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Quantum noise properties of non-ideal optical amplifiers and attenuators

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Abstract
We generalize the standard quantum model (Caves 1982 *Phys. Rev.* D **26** 1817) of the noise properties of ideal linear amplifiers to include the possibility of non-ideal behavior. We find that under many conditions the non-ideal behavior can be described simply by assuming that the internal noise source that describes the quantum noise of the amplifier is not in its ground state. We validate this model by showing that it reproduces the known predictions of two models of non-ideal amplifiers: a laser amplifier with incomplete inversion and a cascade of alternating ideal amplifiers and ideal attenuators. We also describe implications of this model for practical devices.

Keywords: quantum noise, optical amplifiers

(Some figures may appear in colour only in the online journal)

1. Introduction

Optical amplification is required for the operation of many optical systems, including those used in telecommunications, optical information processing, and quantum optics. Examples of optical amplifiers include erbium doped fiber amplifiers [1, 2], semiconductor optical amplifiers [3], and surface plasmon amplifiers [4]. Optical systems also often contain loss elements, which are present either by design or because of the inability to fabricate perfectly transmitting optical components. It is well known that both gain and loss introduce noise into an optical system, and this noise can be highly undesirable under many circumstances.

In many cases of interest, the noise generated by an amplifier or attenuator can be ascribed to a purely quantum-mechanical origin. Indeed, the quantum-mechanical properties of the electromagnetic field pose a fundamental limit to the noise level of an optical beam. An ideal phase-insensitive optical amplifier can be treated by a quantum model that assumes the presence of two input ports, one for the signal field and another that represents the internal noise source of the amplification process [5–7]. In these models, the amplifier noise can be attributed mainly to the beating between the amplified signal field and the amplified vacuum fluctuations.

A non-ideal amplifier or attenuator is a device that introduces more noise than that required by the laws of quantum mechanics. Non-ideal amplifiers or attenuators have been treated theoretically by considering them to be comprised of alternating pairs of ideal amplifiers and attenuators [8], by means of a three-dimensional quantum beam splitter model [9], or by explicit consideration of the propagation of light through a not-totally-inverted laser gain medium [10].

In this paper, we present a generalization of Caves's [5] treatment of the noise properties of linear amplifiers by allowing for non-ideal behavior. We find that a broad class of amplifiers and attenuators can be treated by making just one additional assumption, namely that the internal quantum noise source of the device is not in its ground state. We begin with a brief summary of the usual treatments for ideal amplifiers or attenuators in section 2. Our new model is introduced in section 3 and applied specifically to several types of practical amplifiers. The paper concludes with a discussion and summary in section 4.

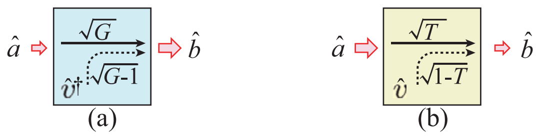

Figure 1. Quantum model of an ideal linear amplifier (a) and an attenuator (b). Here and in figure 2, we use the symbol $\hat{v}$ to designate an internal noise source $\hat{c}$ that is in the vacuum state.

2. Quantum model of an ideal linear amplifier or attenuator

Let us first summarize the quantum description of the noise properties of an ideal amplifier for the case of a single-mode input field propagating through an ideal linear amplifier with intensity gain G. We adopt the standard model shown in figure 1(a) which was introduced by earlier workers [5, 11–13]. This model ascribes the noise properties of the amplifier to an internal quantum noise source that represents the fluctuations that must (by means of the fluctuation-dissipation theorem) accompany any gain or loss mechanism.

The strength of this noise source is adjusted to ensure that the field operator of the output mode possesses standard boson commutation relations. Specifically, the photon annihilation operators for the input and output fields are denoted by $\hat{a}$ and $\hat{b}$, respectively, and are assumed to satisfy the standard commutation relations

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1. \tag{1}$$

The annihilation operator for the output field is expressed in terms of that of the input field and of a Langevin noise source as follows [5]:

$$\hat{b} = G^{1/2}\hat{a} + (G-1)^{1/2}\hat{c}^\dagger. \tag{2}$$

Here $\hat{c}$ is a boson operator, denoting the internal noise field, which also must satisfy the relation $[\hat{c}, \hat{c}^\dagger] = 1$. The internal noise field is assumed to be uncorrelated from the input field, that is,

$$[\hat{a}, \hat{c}^\dagger] = [\hat{a}, \hat{c}] = 0. \tag{3}$$

It is straightforward to verify that under these circumstances the commutation relation $[\hat{b}, \hat{b}^\dagger] = 1$ is satisfied. For the case of an ideal amplifier, one assumes that the internal noise field is in the vacuum state, that is,

$$\langle \hat{n}_c \rangle = \langle \hat{c}^\dagger \hat{c} \rangle = 0. \tag{4}$$

Consequently, the expectation value of the photon number of the output field is given by

$$\langle \hat{n}_b \rangle = \langle \hat{b}^\dagger \hat{b} \rangle = G\langle \hat{n}_a \rangle + (G-1), \tag{5}$$

where $\langle \hat{n}_a \rangle = \langle \hat{a}^\dagger \hat{a} \rangle$ is the average photon number of the input field. One sees that, in addition to the input field being amplified by a factor of G, $(G-1)$ noise photons are added to the output field. The variance of the output photon number is similarly given by

$$\begin{aligned}\langle \Delta\hat{n}_b^2 \rangle &\equiv \langle \hat{n}_b^2 \rangle - \langle \hat{n}_b \rangle^2 \\ &= G^2\langle \Delta\hat{n}_a^2 \rangle + G(G-1)(\langle \hat{n}_a \rangle + 1).\end{aligned} \tag{6}$$

Here, the first term represents the amplification of the fluctuations present in the input field, and the second term represents the added noise.

