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Wave mixing in photorefractive crystals with saturable couplings: stable solutions and instabilities

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Two-wave and four-wave mixing in photorefractive media with saturable gain and saturable absorption is analyzed. Solutions to wave equations are found in terms of quadratures and implicitly given functions. Different models of saturation are compared. It is found that for weak couplings energy transfer between waves is less effective in the strong unsaturated regime as compared to the saturated, while for strong couplings both regimes are equally effective. The existence of multiple solutions in four-wave mixing is established. It is shown that the unsaturated system is more stable than the saturated.

Wave mixing equations in photorefractive (PR) crystals thus far have been solved only in the saturation regime [1]. The redistribution of photo-induced charges is finished in this regime, and the wave coupling constant g and the absorption coefficient α do not depend on the light intensity. However, nowadays the experiments with photorefractives are done at low power levels, and when the pump depletion and absorption are taken into account, the conditions for saturation are never met. Indeed, it is observed that at low beam intensities the wave coupling and the absorption coefficient become intensity dependent [2].

Our goal is to consider degenerate photorefractive mixing of waves with saturable coupling constants, including intensity dependent absorption which is appropriate to PR crystals. We compare different models of saturation. We also investigate the possibilities of unstable operation in the system, in either two-wave (2WM) or four-wave (4WM) mixing.

Wave mixing models employed are the standard two-wave and four-wave mixing arrangements in PR crystals [1]. In 2WM we presume reflection geometry (RG), with the two waves (A_1 and A_2) incident on the crystal from the opposite sides. In 4WM we presume multigrating geometry, and the mixing of pumps is included as the predominant 2WM process. It is also presumed that the diffusion of photo-induced charges is the predominant grating-forming process, so that the coupling constants are real.

First we treat 2WM. Our starting point are the steady-state wave equations for slowly varying envelope intensities in two-wave PR mixing [3],

$$I_1' + \alpha(I) I_1 + 2g(I) I_1 I_2 / I = 0 , \tag{1a}$$

$$I_2' - \alpha(I) I_2 + 2g(I) I_1 I_2 / I = 0 , \tag{1b}$$

where $I = I_1 + I_2$ is the total intensity, and $\alpha(I)$ and $g(I)$ are the intensity dependent absorption and wave coupling coefficients. The prime denotes the derivatives along the propagation (z) direction.

There are many theories of PR effect which include intensity dependent coefficients [2–6]. We opt for a recent theory of Mahgerefteh, Feinberg, and Tayebati [4], which takes into account both deep donors and shallow traps in the crystal. According to this theory the intensity dependent gain coefficient is given by

$$g(I)=\frac{g_s}{k_0^2}\left(k_D^2+\frac{k_T^2 I}{I+\beta/s}\right), \tag{2}$$

where g_s is the value of the saturated gain, $k_0^2=k_D^2+k_T^2$ is the square of the total Debye screening wave vector, and k_D^2 and k_T^2 are its portions coming from the donors and the traps. Further, β is the thermal excitation rate, and s is the excitation cross section for the traps. Thus, when donor contribution to the Debye screening is predominant, there is no intensity dependence. However, when the trap contribution is predominant, or when the two contribute about equally, the following functional dependence is obtained,

$$g(I)=g_s I/(I+C)\,, \tag{3}$$

or

$$g(I)=\frac{g_s}{2}\left(1+\frac{I}{I+C}\right), \tag{4}$$

where C is a material-dependent parameter. We use both forms as models for the intensity dependent gain coefficient in 2WM.

Many different models exist for the saturate absorption coefficient [2,3,5] as well. Again, we consider two models, the PR absorption, found by Brost, Motes and Rotge [2] to account for the observed behavior in PR crystals,

$$\alpha(I)=\alpha_0+\alpha I/(C+I)\,, \tag{5}$$

and the two-level saturable absorption

$$\alpha(I)=\alpha_0/(C+I)\,, \tag{6}$$

where α_0 is the linear absorption, and α a material parameter.

