Partial angular coherence and the angular Schmidt spectrum of entangled two-photon fields

Anand Kumar Jha,[1] Girish S. Agarwal,[2] and Robert W. Boyd[1]
[1]*The Institute of Optics, University of Rochester, Rochester, New York 14627, USA*
[2]*Department of Physics, Oklahoma State University, Stillwater, Oklahoma 74078, USA*
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We study partially coherent fields that have a coherent-mode representation in the orbital-angular-momentum-mode basis. For such fields, we introduce the concepts of the angular coherence function and the coherence angle. Such fields are naturally produced by the process of parametric down-conversion—a second-order nonlinear optical process in which a pump photon breaks up into two entangled photons, known as the signal and idler photons. We show that the angular coherence functions of the signal and idler fields are directly related to the angular Schmidt (spiral) spectrum of the down-converted two-photon field and thus that the angular Schmidt spectrum can be measured directly by measuring the angular coherence function of either the signal or the idler field, without requiring coincidence detection.

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I. INTRODUCTION

Classical coherence theory is a well-established subject. Its modern interpretation is largely due to Wolf and co-workers [1,2]. The study of coherence in the context of entangled fields has revealed a much deeper understanding of entanglement itself [3–6]. One of the central concepts of classical coherence theory is the cross-spectral-density function, which quantifies the field correlations in the space-frequency domain. A cross-spectral-density function always has a unique coherent-mode representation, which is a way to represent a partially coherent field as an incoherent sum of a finite number of completely coherent fields. In this paper, we study the partially coherent fields that have a coherent-mode representation in the orbital-angular-momentum basis. We show that, for such fields, it is very useful to introduce the concepts of the angular coherence function and the coherence angle. In fact, such fields are naturally produced by the process of parametric down-conversion (PDC), owing to the conservation of orbital angular momentum (OAM) in parametric down-conversion [7,8].

The OAM entanglement of PDC photons [8] is a great resource for quantum-information-based protocols due to the fact that the OAM basis provides a discrete but infinite-dimensional Hilbert space, as opposed to the polarization basis, which provides only a two-dimensional Hilbert space [9,10]. For this reason, an accurate measurement of the dimensionality of the OAM-entangled photons is very important. There are two generic ways in which the dimensionality can be measured. The first is by directly measuring the two-photon intensity in coincidence at different values of the OAM mode indices [11,12] and the second is by using the Hong-Ou-Mandel interference technique [13]. In this paper, we propose an alternative way [14] of measuring the dimensionality, by measuring the angular coherence function of either of the down-converted photons. This scheme is different from the existing schemes in that it requires only singles detection, as opposed to the coincidence detection required in the other schemes.

II. PARTIALLY COHERENT FIELDS

A. General representation

In this section, we briefly describe the general representation of partially coherent fields. Let $\{V(\mathbf{r},t)\}$ be an ensemble representing the statistical properties of a partially coherent field that is both stationary, at least in the wide sense, and ergodic. One way to characterize the statistical correlations of such fields is through the mutual coherence function $\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)$, which quantifies the field correlation between the space-time points $(\mathbf{r}_1,t)$ and $(\mathbf{r}_2,t+\tau)$, and is defined as $\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)=\langle V^*(\mathbf{r}_1,t)V(\mathbf{r}_2,t+\tau)\rangle$, where $\langle\cdots\rangle_e$ represents the ensemble average. Another way, which is more convenient, to characterize the field correlations is through the cross-spectral-density function $W(\mathbf{r}_1,\mathbf{r}_2,\omega)$, which quantifies the field correlations in the space-frequency domain and is defined as

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{i\omega\tau}d\tau. \quad (1)$$

For conceptual clarity, we suppress from now on the frequency argument in the definition of the cross-spectral-density function. We also assume that the cross-spectral-density function is a continuous function of $\mathbf{r}_1$ and $\mathbf{r}_2$ within the domain D of interest. The cross-spectral-density function is a bounded function, in the sense that

$$\int_D\int_D|W(\mathbf{r}_1,\mathbf{r}_2)|^2d\mathbf{r}_1d\mathbf{r}_2<\infty. \quad (2)$$