The noise properties of an amplifier are often characterized by a noise figure [14], defined as

$$\mathcal{F} \equiv \frac{(\mathrm{SNR})_{\mathrm{in}}}{(\mathrm{SNR})_{\mathrm{out}}}, \tag{7}$$

where $(\mathrm{SNR})_{\mathrm{in}}$ and $(\mathrm{SNR})_{\mathrm{out}}$ are the signal-to-noise ratios (SNRs) of the input and output fields, respectively, which are defined as follows:

$$(\mathrm{SNR})_{\mathrm{in}} \equiv \frac{\langle \hat{n}_a \rangle^2}{\langle \Delta\hat{n}_a^2 \rangle}, \tag{8}$$

$$(\mathrm{SNR})_{\mathrm{out}} \equiv \frac{G^2\langle \hat{n}_a \rangle^2}{\langle \Delta\hat{n}_b^2 \rangle}. \tag{9}$$

Note that in $(\mathrm{SNR})_{\mathrm{out}}$ we use $G\langle \hat{n}_a \rangle$ for the amplified signal field, not the total output field that includes the noise contributions.

The noise figure of an ideal optical amplifier is then found from the expressions given above to be

$$\mathcal{F} = 1 + \left(1 - \frac{1}{G}\right)\frac{\langle \hat{n}_a \rangle + 1}{\langle \Delta\hat{n}_a^2 \rangle}. \tag{10}$$

When the fluctuations of the input field follow Poisson statistics, so that $\langle \Delta\hat{n}_a^2 \rangle = \langle \hat{n}_a \rangle$, one sees that the value of the noise figure $\mathcal{F}$ of an ideal quantum amplifier approaches the value 2 (or 3 dB) when both G and $\langle \hat{n}_a \rangle$ are much greater than unity [15].

This same model can be used to describe an ideal attenuator of intensity transmission T. We again model this device in terms of an input signal field $\hat{a}$, an output field $\hat{b}$, and an internal noise field $\hat{c}$, as shown in figure 1(b). In order to preserve the commutation relations for the output field $\hat{b}$, we express it as

$$\hat{b} = T^{1/2}\hat{a} + (1-T)^{1/2}\hat{c}. \tag{11}$$

Here, the Langevin operator is again assumed to be uncorrelated from the input operator:

$$[\hat{a}, \hat{c}^\dagger] = [\hat{a}, \hat{c}] = 0. \tag{12}$$

For an ideal attenuator, we again assume that $\hat{c}$ denotes a vacuum state input, i.e.,

$$\langle \hat{n}_c \rangle = \langle \hat{c}^\dagger \hat{c} \rangle = 0. \tag{13}$$

The expected photon number for the output field and its variance are then given by

$$\langle \hat{n}_b \rangle = \langle \hat{b}^\dagger \hat{b} \rangle = T\langle \hat{n}_a \rangle \tag{14}$$

and

$$\begin{aligned}\langle \Delta\hat{n}_b^2 \rangle &\equiv \langle \hat{n}_b^2 \rangle - \langle \hat{n}_b \rangle^2 \\ &= T^2\langle \Delta\hat{n}_a^2 \rangle + T(1-T)\langle \hat{n}_a \rangle,\end{aligned} \tag{15}$$

respectively, for such an ideal attenuator. Here, the first term represents the attenuation of the fluctuations present in the input beam, and the second term represents the noise added due to the random loss of photons from the signal field. The noise figure of the ideal attenuator is then found to be

$$\mathcal{F} = 1 + \left(\frac{1}{T} - 1\right)\frac{\langle\hat{n}_a\rangle}{\langle\Delta\hat{n}_a^2\rangle}. \tag{16}$$

Note that this noise figure can become arbitrarily large as the transmission of the attenuator approaches zero, which can be attributed to the random loss of photons from the signal field.

3. Generalized model for a non-ideal linear amplifier or attenuator

We next present a generalization of the model of 2 that is capable of treating non-ideal amplifiers and attenuators. One example of a non-ideal amplifier is a laser amplifier with incomplete population inversion. Another example is a cascaded assembly of alternating ideal amplifiers and attenuators. Either of these examples can be described completely by standard methods, as we demonstrate below. Our motivation in developing this new model is to be able to treat a general class of non-ideal amplifiers without the need to specify the precise nature of the amplification or attenuation process.