The solution of eqs. (1) with the PR gain (3) and the PR absorption (5), and with $\alpha_0=0$, is given in terms of two functions $F(z)$ and $f(F)$ [7]

$$I_1=\tfrac{1}{2}f\exp(F),\quad I_2=\tfrac{1}{2}f\exp(-F)\,, \tag{7}$$

which are found by evaluating a system of two quadratures:

$$\int_{F_0}^{F}\frac{\mathrm{d}x}{\eta-\tanh x}=\ln\frac{f}{f_0},$$

$$C\int_{F_0}^{F}\frac{\mathrm{d}x}{f(x)(\sinh x-\cosh x)}-\ln\frac{f}{f_0}=g_s z\,, \tag{8}$$

where $\eta=\alpha/g_s$. Equations (8) allow multiple solutions when boundary conditions are applied. It should be noted that we deal here with a two-point boundary value problem. The conditions are that the intensities are given on the opposite faces of the crystal, which extends from $z=0$ to $z=d$. The appearance of multiple solutions complicates the analysis, especially when there is no criterion which would distinguish between the physically relevant and irrelevant solutions. Here two solutions are found, but one leads to negative z, and thereby is discarded. However, in 4WM there is no such criterion and multiple solutions appear, leading to a chaotic response of the system.

The corresponding quadratures for the model with the two-level saturable absorption are given by

$$\ln\frac{f_0}{f}=\frac{v-v_0}{\eta}\,,\quad v-v_0+C\int_{v_0}^{v}\frac{\mathrm{d}x}{u(x)}=-\alpha z\,, \tag{9}$$

where $u(v)=[f^2(v)+v^2]^{1/2}$ is the sum of intensities and v is their difference. The intensities are now

$$I_1=(u+v)/2,\quad I_2=(u-v)/2\,, \tag{10}$$

and no multiple solutions are found this time.

Figure 1 offers a comparison between different models. Also included are the results of Ja [3], who considered the effects of linear absorption. We use the method of Ja (i.e. a shooting procedure) as a checkup on our method. The agreement is pronounced. Our results indicate that the energy transfer between waves is less effective in the unsaturated regime.

The model with PR gain from eq. (4) can not be solved analytically with the absorption included in any form. The same goes for 4WM. Therefore, in what follows absorption is neglected. The variable v is constant then. This is a standard result for 2WM in RG. The solution of eqs. (1) with $\alpha(I)=0$ and $g(I)$ from eq. (4) is given by

$$I_1=\tfrac{1}{2}v_0(\coth F+1),\quad I_2=\tfrac{1}{2}v_0(\coth F-1), \tag{11}$$

where the function $F(z)$ is found from an implicit relation:

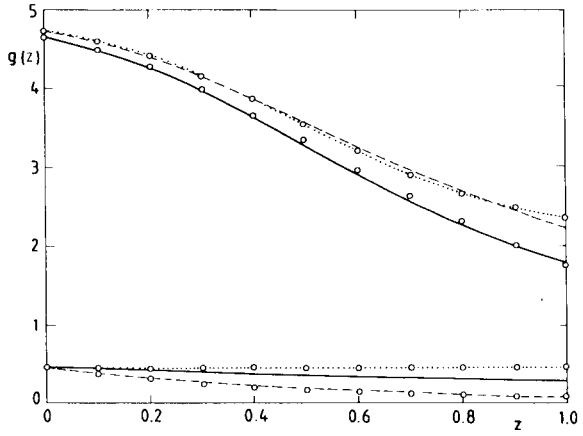

Fig. 1. The comparison of different models. The coupling coefficient g is represented as a function of z. Circles correspond to Ja's results [3]. Absorption mechanisms are different for different pairs of curves. The upper curves are with $C_1=10$, $C_2=0.1$, and the lower with $C_1=1$, $C_2=0.01$, C_1 and C_2 being given boundary conditions for intensities. Full curves correspond to the PR absorption with $\alpha=2\ \mathrm{cm}^{-1}$, dashed to the two-level absorption with $\alpha_0=2\ \mathrm{cm}^{-1}$, and dotted to the case without absorption. $C=1$ throughout.