Further, it is a Hermitian function, that is,

$$W^*(\mathbf{r}_1,\mathbf{r}_2)=W(\mathbf{r}_2,\mathbf{r}_1). \quad (3)$$

And most importantly, it is a non-negative definite function, that is,

$$\int_D\int_D W(\mathbf{r}_1,\mathbf{r}_2)f^*(\mathbf{r}_1)f(\mathbf{r}_2)d\mathbf{r}_1d\mathbf{r}_2\geqslant 0, \quad (4)$$

where $f(\mathbf{r})$ is any square integrable function. The physical interpretation of the non-negative-definiteness condition is that the intensity distribution produced by the field, with an aperture function $f(\mathbf{r})$ in domain D, on a screen is always non-negative. The above conditions, along with the multidimensional version of the Mercer theorem, imply that the cross-spectral-density function $W(\mathbf{r}_1,\mathbf{r}_2)$ is a Hilbert-Schmidt kernel and that it has a coherent-mode representation of the form [2]

$$W(\mathbf{r}_1,\mathbf{r}_2)=\sum_n\alpha_n\psi_n^*(\mathbf{r}_1)\psi_n(\mathbf{r}_2). \quad (5)$$

The functions $\psi_n^*(\mathbf{r})$ are the eigenfunctions and the coefficients α_n are the eigenvalues of the integral equation $\int W(\mathbf{r}_1,\mathbf{r}_2)\psi_n(\mathbf{r}_1)d\mathbf{r}_1 = \alpha_n\psi_n(\mathbf{r}_2)$. The Hermiticity and the non-negative definiteness of $W(\mathbf{r}_1,\mathbf{r}_2)$ ensure that the integral equation has at least one nonzero eigenvalue and that all the eigenvalues are real and non-negative, i.e., $\alpha_n \geqslant 0$. The above equation can be rewritten as $W(\mathbf{r}_1,\mathbf{r}_2) = \sum_n \alpha_n W^{(n)}(\mathbf{r}_1,\mathbf{r}_2)$, where $W^{(n)}(\mathbf{r}_1,\mathbf{r}_2) \equiv \psi_n^*(\mathbf{r}_1)\psi_n(\mathbf{r}_2)$. This representation implies that for any partially coherent field there exists at least one basis in which the cross-spectral-density function can be represented as a superposition of modes that are completely coherent in the space-frequency domain.

B. Partially coherent field in the Laguerre-Gaussian basis

Every type of partially coherent field is characterized by its unique coherent-mode representation. In this paper, we are investigating partially coherent fields that have a coherent-mode representation in the Laguerre-Gaussian (LG) basis. A coherent mode in the LG basis is referred to as an LG mode or an LG beam; these are the exact solutions of the paraxial Helmholtz equation. The normalized field amplitude of these modes at $z = 0$ in the cylindrical coordinate system is given by

$$\big[\mathrm{LG}_p^l(\rho,\phi)\big] \equiv \big[\mathrm{LG}_p^l(\rho)\big]e^{il\phi} = \sqrt{\frac{2p!}{\pi(|l|+p)!}} \times \frac{1}{w_0}\left(\frac{\sqrt{2}\rho}{w_0}\right)^{|l|} L_p^l\left(\frac{2\rho^2}{w_0^2}\right)\exp\left(-\frac{\rho^2}{w_0^2}\right)e^{il\phi}, \tag{6}$$

where w_0 is the beam waist radius at $z = 0$ and l is the azimuthal mode index. Due to the azimuthal phase dependence of $e^{il\phi}$, these modes carry an orbital angular momentum of $l\hbar$ per photon [15]. These modes have been extensively studied in the last few decades. Such fields are very important as they hold promise for many new fascinating applications, especially in quantum-information science. The type of partially coherent fields that we consider in this paper has the following coherent-mode representation:

$$\begin{aligned} W(\mathbf{r}_1,\mathbf{r}_2) \rightarrow\ & W(\rho_1,\phi_1;\rho_2,\phi_2) \\ &= \langle V^*(\rho_1,\phi_1)V(\rho_2,\phi_2)\rangle_e \\ &= \sum_{l,p,p'} \alpha_{lpp'}\big[\mathrm{LG}_p^{*l}(\rho_1,\phi_1)\big]\big[\mathrm{LG}_{p'}^l(\rho_2,\phi_2)\big], \end{aligned} \tag{7}$$

where $V(\rho,\phi)$ is a single realization of the field at location (ρ,ϕ). We note that the field is a coherent superposition of modes carrying different values of the orbital-angular-momentum mode index l. The specific question that we now ask is the following: "For a field that has the above form for the cross-spectral-density function, what is the correlation between the fields at two different angular positions, after the correlations have been integrated over the radial dimensions?" In order to answer this question, we consider the situation as shown in Fig. 1. Consider a partially coherent field passing through a screen in the form of a double angular slit. The two slits are centered at angular positions ϕ_1 and ϕ_2, respectively.