3.1. The new model

To treat a non-ideal amplifier, we use the same model as in section 2, except that now we do not require the internal noise source to be in its ground state. Specifically, we assume that the noise field $\hat{c}$ is not a vacuum field but is a completely incoherent noise field with average photon number $\langle\hat{n}_c\rangle \neq 0$. The output field operator $\hat{b}$ is once again related to the input field operators by

$$\hat{b} = G^{1/2}\hat{a} + (G-1)^{1/2}\hat{c}^\dagger. \tag{17}$$

In this more general case, the average photon number of the output field and its variance are given by

$$\langle\hat{n}_b\rangle = G\langle\hat{n}_a\rangle + (G-1)(\langle\hat{n}_c\rangle + 1) \tag{18}$$

and

$$\begin{aligned}\langle\Delta\hat{n}_b^2\rangle &= G^2\langle\Delta\hat{n}_a^2\rangle + (G-1)^2\langle\Delta\hat{n}_c^2\rangle + G(G-1)\\ &\quad\times[\langle\hat{n}_a\rangle\langle\hat{n}_c\rangle + (\langle\hat{n}_a\rangle+1)(\langle\hat{n}_c\rangle+1)]\\ &= G^2\langle\Delta\hat{n}_a^2\rangle + G(G-1)\langle\hat{n}_a\rangle(2\langle\hat{n}_c\rangle+1)\\ &\quad + (G-1)(\langle\hat{n}_c\rangle+1) + (G-1)^2(\langle\hat{n}_c\rangle+1)^2,\end{aligned} \tag{19}$$

respectively. Here we use our assumption of incoherent noise to obtain the expression in the second line of equation (19), that is, we assume that the variance of the noise field obeys Bose–Einstein statistics in that $\langle\Delta\hat{n}_c^2\rangle = \langle\hat{n}_c\rangle + \langle\hat{n}_c\rangle^2$ [3].

We describe a non-ideal attenuator in an analogous manner. In this case, the annihilation operator for the output field is given by

$$\hat{b} = T^{1/2}\hat{a} + (1-T)^{1/2}\hat{c}, \tag{20}$$

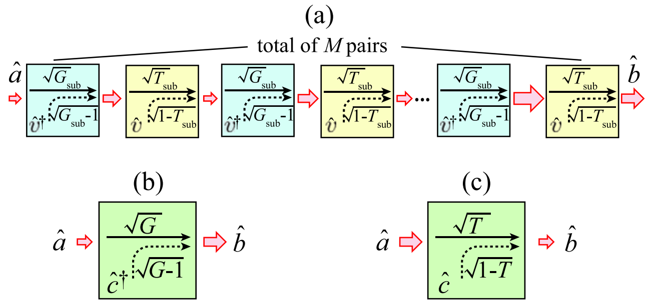

Figure 2. One can describe a non-ideal amplifier using (a) a series of alternating ideal amplifiers and attenuators, or (b) a generalized quantum beam splitter model. Part (c) illustrates the beam splitter model for an attenuator.

where T is the intensity transmission of the attenuator. The expectation value of the photon number and its variance for the output field are then given by

$$\langle\hat{n}_b\rangle = T\langle\hat{n}_a\rangle + (1-T)\langle\hat{n}_c\rangle \tag{21}$$

and

$$\begin{aligned}\langle\Delta\hat{n}_b^2\rangle &= T^2\langle\Delta\hat{n}_a^2\rangle + (1-T)^2\langle\Delta\hat{n}_c^2\rangle + T(1-T)\\ &\quad\times(\langle\hat{n}_a\rangle+1)\langle\hat{n}_c\rangle + T(1-T)(\langle\hat{n}_c\rangle+1)\langle\hat{n}_a\rangle\\ &= T^2\langle\Delta\hat{n}_a^2\rangle + T(1-T)\langle\hat{n}_a\rangle(2\langle\hat{n}_c\rangle+1)\\ &\quad + (1-T)\langle\hat{n}_c\rangle + (1-T)^2\langle\hat{n}_c\rangle^2.\end{aligned} \tag{22}$$

We show in the following subsection that, by means of an appropriate choice for $\langle\hat{n}_c\rangle$, this model can describe various types of realistic amplifiers and attenuators.

3.2. Media with both gain and loss mechanisms

Many practical amplifiers have both amplification and attenuation mechanisms distributed throughout the device. We denote the exponential gain and attenuation coefficients of the gain and loss mechanisms by g_0 and α_0, respectively. The net exponential gain coefficient of the non-ideal amplifier is thus $g = g_0 - \alpha_0$, and the net signal power gain after propagating through the amplifier is $G = \exp(gL)$, where L is the length of the amplifier.

One way to model such a non-ideal amplifier quantum mechanically is to use a series of alternating ideal sub-amplifiers and sub-attenuators (see figure 2(a)), with the gain/loss of each ideal sub-amplifier/attenuator given by

$$G_{\text{sub}} \approx 1 + \frac{g_0 L}{M}, \tag{23}$$

$$T_{\text{sub}} \approx 1 - \frac{\alpha_0 L}{M}, \tag{24}$$

where M is the number of sub-amplifier–attenuator pairs. Note that a large number of ideal sub-amplifier–attenuator pairs is needed to model a non-ideal amplifier with a weak input field, a large net gain G, or when g_0 and α_0 are comparable. Note also that, in most cases, the number of alternating gain and loss elements in the conventional treatment can be taken to be infinitesimal, and the problem can be solved analytically using, e.g., techniques described in [8].