$$g_s z(F)=\frac{b}{1-b^2}(F-F_0)+\ln\left|\frac{\sinh F}{\sinh F_0}\frac{\cosh F+b\sinh F}{\cosh F_0+b\sinh F_0}\right|+\frac{1}{b^2-1}\ln\left|\frac{\mathrm{sch}(F+\tanh^{-1}b)}{\mathrm{sch}(F_0+\tanh^{-1}b)}\right|, \tag{12}$$

where $b=C/2v_0$, v_0 being the constant value of v (to be determined from boundary conditions). Here "sch" stands for the sinh function when $|b|>1$, or for the cosh function when $|b|<1$. The application of boundary conditions raises the possibility of multiple solutions (double), but again only one is found physically reasonable.

Figure 2 depicts the influence of different saturation mechanisms. Unsaturated models from eqs. (3) and (4) are compared with the saturated case $g(I)=g_s$. In general, the saturated case is more effective in extracting energy from one beam to the other. However, for strong enough coupling this difference is lost. All saturation mechanisms then achieve an efficient depletion of the pump and produce the same energy transfer.

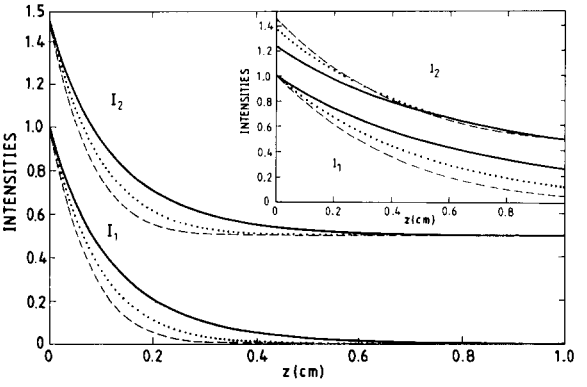

Fig. 2. Presenting the effect of different saturation mechanisms, for $g_s=10\ \mathrm{cm}^{-1}$. Intensities of the two beams are displayed as a function of z. Full lines depict the unsaturated model, eq. (3), while dotted correspond to the model from eq. (4), both with $C=1$. Dashed lines are for the saturated case with $C=0$. The inset shows the situation for $g_s=2\ \mathrm{cm}^{-1}$. While in general saturation enables more efficient energy transfer between the waves, for strong couplings this difference is lost.

The 4WM model employed presumes that the formation of gratings proceeds by more than one grating mechanism. The intensity equations describing 4WM in PR media, with the assumption that the transmission and reflection gratings contribute equally (i.e. have the same g coefficient), are given by [6,7]

$$II_1'=2\gamma(I)I_1I_2-2g(I)I_1(I_4-I_3)\,, \tag{13a}$$

$$II_2'=2\gamma(I)I_1I_2+2g(I)I_2(I_4-I_3)\,, \tag{13b}$$

$$II_3'=2g(I)I_3(I_1+I_2)+4g(I)(I_1I_2I_3I_4)^{1/2}\,, \tag{13c}$$

$$II_4'=2g(I)I_4(I_1+I_2)+4g(I)(I_1I_2I_3I_4)^{1/2}\,, \tag{13d}$$

where, similar to the previous case, the γ term describes 2WM between the pumps, and the g term takes into account the equal strength 4WM. Here I is again the total intensity, so that

$$g(I)=g_s\frac{I}{C+I},\quad \gamma(I)=\gamma_s\frac{I}{C+I}. \tag{14}$$

Absorption is excluded from our 4WM analysis. The presumed geometry is that two counterpropagating pump beams I_1 and I_2 impinge on the crystal, and the signal I_4, coming from the side of the pump I_1 (and using the energy of pumps) generates its own counterpropagating phase conjugate (PC) replica I_3.

The condition for exact phase conjugation has been applied, so that the relative phase $\phi=\phi_4+\phi_3-\phi_2-\phi_1$ is set to zero.

