :  range, 2 ** b (b = wordlength)
C Local Variables:  SGNADN  :  sign flag of ADDITN
C                   SGNT1   :  sign flag of TERM1.
C                   SGNT2   :  sign flag of TERM2.
C                   RN2     :  RN / 2.
C
      REAL     ADDITN, TERM1 , TERM2, RN , RN2
      INTEGER  OVCNT , SGNADN, SGNT1, SGNT2
      ADDITN = TERM1  + TERM2
      ADDITN = ADDITN - RN * AINT (ADDITN / RN)
      RN2 = RN / 2.
      SGNADN = 1 - 2 * INT (ADDITN / RN2)
      SGNT1  = 1 - 2 * INT (TERM1  / RN2)
      SGNT2  = 1 - 2 * INT (TERM2  / RN2)
      IF (SGNT1 .NE. SGNT2 ) RETURN
      IF (SGNT1 .EQ. SGNADN) RETURN
      OVCNT = OVCNT + SGNT1
      RETURN
      END
C -------------------------------------------------------
      SUBROUTINE SATURE (VAR, OVCNT, RN)
C  Variable overflow test and corresponding saturation.
C Input Variables:  OVCNT   :  overflow counter.
C                   RN      :  range, 2 ** b  (b = wordlength).
C                   VAR     :  variable.
      REAL     VAR  , RN
      INTEGER  OVCNT
      IF ( OVCNT ) 10, 40, 20
10    VAR = RN / 2.
      GOTO 30
20    VAR = RN / 2. - 1.
30    OVCNT = 0
40    RETURN
      END
C -------------------------------------------------------------
      FUNCTION  MULTPL (TERM1, TERM2, RNTER1, RN, IRED)
C  Two's complement fixed point multiplication simulation
C Global Variables:  MULTPL : product.
C                    TERM1  : multiplier 1.
C                    TERM2  : multiplier 2.
C                    RNTER1 : 2 ** (b - c), (c=number of integer bits)
C                    RN     : simulation range, 2 ** b (b=wordlenth)
C                    IRED   : quantization flag.
C Local Variables: C   A, AUX, IM : auxiliary variables.
C
      REAL     MULTPL, TERM1, TERM2, RNTER1, RN, A, AUX
      INTEGER  IRED , IM
      MULTPL = TERM1  * TERM2
      IF (TERM1 .GE. RN/2.) MULTPL = MULTPL + RN * (RN - TERM1)
      IF (TERM2 .GE. RN/2.) MULTPL = MULTPL + RN * (RN - TERM2)
      AUX    = RNTER1 * RN
      MULTPL = 2. * MULTPL / AUX
      MULTPL = RN * (MULTPL  -  AINT (MULTPL))
      IM = 1
      A  =  AINT (MULTPL)
      IF (A .EQ. RN/2. - 1.  .OR.  IRED .EQ. 0 )  IM = 0
      MULTPL  =  INT ( MULTPL + IM * .5 )
      RETURN
      END
```

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Tunable Digital Frequency Response Equalization Filters

PHILLIP A. REGALIA AND SANJIT K. MITRA

Abstract—**Tunable digital frequency response equalization filters, which feature adjustable gain at specified frequencies while leaving the remainder of the spectrum unaffected, are advanced. The filter structure is such that the frequency response parameters are independently related to the multiplier coefficients, which permits simple frequency response adjustment by varying the coefficient values. The resulting structure exhibits low coefficient sensitivity characteristics.**

I. Introduction

Design procedures for variable cutoff-frequency digital filters are well known [1]–[4]. In this correspondence, we consider tunable digital equalizer filters which, instead of passing some frequencies and rejecting others, modify one portion of the frequency spectrum while leaving the remainder unaffected. Such filters find application in real-time signal enhancement/correction, tonal control in digital audio systems, etc. First-order and second-order sections are presented, since the cascade of these basic sections allows for quite versatile frequency response manipulation.