Alternatively, one can much more simply use our generalized quantum amplifier model to treat this sort of situation. In fact, as we show in the appendix (see especially equations (A.6) and (A.7)), our model gives the same result as that of calculating explicitly the response of alternating gain and loss segments if one chooses

$$G = \exp(gL) \tag{25}$$

and

$$\langle \hat{n}_c \rangle = \frac{\alpha_0}{g_0 - \alpha_0} \tag{26}$$

in our generalized non-ideal quantum amplifier model.

Another example of an amplifier which is non-ideal because it displays both gain and loss mechanisms is that of a not-entirely-inverted laser gain medium. Here, the excited state population density N_2 provides gain, while the ground state population density N_1 provides loss. The net exponential gain coefficient per unit length g is therefore proportional to $N_2 - N_1$. For such a case, we find that our new model reproduces standard results (as summarized in figure 3) if we take the average number of photons in the noise field to be given (see especially equation (A.7)) by

$$\langle \hat{n}_c \rangle = \frac{\alpha_0}{(g_0 - \alpha_0)} = \frac{N_1}{N_2 - N_1} = n_{\mathrm{sp}} - 1, \tag{27}$$

where the quantity $n_{\mathrm{sp}} = N_2/(N_2 - N_1)$ is often known as the spontaneous emission factor [16] or the inversion factor [17]. Consequently, equations (18) and (19) become

$$\langle \hat{n}_b \rangle = G\langle \hat{n}_a \rangle + (G-1)n_{\mathrm{sp}} \tag{28}$$

and

$$\langle \Delta\hat{n}_b^2 \rangle = G^2\langle \Delta\hat{n}_a^2 \rangle + G(G-1)\langle \hat{n}_a \rangle(2n_{\mathrm{sp}} - 1) + (G-1)n_{\mathrm{sp}} + (G-1)^2 n_{\mathrm{sp}}^2. \tag{29}$$

In the limiting case in which $n_{\mathrm{sp}} = 1$, corresponding to complete population inversion, the noise input field $\langle \hat{n}_c \rangle = 0$ is the vacuum field, and equations (28) and (29) reduce to the results of an ideal amplifier, equations (5) and (6).

The choice $\langle \hat{n}_c \rangle = n_{\mathrm{sp}} - 1$ for a not-entirely-inverted laser medium is also consistent with a semiclassical model [10] in which the noise arises from spontaneous emission of photons into the propagating field mode. The photon number q in this mode varies spatially according to the equation

$$\frac{\mathrm{d}q}{\mathrm{d}z} = gq + R_{\mathrm{spon}}N_2, \tag{30}$$

where R_{spon} is the spontaneous emission rate into the single mode of frequency ν, and I is the intensity. For the present model in which the gain and population densities are uniform within the volume V, the gain is given by $g = (V/c)R_{\mathrm{spon}}(N_2 - N_1)$, and it follows that the photon number q at $z = L$ becomes

$$q(L) = Gq(0) + (G-1)n_{\mathrm{sp}}, \tag{31}$$

exactly as in equation (28).

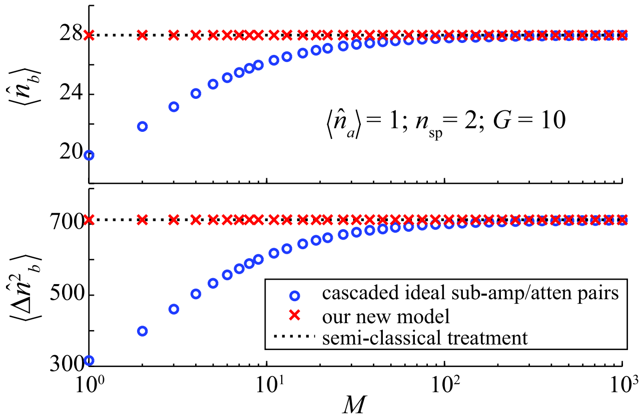

Figure 3. The average photon number (top) and its variance (bottom) for the output field of a not-entirely-inverted amplifier using various models. M is the number of sub-amplifier–attenuator pairs into which the medium has been divided. The average photon number of the input field is $\langle \hat{n}_a \rangle = 1$, the total gain of the amplifier is $G = 10$, and the spontaneous emission factor is $n_{\mathrm{sp}} = 2$. The dotted line is the result given by the semiclassical treatment of [10].

As a numerical example, we next calculate the average photon number and its variance at the output of a not-entirely-inverted laser amplifier modeled by the approaches just described. These results are presented in figure 3. In this example the average photon number of the input field is $\langle \hat{n}_a \rangle = 1$, the total gain of the amplifier is $G = 10$, and the spontaneous emission factor is $n_{\mathrm{sp}} = 2$. The blue circles represent the results obtained by using M pairs of ideal sub-amplifiers and attenuators. One sees that the calculated results obtained using cascaded ideal sub-amplifier/attenuator pairs converge to the correct values [9, 10] (that is, the values given by a semiclassical treatment) when the number of pairs M is greater than 100. On the other hand, our new approach gives the same correct answer using just a single step. For comparison purposes, we have also calculated the output of M cascaded generalized quantum sub-amplifiers, each with the same value of n_{sp} and gain $G_{\mathrm{sub}} = G^{1/M}$. In fact, the output of two cascaded generalized quantum amplifiers with the same value of n_{sp} but with different gains G_1 and G_2 is identical to that of a single generalized quantum amplifier with n_{sp} and intensity gain $G = G_1G_2$. A detailed proof of this result is given in appendix B. This result can also be seen from figure 3 in that the results obtained using our new treatment, indicated by the red crosses, do not change as M increases. Our model, in contrast to the model based on alternating sub-amplifier/attenuator pairs, requires only a single calculation step, and thus is computationally efficient.