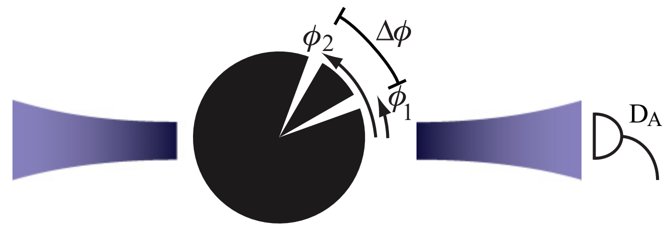

FIG. 1. (Color online) A scheme for studying the angular coherence properties of a partially coherent beam.

The separation between the slits is $\Delta\phi = \phi_1 - \phi_2$. The field $\Psi(\rho,\phi)$ immediately after the aperture is given by

$$\Psi(\rho,\phi) = V(\rho,\phi)\Phi(\phi), \tag{8}$$

where $V(\rho,\phi)$ is the incoming field and $\Phi(\phi)$ is the amplitude transmission function of the aperture. We decompose the above field in the LG basis as

$$\Psi(\rho,\phi) = \sum_{l,p} A_{lp}\big[\mathrm{LG}_p^l(\rho,\phi)\big], \tag{9}$$

where

$$A_{lp} = \iint \rho\, d\rho\, d\phi \big[\mathrm{LG}_p^{*l}(\rho,\phi)\big]V(\rho,\phi)\Phi(\phi) \tag{10}$$

is the probability amplitude for the field to be found in mode $\mathrm{LG}_p^l(\rho,\phi)$. Since we are interested in field correlations at different angular positions, we sum over all the p modes and obtain the intensity I_l of the field for a given value of l as

$$I_l = \sum_{p=0}^{\infty} I_{lp} \equiv \sum_{p=0}^{\infty}\langle A_{lp}^* A_{lp}\rangle_e. \tag{11}$$

Using Eqs. (7) and (10), we write I_l as

$$\begin{aligned} I_l = \sum_{l',p',p''} \alpha_{l'p'p''} \sum_p \iint \rho\rho' d\rho\, d\rho' \big[\mathrm{LG}_p^l(\rho)\big] \\ \times \big[\mathrm{LG}_p^{*l}(\rho')\big]\big[\mathrm{LG}_{p'}^{*l'}(\rho)\big]\big[\mathrm{LG}_{p''}^{l'}(\rho')\big] \\ \times \iint d\phi\, d\phi' e^{i(l-l')(\phi-\phi')}\Phi^*(\phi)\Phi(\phi'), \end{aligned} \tag{12}$$

where we have substituted for $\langle V^*(\rho_1,\phi_1)V(\rho_2,\phi_2)\rangle_e$ from Eq. (7). The summation over p can be evaluated by using the identity $\sum_p[\mathrm{LG}_p^l(\rho)][\mathrm{LG}_p^{*l}(\rho')] = (1/\pi)\delta(\rho^2 - \rho'^2)$ (see Appendix B for the derivation), which gives

$$\begin{aligned} I_l = \sum_{l',p',p''} \alpha_{l'p'p''}\frac{1}{2\pi}\int \rho\, d\rho\, \big[\mathrm{LG}_{p'}^{*l'}(\rho)\big]\big[\mathrm{LG}_{p''}^{l'}(\rho)\big] \\ \times \iint d\phi\, d\phi' e^{i(l-l')(\phi-\phi')}\Phi^*(\phi)\Phi(\phi'). \end{aligned} \tag{13}$$