II. First-Order Equalizer

We begin by considering a first-order analog transfer function prototype:

$$F(s) = \frac{s + Kp}{s + p}. \tag{1}$$

The low-frequency and high-frequency gains are, respectively,

$$F(0) = K \tag{2a}$$

$$F(\infty) = 1. \tag{2b}$$

The parameter K controls the low-frequency gain ultimately achieved, while p controls the bandwidth of frequencies which are boosted or attenuated. Equation (1) may conveniently be decomposed as the linear sum of bounded real functions:

$$F(s) = \frac{s}{s + p} + K\frac{p}{s + p} \tag{3}$$

where

$$\frac{s}{s + p} = \frac{1}{2}\left[1 + \frac{s - p}{s + p}\right] \tag{4a}$$

$$\frac{p}{s + p} = \frac{1}{2}\left[1 - \frac{s - p}{s + p}\right]. \tag{4b}$$

Manuscript received November 15, 1985; revised June 24, 1986. This work was supported by the National Science Foundation under Grant ECS 85-08017.

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IEEE Log Number 8610887.

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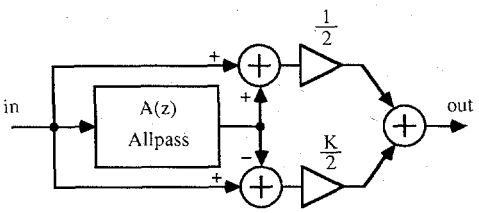


Fig. 1. Equalizer filter structure.

Denote

$$A(s) = \frac{s - p}{s + p}$$

which is recognized as a first-order all-pass function. Then (3) may be expressed as

$$F(s) = \tfrac{1}{2}[1 + A(s)] + \tfrac{1}{2}K[1 - A(s)]. \tag{5}$$

By applying the bilinear transform, (5) becomes

$$F(z) = \tfrac{1}{2}[1 + A(z)] + \tfrac{1}{2}K[1 - A(z)], \tag{6}$$

where

$$A(z) = -\frac{z^{-1} + a}{1 + az^{-1}}. \tag{7}$$

This suggests the realization of Fig. 1, where the all-pass filter can be implemented using well-known structures which remain all-pass in spite of coefficient quantization [5]–[7].

Letting Ω denote the transition frequency between the modified and unmodified portions of the frequency response,[1] the parameter a becomes

$$a = \frac{\tan(\Omega/2) - 1}{\tan(\Omega/2) + 1}. \tag{8}$$

Thus, for a low-frequency equalizer with variable gain below $\Omega = 0.0046\pi$, (8) gives $a = -0.9858$. The frequency response for various values of K is shown in Fig. 2(a), which illustrates the single parameter adjustment of the low-frequency gain. Fig. 2(b) shows the frequency response obtained with K held constant and a varied, with a restricted to a five bit wordlength (including the sign bit), and demonstrates the single parameter adjustment of the boost or cut bandwidth.

III. Second-Order Equalizer

By using a second-order equalizer structure, the magnitude response can be adjusted about a specified center frequency. This can be obtained from the first-order equalizer by using a low-pass-to-band-pass spectral transformation:

$$z^{-1} \rightarrow -z^{-1}\frac{z^{-1} + b}{1 + bz^{-1}}. \tag{9}$$

This transformation maps the behavior at $\omega = 0$ of a reference filter to a new center frequency $\omega = \omega_0$, where $b = -\cos\omega_0$. Applying the transformation of (9) to the first-order all-pass filter of (7) we obtain

$$A(z) = \frac{z^{-2} + b(1 + a)z^{-1} + a}{1 + b(1 + a)z^{-1} + az^{-2}}. \tag{10}$$

The all-pass function of (10) may be realized using a Gray and Markel all-pass lattice filter [5], where a and b denote the lattice coefficients.