In the opposite situation, in which the loss mechanism prevails over the gain mechanism, i.e., $\alpha_0 > g_0$, the medium becomes an attenuator. In this case, one can use our modified quantum attenuator model, i.e., equations (21) and (22), with $T = \exp(-\alpha L)$ and $\langle \hat{n}_c \rangle = g_0/(\alpha_0 - g_0)$. We again use a not-entirely-inverted laser medium as an example, in which the ground state population is larger than the excited state population. In such a case, $\langle \hat{n}_c \rangle = N_2/(N_1 - N_2) = -n_{\mathrm{sp}}$,

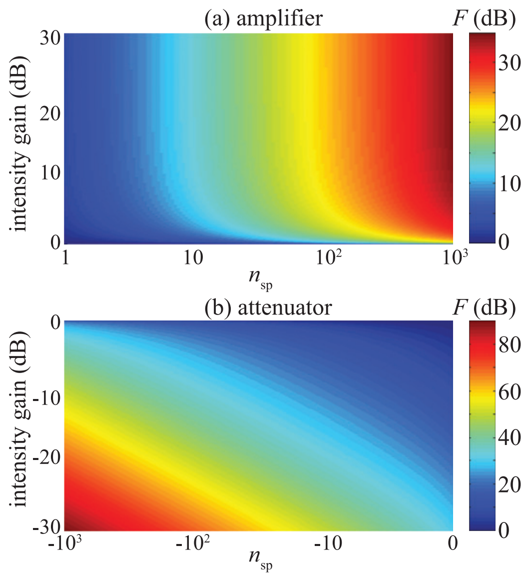

Figure 4. The noise figure of a laser medium as a function of intensity gain G and spontaneous emission factor $n_{\rm sp}$ when it operates as (a) an amplifier and (b) an attenuator. Here, the average photon number of the input field is 1000.

and the average value and variance of the output field are

$$\langle \hat{n}_b \rangle = T\langle \hat{n}_a \rangle + n_{\rm sp}(T-1) \tag{32}$$

and

$$\langle \Delta \hat{n}_b^2 \rangle = T^2 \langle \Delta \hat{n}_a^2 \rangle + T(T-1)\langle \hat{n}_a \rangle (2n_{\rm sp}-1) + (T-1)n_{\rm sp} + (1-T)^2 n_{\rm sp}^2. \tag{33}$$

Note that all four terms in equation (33) are positive, as $n_{\rm sp}$ is a negative number for $T < 1$. Again, in the limiting case for which $n_{\rm sp} = 0$, indicating that all the population is in the ground state, the noise field $\langle \hat{n}_c \rangle = 0$ becomes the vacuum field, and the results of equations (32) and (33) reduce to those for an ideal attenuator, given by equations (14) and (15).

Note also that the results given by equations (32) and (33) are actually of the same form as those of equations (28) and (29). This indicates that, although we start from two different mathematical descriptions for amplifiers and attenuators, we have obtained a consistent description for a non-ideal laser medium which can act as either an amplifier or an attenuator.

The noise figure of such a lossy laser medium is therefore

$$\mathcal{F} = 1 + \frac{G-1}{G}\frac{\langle \hat{n}_a \rangle}{\langle \Delta \hat{n}_{\rm a}^2 \rangle}(2n_{\rm sp}-1) + \frac{G-1}{G^2}\frac{n_{\rm sp}}{\langle \Delta \hat{n}_{\rm a}^2 \rangle} + \frac{(G-1)^2}{G^2}\frac{n_{\rm sp}^2}{\langle \Delta \hat{n}_{\rm a}^2 \rangle}. \tag{34}$$

G can of course have any positive value, while $n_{\rm sp}$ is greater than one for $G > 1$ and is negative for $G < 1$.

The noise figure $\mathcal{F}$ of a laser medium as a function of intensity gain G and spontaneous emission factor $n_{\rm sp}$ when it operates as either an amplifier or an attenuator is shown in figure 4. The average photon number of the input field used in the calculation is $\langle \hat{n}_a \rangle = 1000$, and we assume that its fluctuations obey Poisson statistics. For a given value of $n_{\rm sp}$, the noise figure of a laser amplifier increases as the gain becomes larger, but it also saturates to $2n_{\rm sp}$ when the input signal is strong compared to $n_{\rm sp}$. On the other hand, the noise figure for an attenuator increases without bound as the transmission decreases. Furthermore, besides the linear term $(2n_{\rm sp}-1)/T$, there is a second-order term proportional to $n_{\rm sp}^2/T^2$. Thus, the noise figure can exceed 80 dB for an attenuator with $T = 10^{-3}$ and $n_{\rm sp} = 10^3$.

4. Discussion and summary

In summary, we have presented a general quantum beam splitter model for a realistic amplifier or attenuator. The model has two inputs, the signal field and a noise field. By choosing appropriately the input noise field, our model can describe various types of realistic amplifiers and attenuators. For laser amplifiers, the smallest noise figure is obtained at largest gain. Our proposed model can be experimentally tested for, e.g., erbium doped fiber amplifiers, using configurations described in, e.g., [18].