The radial integral is evaluated by noting that the radial LG modes with a fixed value for the angular-momentum-mode index form a complete basis, that is, $\int \rho\, d\rho\, [\mathrm{LG}_{p'}^{*l'}(\rho)][\mathrm{LG}_{p''}^{l'}(\rho)] = \delta_{p'p''}/2\pi$. Using this formula, we obtain

$$I_l = \sum_{l'} \frac{C_{l'}}{2\pi}\iint d\phi\, d\phi' e^{i(l-l')(\phi-\phi')}\Phi^*(\phi)\Phi(\phi'), \tag{14}$$

where $C_{l'} = (1/2\pi)\sum_{p'} \alpha_{l'p'p'}$. Next, we substitute the expression for the aperture function $\Phi(\phi) = k_1\delta(\phi - \phi_1) + k_2\delta(\phi - \phi_2)$. The intensity I_l then assumes the following form:

$$I_l = \frac{k_1^2}{2\pi}\sum_{l'=-\infty}^{\infty} C_{l'} + \frac{k_2^2}{2\pi}\sum_{l'=-\infty}^{\infty} C_{l'} + \frac{k_1k_2}{2\pi}\sum_{l'=-\infty}^{\infty} C_{l'}e^{-il'\Delta\phi}e^{il\Delta\phi} + \text{c.c.}, \tag{15}$$

Equation (15) can be seen to be the angular interference law, since it quantifies the interference between the fields coming from two separate angular positions.

The function

$$W(\phi_1,\phi_2) = \sum_{l'=-\infty}^{\infty} C_{l'}e^{-il'\Delta\phi} \tag{16}$$

represents the correlation that exists between the fields at ϕ_1 and ϕ_2. We refer to $W(\phi_1,\phi_2)$ as the angular coherence function. We note that in Ref. [16] Paterson introduced the "rotational coherence function," which describes correlation between two field points with the same radial but different angular positions. The angular correlation function constructed above is integrated over the radial dimensions and thus describes only the correlation between field points with different angular positions, without any reference to their radial positions. The field represented by $W(\phi_1,\phi_2)$ is completely coherent if there is only one term in the above expansion. However, when there is more than one term in the expansion, the field is only partially coherent, and, as a consequence, two field points are mutually coherent over only a finite range of angular separation $\Delta\phi$. In order to quantify this thought, we rearrange the above equation to write it as

$$I_l = \frac{1}{2\pi}\sum_{l'=-\infty}^{\infty} C_{l'}\left[k_1^2 + k_2^2 + 2k_1k_2\lambda(\Delta\phi)\cos(l\Delta\phi + \theta)\right], \tag{17}$$

where

$$\lambda(\Delta\phi) = \frac{|W(\phi_1,\phi_2)|}{\sum_{l'=-\infty}^{\infty} C_{l'}} \tag{18}$$

is the degree of angular coherence and θ the argument of $W(\phi_1,\phi_2)$. For a completely coherent field, the degree of coherence $\lambda(\Delta\phi)$ is equal to unity. The width of $\lambda(\Delta\phi)$ is a measure of the angular separation over which the field remains coherent. We note that $\lambda(\Delta\phi)$ involves a discrete Fourier transform and that therefore it is a periodic function of the argument $\Delta\phi$. For this reason, one has to be careful in defining the width of $\lambda(\Delta\phi)$. However, when $C_{l'}$ has a broad distribution in l' such that the spread of $\lambda(\Delta\phi)$ as a function of $\Delta\phi$ is well within the range $[0,2\pi]$, the width of $\lambda(\Delta\phi)$ can be defined unambiguously, and this width can, to a very good approximation, be taken as the coherence angle of the field. As shown in Appendix A, when $C_{l'}$ has a broad Gaussian distribution in l' with σ being the standard deviation of the distribution, $\lambda(\Delta\phi)$ assumes, to within a very good approximation, the following functional form:

$$\lambda(\Delta\phi) = \exp\left(-\frac{\sigma^2\Delta\phi^2}{2}\right). \tag{19}$$

