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PAPER

Divergence of an orbital-angular-momentum-carrying beam upon propagation

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Abstract

There is recent interest in the use of light beams carrying orbital angular momentum (OAM) for creating multiple channels within free-space optical communication systems. One crucial issue is that, for a given beam size at the transmitter, the beam divergence angle increases with increasing OAM. Therefore the larger the value of OAM, the larger the aperture required at the receiving optical system if the efficiency of detection is to be maintained. Confusion exists as to whether this divergence scales linearly with, or with the square root of, the beam's OAM. We clarify how both these scaling laws are valid, depending upon whether it is the radius of the waist of the beam's Gaussian term or the radius of rms intensity of the beam that is kept constant while varying the OAM.

1. Introduction

Over the past 20 years there has been a growing interest in the orbital angular momentum (OAM) of light, which is carried by any optical beam possessing helical phase fronts [1]. For phase fronts described by an azimuthally dependent phase factor $\exp(i\ell\phi)$, the OAM is equivalent to $\ell\hbar$ per photon [2]. This OAM of a light beam can be derived in a number of ways, but for our purposes it can be seen to arise from the local direction of the Poynting vector. At all points within the beam, the Poynting vector is perpendicular to the phase front and hence possesses an azimuthal momentum component [3]. Throughout this paper we consider only the case of free-space propagation of light beam. As confirmed by measurement [4], the helical phase fronts of an OAM-carrying beam lead to the local Poynting vector being skewed with respect to the beam axis by an angle β, which is a function of r, the radial distance from the beam axis:

$$\beta = \frac{|\ell|}{k_0 r}, \tag{1}$$

where $k_0 = 2\pi/\lambda$. One analytically simple and widely used form of helically phased beams is the Laguerre–Gaussian (LG) modes which have a field amplitude given as the product of a Gaussian term with beam waist w_0 (defined as the radius at which the electric field of the Gaussian term is lower by a factor $1/e$), with a generalized Laguerre polynomial with indices $|\ell|$ and p. The radial index p gives the number of off-axis nodes in the radial direction.

In this work we restrict ourselves to the small numerical aperture case where the size of any spatial features in the modes or their superpositions is significantly larger than the optical wavelength. For $p = 0$ and $\ell = 0$ one has the fundamental Gaussian mode. For $p = 0$ and $\ell > 0$ the LG modes are single-ringed annular modes with an azimuthal phase term $\exp(i\ell\phi)$ and a radius of maximum optical intensity, $r(I_{\max})$, given as [3]

$$r\left(I_{\max}\right) = \sqrt{\frac{|\ell|}{2}}\, w(z), \tag{2}$$

where $w(z) = w_0\sqrt{1 + z^2/z_R^2}$ and $z_R = \frac{1}{2}kw_0^2$ is the Rayleigh range (here $k = 2\pi/\lambda$). One notes that for high OAM values the size of the LG mode is much larger than the beam waist. The restriction of our study to the paraxial regime also implies that $w(z) \gg \lambda$ and hence that β (see equation (1)) remains small for all radii at which the beam has a non-negligible intensity.

In terms of their complex amplitudes, LG beams with different mode indices are orthogonal to each other and this leads to their potential use in communication applications where the orthogonality of the modes allows for multiple states within a channel or independent channels. A simple optical communication system based on OAM was demonstrated over 10 years ago [5], but recently the idea has been revisited with reports of both high data rates [6] and outdoor demonstrations [7]. One issue of critical importance is the degree to which the use of OAM-carrying beams require the use of larger aperture optics. Confusion exists as to whether the divergence and hence the far-field size of a beam carrying OAM scales linearly or with the square root of its angular momentum.

2. Divergence of an OAM carrying beam

The the geometrically calculated skew angle of the Pointing vector calculated with respect to the beam axis given by equation (1) might be thought to approximate the angular divergence of the beam. However, an additional contribution to the divergence that has to be considered is the normal diffractive spreading arising from the finite beam diameter. Rather than the beam waist w_0, or the radius of maximum intensity $r(I_{\max})$, the divergence of a light beam is governed by the standard deviation of its spatial distribution, r_{rms} [8, 9]. The calculation of these quantities is straightforward. We start from the definition of a LG mode which gives the intensity distribution for the lowest-order radial mode ($p = 0$) as

$$I_\ell(r, \phi, z) = \frac{2}{w(z)^2\pi\ |\ell|!}\left(\frac{\sqrt{2}\, r}{w(z)}\right)^{2|\ell|} \exp\left(-\frac{2r^2}{w(z)^2}\right), \tag{3}$$

which is normalized such that $\int I_\ell(r, \phi, z)\, r\mathrm{d}r\mathrm{d}\phi = 1$. From this distribution, we can calculate the radial position of the maximum of intensity, by solving the equation $\partial I_\ell/\partial r = 0$, obtaining (as anticipated in equation (2) [3])