In a recent publication [7], two of the current authors showed that the theory of the ideal quantum amplifier can be used to establish the limiting noise figure of an optical delay (or advancement) line that operates by means of slow-light (or fast-light) effects [19–23]. The treatment of non-ideal amplifiers given in the present paper can serve as a starting point for generalizing the treatment given in this earlier paper [7]. While the maximum achievable fractional delay or advancement has been the primary figure of merit used to evaluate the performance of a slow- or fast-light device, the noise properties of the device are also important for many practical applications such as the use of slow-light methods in telecommunications and in spectroscopy [24, 25]. For example, the SNR of the output field from a slow-light tunable time-delay element is not only determined by the signal distortion it introduces, but by the noise property of the slow-light medium as well. Thus, it is important to take the noise properties into account in the design of an optimized gain medium for telecommunications, especially from a system performance point of view [26]. Thus, investigation of the relation between the noise properties of slow- and fast-light devices and of the quantum theory of non-ideal amplifiers is a worthwhile topic for future study.

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Appendix A. Equivalence between a sub-amplifier–attenuator pair and a non-ideal amplifier/attenuator

In these appendices, we show mathematically that the noise properties of propagation through a cascade of alternating sub-amplifiers and sub-attenuators are given equivalently by the generalized quantum amplifier model presented in this paper. In this appendix, we establish this result for propagation through a single sub-amplifier followed by a single sub-attenuator. In appendix B we show that this result also holds for an arbitrarily large cascaded sequence of sub-amplifiers and sub-attenuators. In this manner, we establish the result in general.

Suppose the intensity transmissions through the sub-amplifier and sub-attenuator, each with length δl, are given by

$$G_0 = \exp(g_0\delta l) \approx 1 + g_0\delta l, \tag{A.1}$$

$$T_0 = \exp(-\alpha_0\delta l) \approx 1 - \alpha_0\delta l, \tag{A.2}$$

where g_0 and α_0 are the gain and attenuation coefficients, respectively. We assume that δl is small enough that $g_0\delta l$ and $|\alpha_0\delta l|$ are both much smaller that unity. The expected output photon number and its variance after the first sub-amplifier are given according to equations (5) and (6) by

$$\langle\hat{n}_{b,G}\rangle = G_0\langle\hat{n}_a\rangle + (G_0 - 1) \tag{A.3}$$

and

$$\langle\Delta\hat{n}^2_{b,G}\rangle = G_0^2\langle\Delta\hat{n}_a^2\rangle + G_0(G_0-1)(\langle\hat{n}_a\rangle + 1). \tag{A.4}$$

The expected output photon number after the beam has passed through the sub-amplifier/sub-attenuator pair is given by

$$\begin{aligned}\langle\hat{n}_b\rangle &= T_0\langle\hat{n}_{b,G}\rangle\\ &= T_0G_0\langle\hat{n}_a\rangle + T_0(G_0-1)\\ &= G\langle\hat{n}_a\rangle + (1-\alpha_0\delta l)g_0\delta l\\ &\approx G\langle\hat{n}_a\rangle + g_0\delta l\\ &= G\langle\hat{n}_a\rangle + \frac{g_0}{g_0-\alpha_0}(g_0\delta l - \alpha_0\delta l)\\ &\approx G\langle\hat{n}_a\rangle + (\langle\hat{n}_c\rangle + 1)(G-1),\end{aligned} \tag{A.5}$$

where we have introduced the quantities

$$G = T_0G_0 \tag{A.6}$$

and

$$\langle\hat{n}_c\rangle = \alpha_0/(g_0-\alpha_0). \tag{A.7}$$

Here we assume that the terms higher than the first power in δl are negligible.

The variance of the output photon number is likewise given by

$$\begin{aligned}\langle\Delta\hat{n}_b^2\rangle &= T_0^2\langle\Delta\hat{n}^2_{b,G}\rangle + T_0(1-T_0)\langle\hat{n}_{b,G}\rangle = T_0^2[G_0^2\langle\Delta\hat{n}_a^2\rangle\\ &\quad + G_0(G_0-1)(\langle\hat{n}_a\rangle+1)] + T_0(1-T_0)(G_0\langle\hat{n}_a\rangle\\ &\quad + (G_0-1)) = G^2\langle\Delta\hat{n}_a^2\rangle + G(G-1)\left(1 + 2\frac{1-T_0}{G-1}\right)\\ &\quad \times\langle\hat{n}_a\rangle + G_0(G_0-1) + T_0(1-T_0)(G_0-1)\\ &\approx\ G^2\langle\Delta\hat{n}_a^2\rangle + G(G-1)\left(1+2\langle\hat{n}_c\rangle\right)\langle\hat{n}_a\rangle\\ &\quad + (G-1)(\langle\hat{n}_c\rangle+1).\end{aligned} \tag{A.8}$$

Again, we assume here that terms higher than the first power in δl are negligible. These results are consistent with those (equations (18) and (19)) of a single non-ideal amplifier with intensity gain G and with a completely incoherent noise input field with average photon number $\langle\hat{n}_c\rangle$.