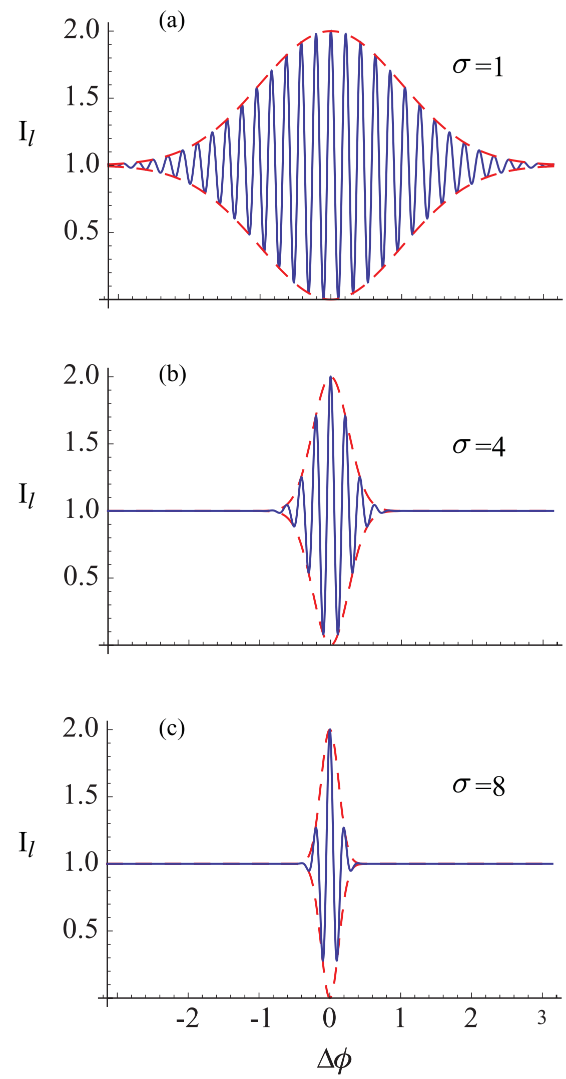

FIG. 2. (Color online) Intensity I_l, with $l = 30$ and $k_1 = k_2$, as a function of the angular separation $\Delta\phi$ for three different values of σ: (a) $\sigma = 1$, (b) $\sigma = 4$, and (c) $\sigma = 8$.

We note that $1/\sigma$ is a measure of the angular width over which the fields at the two angular positions remain mutually coherent. Therefore, $\phi_{\text{coh}} \equiv 1/\sigma$ can be defined as the coherence angle of the beam. Figure 2 shows plots of the detection probability I_l as a function of the angular separation $\Delta\phi$ for three different values of σ. We see that as the width σ of the OAM-mode distribution increases, the coherence angle decreases. The visibility of angular interference is given by

$$V(\Delta\phi) = \frac{2k_1k_2}{k_1^2 + k_2^2}\lambda(\Delta\phi), \tag{20}$$

and when $|k_1| = |k_2|$, we get $V(\Delta\phi) = \lambda(\Delta\phi)$.

III. ANGULAR COHERENCE AND OAM ENTANGLEMENT

In this section, we study a process known as parametric down-conversion that produces fields of the type considered in the previous section. We also show how the concept of angular coherence can be useful for characterizing OAM entanglement of the PDC photons [8].

A. Field produced by parametric down-conversion

Parametric down-conversion is a nonlinear optical process in which a pump photon is broken up into two entangled photons known as the signal photon and the idler photon. When the pump field is of the form of a Gaussian beam, that is, an LG beam with $l = 0$ and $p = 0$, the state $|\psi_2\rangle$ of the down-converted two-photon field is given by [17–21]

$$|\psi_2\rangle = \sum_{l_s,p_s,p_i} \int d\omega_s \chi_{l_s p_s p_i}(\omega_s) \times |l_s,p_s,\omega_s\rangle_s |-l_s,p_i,\omega_0-\omega_s\rangle_i, \quad (21)$$

where $\chi_{l_s p_s p_i}(\omega_s)$ is the probability amplitude that the signal and idler photons are in the LG modes characterized by indices (l,p_s) and $(-l,p_i)$, respectively. We note that, due to the conservation of OAM in PDC, the signal and idler photons have equal but opposite OAMs. In writing the above state, we have assumed that the pump field is monochromatic with frequency ω_0. We have also assumed perfect frequency phase matching such that $\omega_s + \omega_i = \omega_0$, where ω_s and ω_i denote the frequencies of the signal and idler photons. The density operator corresponding to the above two-photon state is $\hat{\rho}_2 = |\psi_2\rangle\langle\psi_2|$. The density operator $\hat{\rho}_s$ corresponding to the signal field can be calculated by taking a partial trace over the idler modes, which gives

$$\hat{\rho}_s = \mathrm{tr}_i\,\hat{\rho}_2 = \sum_{l_s,p_s,p_s'} \int d\omega_s C_{l_s p_s p_s'}(\omega_s) |l_s,p_s,\omega_s\rangle_{s\,s}\langle l_s,p_s',\omega_s|, \quad (22)$$

where

$$C_{l_s p_s p_s'}(\omega_s) = \sum_{p_i''} \chi_{l_s p_s p_i''}(\omega_s)\chi^*_{l_s p_s' p_i''}(\omega_s). \quad (23)$$