$$r\left(I_{\max}\right) = \sqrt{\frac{|\ell|}{2}}\, w(z). \tag{4}$$

However, as $r(I_{\max})$ is zero for $\ell = 0$, clearly one cannot generally use this quantity to derive the angular spread of the beam as it propagates. As has been reported previously, a better quantity on which to base a calculation of the divergence of a light beam is the standard deviation of the spatial distribution of the beam, r_{rms}. This radius is a function of z and it is given by the square root of the radial variance of the intensity distribution [10]:

$$r_{\mathrm{rms}}(z) = \sqrt{2\pi\int_0^\infty r^2 I_\ell(r, \phi, z)\, r\mathrm{d}r} \tag{5}$$

$$= \sqrt{\frac{|\ell| + 1}{2}}\, w(z). \tag{6}$$

In the plane of the beam waist, $z = 0$, this radius is

$$r_{\mathrm{rms}}(0) = \sqrt{\frac{|\ell| + 1}{2}}\, w_0. \tag{7}$$

In any given plane, figure 1 shows the ℓ dependence of $r(I_{\max})$ and r_{rms}. One notes that

$$\frac{r_{\mathrm{rms}}}{r\left(I_{\max}\right)} = \sqrt{\frac{|\ell| + 1}{|\ell|}}, \tag{8}$$

which tends to 1 as $|\ell|$ increases, so these two radii become equal for large ℓ, see figure 1.

Given the radius $r_{\mathrm{rms}}(z)$ as a function of propagation distance, z, we can calculate the corresponding divergence angle of a $p = 0$ LG mode, α_ℓ, to be

$$\alpha_\ell(z) = \arctan\left(\frac{\partial r_{\mathrm{rms}}(z)}{\partial z}\right) \tag{9}$$

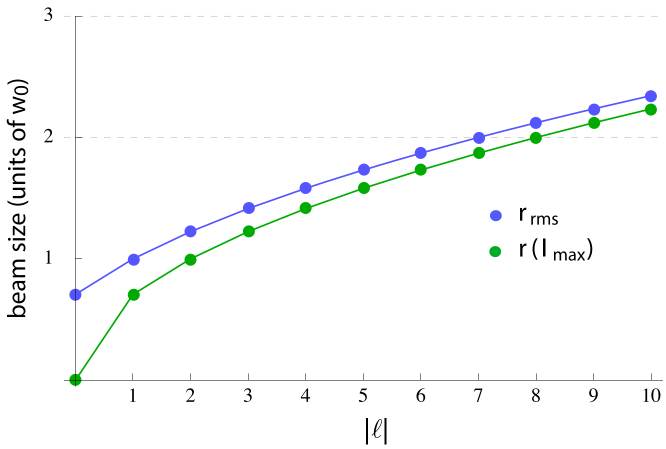

Figure 1. A comparison between the radii of maximum intensity, $r\,(I_{\max})$, and the rms radius, $r_{\rm rms}$, of the spatial distribution of a $p = 0$ LG mode. For large ℓ, the two radii get asymptotically close, as $r\,(I_{\max})$ is proportional to $\sqrt{|\ell|}$, while $r_{\rm rms}$ is proportional to $\sqrt{|\ell| + 1}$.

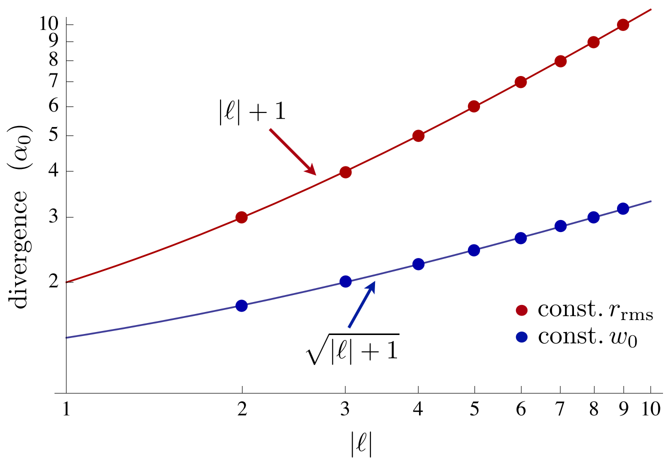

Figure 2. The divergence of an OAM beam obtained by keeping the $r_{\rm rms}$ radius in the launch plane constant is larger than for an OAM beam obtained by keeping the Gaussian waist w_0 constant. Moreover, the first scales as $|\ell| + 1$ whereas the second scales as $\sqrt{|\ell| + 1}$. This log–log plot is given in units of the divergence angle $\alpha_0 = w_0/\sqrt{2}\,z_R$ of a fundamental Gaussian beam. The lines serve purely as a guide to the eye.