In the case in which the net transmission through the sub-amplifier–attenuator pair is less then unity, the expected output photon number through the sub-attenuator can now be expressed as

$$\begin{aligned}\langle\hat{n}_b\rangle &= T_0\langle\hat{n}_{b,G}\rangle\\ &= T\langle\hat{n}_a\rangle + (1-\alpha_0\delta l)g_0\delta l\\ &\approx T\langle\hat{n}_a\rangle + g_0\delta l\\ &= T\langle\hat{n}_a\rangle + \frac{g_0}{\alpha_0-g_0}(\alpha_0\delta l - g_0\delta l)\\ &\approx T\langle\hat{n}_a\rangle + \langle\hat{n}_c\rangle(1-T),\end{aligned} \tag{A.9}$$

where $T = T_0G_0$, and $\langle\hat{n}_c\rangle = g_0/(\alpha_0-g_0)$. Similarly, its variance can be expressed in a different way as follows:

$$\begin{aligned}\langle\Delta\hat{n}_b^2\rangle &= T_0^2[G_0^2\langle\Delta\hat{n}_a^2\rangle + G_0(G_0-1)(\langle\hat{n}_a\rangle+1)]\\ &\quad + T_0(1-T_0)(G_0\langle\hat{n}_a\rangle + (G_0-1)) = T^2\langle\Delta\hat{n}_a^2\rangle\\ &\quad + T(1-T)\left(1+2\frac{1-T_0}{1-T}\right)\langle\hat{n}_a\rangle + G_0(G_0-1)\\ &\quad + T_0(1-T_0)(G_0-1) \approx T^2\langle\Delta\hat{n}_a^2\rangle + T(1-T)\\ &\quad \times(1+2\langle\hat{n}_c\rangle)\langle\hat{n}_a\rangle + (1-T)\langle\hat{n}_c\rangle.\end{aligned} \tag{A.10}$$

The above results are consistent with those (equations (21) and (22)) of a single non-ideal amplifier with intensity transmission T and a completely incoherent noise input field with average photon number $\langle\hat{n}_c\rangle$.

Appendix B. Proof of cascadability of non-ideal quantum amplifiers

To model a non-ideal amplifier with both amplification and attenuation mechanisms, one typically needs a large number of sub-amplifier–attenuator pairs. In appendix A, we have shown that a pair consisting of an ideal sub-amplifier and sub-attenuator is equivalent to a single non-ideal amplifier or attenuator with an appropriate choice of the internal noise field. Here, we show mathematically that the transmission through two cascaded non-ideal amplifiers with different intensity gains G_1 and G_2 but with the same noise input field $\langle\hat{n}_c\rangle$ is equivalent to propagation through a single non-ideal amplifier with intensity gain $G = G_1G_2$ and the same noise input field $\langle\hat{n}_c\rangle$. Thus, we can use a single step model to describe a non-ideal amplifier with both amplification and attenuation mechanisms.

The average photon number of the output field and its variance after passing the first amplifier are given by

$$\langle\hat{n}_{b,1}\rangle = G_1\langle\hat{n}_a\rangle + (G_1-1)(\langle\hat{n}_c\rangle + 1) \tag{B.1}$$

and

$$\langle\Delta\hat{n}_{b,1}^2\rangle = G_1^2\langle\Delta\hat{n}_a^2\rangle + G_1(G_1-1)\langle\hat{n}_a\rangle(2\langle\hat{n}_c\rangle+1) + (G_1-1)(\langle\hat{n}_c\rangle+1) + (G_1-1)^2(\langle\hat{n}_c\rangle+1)^2. \quad \text{(B.2)}$$

The average photon number of the output field after passing the second amplifier is given by

$$\begin{aligned}\langle\hat{n}_b\rangle &= G_2\langle\hat{n}_{b,1}\rangle + (G_2-1)(\langle\hat{n}_c\rangle+1) = G_2[G_1\langle\hat{n}_a\rangle + (G_1-1)(\langle\hat{n}_c\rangle+1)] + (G_2-1)(\langle\hat{n}_c\rangle+1)\\ &= G_1G_2\langle\hat{n}_a\rangle + (G_1G_2-G_2+G_2-1)(\langle\hat{n}_c\rangle+1)\\ &= G\langle\hat{n}_a\rangle + (G-1)(\langle\hat{n}_c\rangle+1), \end{aligned}\quad \text{(B.3)}$$

where $G = G_1G_2$ is the overall gain through both amplifiers. The variance of the average photon number of the output field after passing through both amplifiers is given by