Next, by using Glauber's method [22], we calculate the classical correlation function $G_s(\rho_1,\phi_1;\rho_2,\phi_2;\tau)$ corresponding to the density matrix for the signal photon:

$$G_s(\rho_1,\phi_1;\rho_2,\phi_2;\tau) = \mathrm{tr}[\rho_s \hat{E}^{(-)}(\rho_1,\phi_1,t)\hat{E}^{(+)}(\rho_2,\phi_2,t+\tau)], \quad (24)$$

where

$$\hat{E}^{(-)}(\rho_1,\phi_1,t) = \sum_{l_1,p_1} \int d\omega \hat{s}^{\dagger}_{l_1 p_1}(\omega)\left[\mathrm{LG}^{*l_1}_{p_1}(\rho_1,\phi_1)\right]e^{i\omega t},$$

etc., and where $\hat{s}^{\dagger}_{l_1 p_1}(\omega)$ is the signal-photon annihilation operator for mode $[\mathrm{LG}^{*l_1}_{p_1}(\rho_1,\phi_1)]e^{i\omega t}$. Carrying out the above trace, we obtain

$$G_s(\rho_1,\phi_1;\rho_2,\phi_2;\tau) = \sum_{l_s,p_s,p_s'} \int d\omega_s C_{l_s p_s p_s'}(\omega_s)\left[\mathrm{LG}^{*l_s}_{p_s'}(\rho_1,\phi_1)\right] \times \left[\mathrm{LG}^{l_s}_{p_s}(\rho_2,\phi_2)\right]e^{-i\omega_s\tau}. \quad (25)$$

Finally, by taking the Fourier transforms of both sides of the above equation and using the definition in Eq. (1), we obtain the frequency-domain correlation function $W_s(\rho_1,\phi_1;\rho_2,\phi_2;\omega_s)$ for the signal photon:

$$W_s(\rho_1,\phi_1;\rho_2,\phi_2;\omega) = \sum_{l_s,p_s,p_s'} C_{l_s p_s p_s'}(\omega)\left[\mathrm{LG}^{*l_s}_{p_s'}(\rho_1,\phi_1)\right] \times \left[\mathrm{LG}^{l_s}_{p_s}(\rho_2,\phi_2)\right]. \quad (26)$$

We see at once that the correlation function for the signal photon has the same functional form as the cross-spectral-density function considered in Eq. (7). Therefore, it follows that, with respect to a detection system that is sensitive only to the azimuthal mode index, the angular coherence function for the signal photon has the same functional form as that of $W(\phi_1,\phi_2)$ in Eq. (16). From now on, we suppress the frequency argument in writing the correlation functions for the signal photon. Starting from Eq. (26) and using the procedure of Sec. II B, one can show that the angular coherence function $W_s(\phi_1,\phi_2)$ corresponding to the signal photon is

$$W_s(\phi_1,\phi_2) = \sum_{l_s=-\infty}^{\infty} C_{l_s} e^{-il_s\Delta\phi}, \quad (27)$$

where $C_{l_s} = (1/2\pi)\sum_{p_s} C_{l_s p_s p_s}$. We note that the signal photon field is an incoherent superposition of modes carrying different OAMs. Also, from our discussions in the previous section, we find that if C_{l_s} has a broad distribution in l_s, its width can be measured directly by measuring the coherence angle of the signal field. This fact has one very important implication which we discuss in the next section.

B. The angular Schmidt spectrum and the coherence angle

A complete characterization of OAM entanglement of the two-photon state shown in Eq. (21) can be performed through Schmidt decomposition, which yields the Schmidt modes, a natural set of biorthogonal mode pairs that constitute the two-photon state [23]. We note that the two-photon state of Eq. (21) is not in the Schmidt-decomposed form since therein we have summation over three different indices. However, in many quantum-information protocols, such as those based on OAM entanglement, one is concerned with only the OAM-mode index of the photons. In such cases, the detection system is sensitive only to the OAM-mode index and therefore the two-photon state of Eq. (21) can be written in a Schmidt decomposed form that is only one dimensional:

$$|\psi_2\rangle = \sum_{l=-\infty}^{\infty} \sqrt{C_l}|l\rangle_s|-l\rangle_i. \quad (28)$$