For $K = 0$, the filter structure degenerates to that of a digital notch filter [8]. Thus, if we let Ω denote the -3 dB notch bandwidth obtained for $K = 0$, and ω_0 the center frequency of the equalizer, the design equations become:

$$b = -\cos\omega_0 \tag{11a}$$

$$K = F(e^{j\omega_0}) \tag{11b}$$

$$a = \frac{1 - \tan(\Omega/2)}{1 + \tan(\Omega/2)}. \tag{11c}$$

The parametric tuning provided by the filter is illustrated with a design example. Consider the design of an equalizer with variable gain at $\omega_0 = 0.38\pi$ (normalized frequency), with a -3 dB notch bandwidth $\Omega = 0.06\pi$ obtained for $K = 0$. Then

$$a = \frac{1 - \tan(0.06\pi/2)}{1 + \tan(0.06\pi/2)} = 0.8273$$

$$b = -\cos(0.38\pi) = -0.3681.$$

For simplicity, we restrict the coefficient wordlength to five bits (including the sign bit), whence $a = 0.8125$ and $b = -0.375$. The value of K then determines the gain at $\omega_0 = \cos^{-1}(0.375) = 0.3776\pi$.

Fig. 3(a) shows the frequency response obtained for different values of K, which demonstrates the variable gain at the specified center frequency. Fig. 3(b) demonstrates the effect of varying only b (keeping a and K constant), which effectively tunes the center frequency. Fig. 3(c) shows how the peak (or notch) bandwidth may be varied by adjusting only a. In all cases, the coefficients have been restricted to five bits.

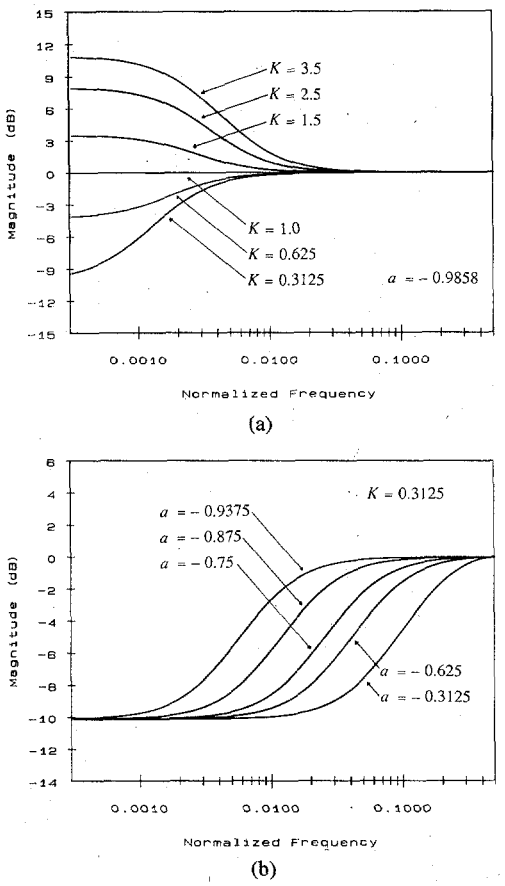
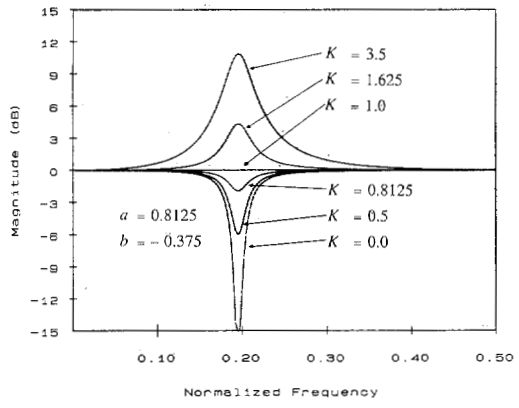


Fig. 2. (a) Low-frequency gain adjusted by varying K. (b) Filter bandwidth adjusted by varying a.