$$\begin{aligned}\langle\Delta\hat{n}_b^2\rangle &= G_2^2\langle\Delta\hat{n}_{b,1}^2\rangle + G_2(G_2-1)\langle\hat{n}_{b,1}\rangle(2\langle\hat{n}_c\rangle+1) + (G_2-1)(\langle\hat{n}_c\rangle+1) + (G_2-1)^2(\langle\hat{n}_c\rangle+1)^2\\ &= G_2^2[G_1^2\langle\Delta\hat{n}_a^2\rangle + G_1(G_1-1)\langle\hat{n}_a\rangle(2\langle\hat{n}_c\rangle+1) + (G_1-1)(\langle\hat{n}_c\rangle+1) + (G_1-1)^2(\langle\hat{n}_c\rangle+1)^2]\\ &\quad + G_2(G_2-1)[G_1\langle\hat{n}_a\rangle + (G_1-1)(\langle\hat{n}_c\rangle+1)](2\langle\hat{n}_c\rangle+1)\\ &\quad + (G_2-1)(\langle\hat{n}_c\rangle+1) + (G_2-1)^2(\langle\hat{n}_c\rangle+1)^2\\ &= (G_1G_2)^2\langle\Delta\hat{n}_a^2\rangle + G_1G_2(G_1G_2-1)\times\langle\hat{n}_a\rangle(2\langle\hat{n}_c\rangle+1) + (G_1G_2-1)(\langle\hat{n}_c\rangle+1)\\ &\quad + [G_2^2G_1^2 - 2G_2G_1 + 1](\langle\hat{n}_c\rangle+1)^2\\ &= G^2\langle\Delta\hat{n}_a^2\rangle + G(G-1)\langle\hat{n}_a\rangle(2\langle\hat{n}_c\rangle+1) + (G-1)(\langle\hat{n}_c\rangle+1) + (G-1)^2(\langle\hat{n}_c\rangle+1)^2. \end{aligned}\quad \text{(B.4)}$$

Thus, one sees that the property of the output field after propagating through two cascaded non-ideal amplifiers with different gain but the same noise characteristics is equivalent to that obtained in propagating through a single non-ideal amplifier with the same noise property and total gain.

The situation for cascaded attenuators is very similar. Suppose we have two cascaded attenuators with transmissions of T_1 and T_2, respectively, and they both have the noise input field $\langle\hat{n}_c\rangle$. The average photon number of the output field and its variance after passing the first attenuator are given by

$$\langle\hat{n}_{b,1}\rangle = T_1\langle\hat{n}_a\rangle + (1-T_1)\langle\hat{n}_c\rangle \quad \text{(B.5)}$$

and

$$\langle\Delta\hat{n}_{b,1}^2\rangle = T_1^2\langle\Delta\hat{n}_a^2\rangle + T_1(1-T_1)\langle\hat{n}_a\rangle(2\langle\hat{n}_c\rangle+1) + (1-T_1)\langle\hat{n}_c\rangle + (1-T_1)^2\langle\hat{n}_c\rangle^2. \quad \text{(B.6)}$$

After passing through the second attenuator, the average photon number of the output field becomes

$$\begin{aligned}\langle\hat{n}_b\rangle &= T_2\langle\hat{n}_{b,1}\rangle + (1-T_2)\langle\hat{n}_c\rangle = T_2[T_1\langle\hat{n}_a\rangle + (1-T_1)\langle\hat{n}_c\rangle]\\ &\quad + (1-T_2)\langle\hat{n}_c\rangle = T_1T_2\langle\hat{n}_a\rangle + (T_2 - T_1T_2 + 1 - T_2)\langle\hat{n}_c\rangle\\ &\quad \times T\langle\hat{n}_a\rangle + (1-T)\langle\hat{n}_c\rangle, \end{aligned}\quad \text{(B.7)}$$

where $T = T_1T_2$ is the overall transmission through both attenuators. The variance of the final output photon number is given by

$$\begin{aligned}\langle\Delta\hat{n}_b^2\rangle &= T_2^2\langle\Delta\hat{n}_{b,1}^2\rangle + T_2(1-T_2)\langle\hat{n}_{b,1}\rangle(2\langle\hat{n}_c\rangle+1) + (1-T_2)\langle\hat{n}_c\rangle + (1-T_2)^2\langle\hat{n}_c\rangle^2 = T_2^2[T_1^2\langle\Delta\hat{n}_a^2\rangle\\ &\quad + T_1(1-T_1)\langle\hat{n}_a\rangle(2\langle\hat{n}_c\rangle+1) + (1-T_1)\langle\hat{n}_c\rangle + (1-T_1)^2\langle\hat{n}_c\rangle^2] + T_2(1-T_2)[T_1\langle\hat{n}_a\rangle + (1-T_1)\langle\hat{n}_c\rangle]\\ &\quad \times (2\langle\hat{n}_c\rangle+1) + (1-T_2)\langle\hat{n}_c\rangle + (1-T_2)^2\langle\hat{n}_c\rangle^2\\ &= (T_1T_2)^2\langle\Delta\hat{n}_a^2\rangle + [T_2^2T_1(1-T_1) + T_2T_1(1-T_2)]\times\langle\hat{n}_a\rangle(2\langle\hat{n}_c\rangle+1) + [T_2^2(1-T_1) + T_2(1-T_2)(1-T_1)\\ &\quad + 1 - T_2]\langle\hat{n}_c\rangle + [T_2^2(1-T_1)^2 + 2T_2(1-T_2)(1-T_1) + (1-T_2)^2]\langle\hat{n}_c\rangle^2 = T^2\langle\Delta\hat{n}_a^2 + T(1-T)\langle\hat{n}_a\rangle\\ &\quad \times (2\langle\hat{n}_c\rangle+1) + (1-T)\langle\hat{n}_c\rangle + (1-T)^2\langle\hat{n}_c\rangle^2. \end{aligned}\quad \text{(B.8)}$$

Thus, we have proven mathematically that one can use a single generalized quantum amplifier/attenuator model to describe an amplifier or attenuator with both distributed gain and attenuation mechanisms.

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