$$= \arctan\left(\sqrt{\frac{|\ell| + 1}{2}}\,\frac{w_0^2}{z_R^2}\,\frac{z}{w\,(z)}\right). \tag{10}$$

This divergence angle is a function of z, but it asymptotically reaches a limiting value (when the beam is sufficiently far from the waist). Within the paraxial regime, for small α_ℓ, we can equate the tangent of the angle with the angle and for $\lim_{z\to\infty}$, $z/w\,(z) = z_R/w_0$. Hence in the far field, the divergence angle of a $p = 0$ LG mode is given by

$$\alpha_\ell = \sqrt{\frac{|\ell| + 1}{2}}\,\frac{2}{k_0 w_0}. \tag{11}$$

Equation (11) is a convenient form for expressing the ℓ dependence of the beam divergence while holding w_0 of the LG mode constant, revealing the approximate square root scaling of the far-field beam size; see figure 2.

Alternatively the divergence angle of the beam can be written in terms of other quantities; for instance, using equation (7), one can instead express the ℓ dependence of the beam divergence for a constant $r_{\rm rms}\,(0)$ as

$$\alpha_\ell = \frac{|\ell| + 1}{k_0 r_{\rm rms}\,(0)}. \tag{12}$$

Under this condition, of a fixed $r_{\rm rms}\,(0)$, we see that the beam divergence scales linearly with ℓ, see figure 2. Note also for a fixed $r_{\rm rms}\,(0)$ that w_0 itself becomes a function of ℓ and scales as $(|\ell| + 1)^{-1/2}$.

3. Discussion

Typical of a practical optical system is that the aperture of the optics is fixed by design, with a limiting radius R. Since the radius of maximum intensity of an LG beam can be much larger than w_0, in order to design a system capable of using a wide range of different OAM modes one must choose a waist $w_0 \ll R$.

One example of an approach that produces modes of a fixed beam waist is the cylindrical-lens mode converter that converts any incoming Hermite Gaussian mode into the corresponding LG mode with the same beam waist [11]. When using a mode converter of this type to produce the LG modes one should observe a square root scaling of the far-field beam divergence, equation (11), i.e. a scaling approximately proportional to $\sqrt{|\ell|}$.

A more frequently used and flexible method of producing beams carrying OAM is the illumination of a diffractive optical element (computer generated hologram) with an expanded Gaussian mode from a conventional laser. If the diffractive element is a spatial light modulator (or similar) then the displayed kinoform can be updated to change the beam type allowing a switching of the system between different OAM states. Most common of such kinoforms is a forked diffraction grating that produces a helically phased beam in the first diffraction order [12]. In the plane of the diffractive element, the beam has the required helical phase structure, but possesses the Gaussian intensity distribution of the illuminating beam. This beam corresponds to a superposition of different LG modes, all of the same OAM ℓ, but with various values of p. The precise p-weighting of this superposition is a complicated function of both $|\ell|$ and the value of beam waist w_0 chosen to perform the decomposition [13]. An obvious choice of beam waist for the decomposition is the one which maximizes the amount of power in the $p = 0$ mode, as referred to above this optimum beam waist scales $(|\ell| + 1)^{-1/2}$. Therefore rather than the resulting beam having a particular $|\ell|$-dependent beam waist, the $|\ell|$-invariant quantity is simply the size of the beam incident on the diffraction grating and hence the system has a fixed $r_{\rm rms}(0)$. Therefore when using a forked diffraction grating as the beam generation component within an optical system to produce LG-like modes one should observe the approximately linear $|\ell|$-scaling of the far-field beam divergence, equation (12). This linear scaling of the beam size as a function of $|\ell|$ has previously been noted within optical tweezers and microscope systems [14].

We note that the reasoning we have followed above is consistent with an understanding of the scaling in terms of the spatial resolution of an optical system. For example, if one considers a transmitted beam comprising a superposition of $\pm\ell$ modes, then the resulting beam profile is an annular ring of $2|\ell|$ maxima, or 'petals'. In the far field the spacing between these petals cannot be smaller than the resolving power of the optical system. Since the number of petals scales with $|\ell|$, this implies that the beam size and hence the beam divergence must also scale with $|\ell|$. A square root scaling of the beam divergence α_ℓ and corresponding decrease in petal separation would seemingly be in conflict with this simple resolution argument. However, a square root scaling requires a constant beam waist w_0, which implies a beam size $r_{\rm rms}(0)$ and hence a minimum transmission aperture that must increase with the square root of $|\ell|$. This increasing aperture itself supports a higher spatial resolution, meaning that the reduced spacing of the resulting petals remains compatible with the resolution limit.