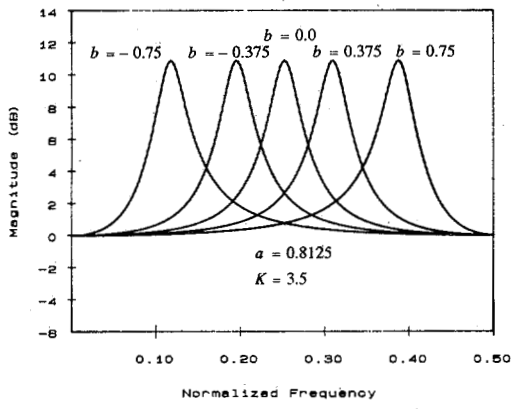
IV. Phase Response

The above discussion has considered magnitude response equalization, although the equalizers presented here have the important property of minimum phase. Thus, given any equalizer function

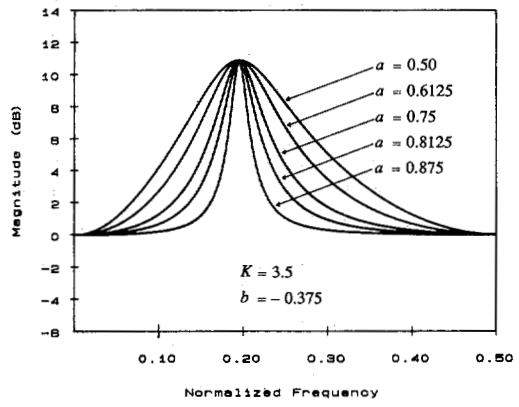
[1] In particular, from (8), one can show that for $K \gg 1$, $|F(e^{j\Omega})| \approx 0.707K$, and for $K \ll 1$, $|F(e^{j\Omega})| \approx 0.707$.



(a)



(b)



(c)

Fig. 3. (a) Center frequency gain versus K. (b) Center frequency versus b. (c) Filter bandwidth versus a.

presented above, a stable inverse transfer function exists such that the cascade achieves zero phase. In addition, when used to correct for (minimum phase) frequency response deficiencies in some part of the system, minimum phase equalization is to be preferred over linear phase, as linear phase equalization cannot correct for phase distortion.

To show the minimum phase property of the equalizers, (6) can be rewritten in the form

$$F(z) = \frac{N(z)}{D(z)} = \frac{1}{2}(1 + K) + \frac{1}{2}(1 - K)\,A(z). \tag{12}$$

As $A(z)$ is all-pass, it may be expressed as

$$A(z) = \frac{\tilde{D}(z)}{D(z)} \tag{13}$$

where $D(z)$ is minimum phase for stability, and $\tilde{D}(z) = z^{-N}D(z^{-1})$ is the corresponding causal maximum phase polynomial, with $|\tilde{D}(e^{j\omega})| = |D(e^{j\omega})|$ for all ω. The numerator polynomial in (12) may then be written

$$\frac{2}{1 + K}N(z) = D(z) + \frac{1 - K}{1 + K}\tilde{D}(z). \tag{14}$$

Now, for $0 < K < \infty$, then $|(1 - K)/(1 + K)| < 1$. Thus, by application of Rouche's theorem [9] along the unit circle, it follows that $N(z)$ is minimum phase for $0 < K < \infty$. In a similar manner, one can show that replacing K by $-K$ leaves the magnitude response unchanged, with the equalizer now maximum phase.

V. Concluding Remarks

The filter structure presented here provides a simple tunable equalizer realization. The frequency selective portion of the circuit [i.e., $A(z)$] can be realized in a structurally passive form, which leads to a low sensitivity realization [10]. In addition, due to the parametric decoupling of the coefficient values, quantizing one coefficient leaves the remaining frequency response parameters unaffected. Cascading a few such circuits allows for a convenient parametric frequency response equalizer.

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A New Method for Synthesis of Low-Peak-Factor Signals

A. van den Bos

***Abstract*—For a specified power spectrum, the minimum-to-maximum amplitude range of a periodic signal depends on the phase angles of the harmonics. A low value of this range, which as a fraction of the root-mean-square value is usually called the *peak factor*, is often desirable. A new phase angle adjusting method is proposed producing lower peak factors than the conventional method.**

Manuscript received November 15, 1985; revised May 5, 1986.

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IEEE Log Number, 8611231.

0096-3518/87/0100-0120$01.00 © 1987 IEEE