Here s and i stand for signal and idler photons, respectively, and $|l\rangle$ represents an eigenmode of order l, corresponding to an azimuthal phase $e^{il\phi}$. C_l is the angular Schmidt coefficient, which is the probability that the signal and idler photons are generated in modes of order l and $-l$, respectively. The distribution of this mode probability is referred to as the angular Schmidt spectrum or the spiral spectrum of the PDC photons [13,24]. For the two-photon state in Eq. (28), the corresponding angular coherence function of the signal photon is still given by Eq. (27). The angular Schmidt spectrum is directly related to the entanglement of the two-photon field through an entanglement measure known as the Schmidt number K, which for the normalized angular Schmidt spectrum is defined as [23,25,26]:

$$K \equiv \frac{1}{\sum_{l=-\infty}^{\infty} C_l^2}. \tag{29}$$

There are two generic ways in which the angular Schmidt spectrum of the two-photon field can be measured. First is by directly measuring the two-photon intensity in coincidence at different values of the OAM-mode index [11,12], and the second is by using the Hong-Ou-Mandel interference technique [13]. However, comparing Eqs. (28) and (27), we find that the OAM-mode spectrum of the signal photon is identically equal to the angular Schmidt spectrum of the two-photon field. Therefore, it follows that by measuring the angular coherence function, as shown in the previous section, of the signal field, one can construct the angular Schmidt spectrum of the two-photon field. We note that in this method one calculates the angular Schmidt (spiral) spectrum without doing coincidence measurements, in contrast to the above-mentioned methods, which require coincidence detection. In situations in which C_l has a broad, Gaussian distribution, it can be shown that $K \approx 2\sqrt{\pi}\sigma$, where σ is the standard deviation of the distribution. This approximate equality becomes an exact equality in the limit in which the distribution becomes infinitely broad. Now, for the Gaussian distribution, σ is equal to $1/\phi_{\rm coh}$, where $\phi_{\rm coh}$ is the coherence angle of the beam. Thus the Schmidt number in this case is inversely proportional to the coherence angle of the signal or the idler field: $K \approx 2\sqrt{\pi}/\phi_{\rm coh}$. We thus find that as the entanglement of the two-photon field increases the coherence angle of the signal and idler fields decreases. We note that the above method of estimating the entanglement of the two-photon field by performing measurements on the one-photon signal or idler field is applicable only when the two-photon field can be assumed to be in the Schmidt decomposed form of Eq. (28) and fortunately the field produced by the down-converter does have this form. In situations in which this assumption is not valid, the entanglement has to be estimated by doing measurements on the entire two-photon field.

IV. CONCLUSIONS

In conclusion, we have studied partially coherent fields that have coherent-mode representations in the OAM-mode basis. We have introduced the concepts of the angular correlation function and the coherence angle, and by utilizing the concept of partial angular coherence we have also proposed a method to measure the angular Schmidt spectrum of the entangled two-photon field produced by parametric down-conversion. This proposed method may have important implications as it requires only singles measurement, as opposed to the other methods which are based on coincidence measurements.

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APPENDIX A: ANGULAR COHERENCE OF FIELDS WITH A BROAD DISTRIBUTION FOR C_l

In this Appendix, we consider partially coherent fields with a broad distribution for C_l. For such fields, we can calculate the exact functional form of the angular correlation function $W(\phi_1,\phi_2)$. First, we write the angular correlation function in the following form:

$$\begin{aligned} W(\phi_1,\phi_2) &= \frac{1}{2\pi}\sum_{l=-\infty}^{\infty} C_l e^{-il\Delta\phi} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(l)\mathrm{comb}(l)e^{-il\Delta\phi}dl, \end{aligned} \tag{A1}$$

where $C(l)$ is a continuous function of l. The comb function is defined as $\mathrm{comb}(l) = \sum_{n=-\infty}^{\infty}\delta(l-n)$. In the above equation, the angular correlation function $W(\phi_1,\phi_2)$ is, up to a constant, the Fourier transform of the product of $C(l)$ and $\mathrm{comb}(l)$. We can therefore write it as the convolution of the Fourier transforms of $C(l)$ and $\mathrm{comb}(l)$, that is,