4. Conclusions

In conclusion, we have explained how optical systems exploiting the propagation of light beams carrying OAM can produce a far-field beam divergence that scales either with the square root of the OAM or linearly with the OAM, see figure 3. The square root scaling applies to systems where the beam waist of the Gaussian term, w_0, is held constant for all values of ℓ, such as the case where the beams are produced using a cylindrical lens mode converter [11]. The linear scaling of the beam divergence applies to systems where the rms radius of the launch beam is held constant, such as the case where the beam is produced by a simple forked diffraction grating implemented on a spatial light modulator illuminated with a Gaussian beam of fixed beam[6].

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[6] One notes that more sophisticated hologram designs can produce arbitrary combinations of LG modes with any values of ℓ, p and w_0 (see for instance M Dennis *et al Nat. Phys.* **6**, 118 (2010)), albeit at greatly reduced efficiency compared to the simple fork design that is widely used.

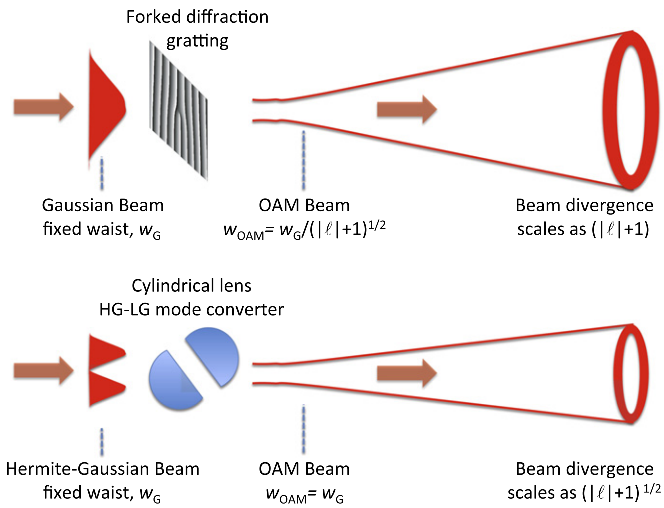

Figure 3. A schematic diagram comparing the far-field divergence for OAM carrying beams where the beams are created with a forked diffraction grating (top) illuminated with a Gaussian beam of fixed beam waist, with the case of a cylindrical lens mode converter (bottom) to transform a Hermite–Gaussian mode also of fixed beam waist.

References

[1] Yao A M and Padgett M J 2011 Orbital angular momentum: origins, behavior and applications *Adv. Opt. Photonics* **3** 161

[2] Allen L, Beijersbergen M W, Spreeuw R J C and Woerdman J P 1992 Orbital angular momentum of light and the transformation of Laguerre–Gaussian laser modes *Phys. Rev.* A **45** 8185–9

[3] Padgett M J and Allen L 1995 The poynting vector in Laguerre–Gaussian laser modes *Opt. Commun.* **121** 36–40

[4] Leach J, Keen S, Padgett M J, Saunter C and Love G D 2006 Direct measurement of the skew angle of the poynting vector in a helically phased beam *Opt. Express* **14** 11919–24

[5] Gibson G, Courtial J, Padgett M, Vasnetsov M, Pas'ko V, Barnett S and Franke-Arnold S 2004 Free-space information transfer using light beams carrying orbital angular momentum *Opt. Express* **12** 5448–56

[6] Wang J *et al* 2012 Terabit free-space data transmission employing orbital angular momentum multiplexing *Nat. Photonics* **6** 488–96

[7] Krenn M, Fickler R, Fink M, Handsteiner J, Scheidl T, Ursin R and Zeilinger A 2014 Twisted light communication through turbulent air across vienna *New J. Phys.* **16** 113028

[8] Stelzer E H K and Grill S 2000 The uncertainty principle applied to estimate focal spot dimensions *Opt. Commun.* **173** 51–56

[9] Padgett M 2008 On the focussing of light, as limited by the uncertainty principle *J. Mod. Opt.* **55** 3083–9

[10] Phillips R L and Andrews L C 1983 Spot size and divergence for Laguerre–Gaussian beams of any order *Appl. Opt.* **22** 643–4

[11] Beijersbergen M W, Allen L, van der Veen H E L O and Woerdman J P 1993 Astigmatic laser mode converters and transfer of orbital angular momentum *Opt. Commun.* **96** 123–32

[12] Bazhenov V Y, Vasnetsov M V and Soskin M S 1990 Laser-beams with screw dislocations in their wave-fronts *J. Exp. Theor. Phys. Lett.* **52** 429–31

[13] Dennis M R, O'Holleran K and Padgett M J 2009 Singular optics: optical vortices and polarization singularities *Prog. Opt.* **53** 293–363

[14] Curtis J E and Grier D G 2003 Structure of optical vortices *Phys. Rev. Lett.* **90** 133901