$$\begin{aligned} W(\phi_1,\phi_2) &= \frac{1}{\sqrt{2\pi}}\mathcal{F}[C(l)]\otimes\mathcal{F}[\mathrm{comb}(l)] \\ &= \frac{1}{\sqrt{2\pi}}C(\Delta\phi)\otimes\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(l-n)e^{-il\Delta\phi}dl \\ &= \frac{1}{2\pi}C(\Delta\phi)\otimes\sum_{n=-\infty}^{\infty}e^{-i\Delta\phi n}, \end{aligned} \tag{A2}$$

where $\otimes$ represents the convolution and $\mathcal{F}[\cdots]$ the Fourier transformation; $C(\Delta\phi)$ is the Fourier transform of the OAM-mode distribution $C(l)$. Using the formula $\sum_n e^{-i\Delta\phi n} = 2\pi\sum_k \delta(\Delta\phi - 2\pi k)$, we write $W(\phi_1,\phi_2)$ as

$$\begin{aligned} W(\phi_1,\phi_2) &= C(\Delta\phi)\otimes\sum_{k=-\infty}^{\infty}\delta(\Delta\phi-2\pi k) \\ &= \sum_{k=-\infty}^{\infty}C(\Delta\phi-2\pi k). \end{aligned} \tag{A3}$$

Now we restrict the values of $\Delta\phi$ to be between 0 and 2π and assume that the width of $C(\Delta\phi)$ is much smaller than π. This is justified as we have already assumed that C_l has a broad distribution. We thus find that the only significant contribution to $W(\phi_1,\phi_2)$ comes from the $k=0$ term, and thus we obtain $W(\phi_1,\phi_2) = C(\Delta\phi)$, that is, the angular correlation function is the Fourier transform of the OAM-mode distribution. In the case in which $C(l)$ is Gaussian, that is, $C(l) = 1/(\sqrt{2\pi}\sigma)\exp[-l^2/(2\sigma^2)]$, where σ is the standard deviation of the distribution, the degree of coherence is given by $\lambda(\Delta\phi) = \exp(-\frac{\sigma^2\Delta\phi^2}{2})$.

APPENDIX B: EVALUATION OF THE SUMMATION IN Eq. (12)

Rearranging the equation

$$\sum_p [\mathrm{LG}_p^l(\rho)][\mathrm{LG}_p^{*l}(\rho')] = \sum_p \frac{2p!}{\pi(|l|+p)!}\frac{1}{w_0^2}\left(\frac{2\rho\rho'}{w_0^2}\right)^{|l|} \exp\left(-\frac{\rho^2+\rho'^2}{w_o^2}\right) L_p^{|l|}\left(\frac{2\rho^2}{w_0^2}\right) L_p^{|l|}\left(\frac{2\rho'^2}{w_0^2}\right), \tag{B1}$$

we can write the above equation as

$$\sum_p [\mathrm{LG}_p^l(\rho)][\mathrm{LG}_p^{*l}(\rho')] = \frac{2}{\pi w_0^2}\left(\frac{2\rho\rho'}{w_0^2}\right)^{|l|} \exp\left(-\frac{\rho^2+\rho'^2}{w_o^2}\right)\sum_p \frac{\Gamma(p+1)}{\Gamma(|l|+p+1)} L_p^{|l|}\left(\frac{2\rho^2}{w_0^2}\right) L_p^{|l|}\left(\frac{2\rho'^2}{w_0^2}\right). \tag{B2}$$

The summation on the right-hand side is a standard result for Laguerre polynomials, using which we get

$$\sum_p [\mathrm{LG}_p^l(\rho)][\mathrm{LG}_p^{*l}(\rho')] = \frac{2}{\pi w_0^2}\left(\frac{2\rho\rho'}{w_0^2}\right)^{|l|} \exp\left(-\frac{\rho^2+\rho'^2}{w_o^2}\right)\left(\frac{2\rho^2}{w_0^2}\frac{2\rho'^2}{w_0^2}\right)^{-|l|/2} \exp\left(\frac{\rho^2+\rho'^2}{w_0^2}\right)\delta\left(\frac{2\rho^2}{w_0^2}-\frac{2\rho'^2}{w_0^2}\right). \tag{B3}$$

Finally after rearranging, we get the desired result

$$\sum_p [\mathrm{LG}_p^l(\rho)][\mathrm{LG}_p^{*l}(\rho')] = \frac{1}{\pi}\delta(\rho^2-\rho'^2). \tag{B4}$$

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