The solution of eqs. (13) is similarly written in terms of two functions v and w [7],

$$I_1=I_{4d}\frac{u(v)-(v+\delta)}{2}, \quad I_2=I_{4d}\frac{u(v)+(v+\delta)}{2}, \tag{15a}$$

$$I_3=I_{4d}v\frac{\cosh w(v)-1}{2}, \quad I_4=I_{4d}v\frac{\cosh w(v)+1}{2}, \tag{15b}$$

which are also given as quadratures:

$$w(v)=\int_1^v \frac{f(x)}{xu(x)}\,\mathrm{d}x, \tag{16a}$$

$$\ln v(z)+c\int_1^v \frac{\mathrm{d}x}{xu(x)}+\int_1^v \frac{\cosh w(x)}{xu(x)}\,\mathrm{d}x=2g_s(z-d), \tag{16b}$$

where

$$c=C/I_{4d}, \quad f(v)=av^{\xi/2},$$

$$u(v)=[(v+\delta)^2+f^2]^{1/2}, \quad \text{and } \xi=\gamma_s/g_s.$$

In this manner the original two-point boundary value problem is written as an initial value problem. The value of both variables are known on the $z=d$ face of the crystal, $v_d=1$, $w_d=0$. However, the input parameters a and δ, which figure in eqs. (16), depend on the missing boundary values. Their evaluation is provided by a self-consistent iterative map procedure. Sometimes during this procedure unstable situations arise, leading to a chaotic output. The procedure is described in our recent publication [7]. Here we provide an outline.

The values of a and δ could be given in terms of the missing values I_{1d} and I_{4d} on the $z=d$ face of the crystal, or in terms of I_{20} and I_{30} (equivalently, v_0 and w_0), missing on the $z=0$ face. The problem of fitting boundary conditions is resolved by the following procedure. We start by choosing arbitrary initial values a_0 and δ_0; from these I_{1d} and I_{4d} are calculated. I_{20} and I_{30} (i.e. v_0 and w_0) are found by evaluating integrals in eqs. (16), or by solving eqs. (13) on computer. This enables the calculation of the new values for a_1 and δ_1, and the procedure is repeated till convergence. In this manner a map is defined in the parameter plane, and the procedure corresponds to the evaluation of the fixed points of the map. An interesting question is whether the map can become unstable, and what happens when it becomes unstable.

The procedure is stable for g negative, and for arbitrary other parameters. For g positive the instabilities set in, and we investigate these instabilities by standard methods of nonlinear dynamics. This is done by evaluating the fixed points of the arbitrary composition power of the map. Such fixed points correspond to different periodic cycles that may exist in the map, and may reveal the nature of the transition to chaos, if there is one in the system. However, seeing such instabilities in a mode map does not mean that they exist in a real crystal. Instabilities following from a steady-state analysis and not from a time-dependent (dynamical) treatment of the process could be spurious. An illuminating discussion of this point can be found among von Neumann's collected works [8]. The existence of such instabilities should be verified experimentally.

Figures 3 and 4 display unstable situations that may arises in the system. Figure 3 presents bifurcation diagrams of the reflectivity when the 4WM coupling g is varied, for saturated and unsaturated crystals. In the saturated case complicated behavior is observed, including transitions to chaos via direct and inverse period doubling. The corresponding unsaturated case is simpler, displaying only quasiperiodic behavior and one period doubling transition to chaos, at the unphysical border. Unphysical means that for higher values of g the wave-mixing model is inappropriate: the PC wave intensity becomes negative.

Also, the process of 4WM is more stable in the unsaturated regime, i.e. stronger couplings are needed in order to destabilize an unsaturated system. This conclusion is further corroborated in fig. 4, which depicts the bifurcation diagram obtained by sampling the constant C as the control parameter. Evidently, the system becomes more stable as the constant C is increased, which is equivalent to having less and less saturated crystal.

In summary, we have investigated 2WM and 4WM processes in photorefractive crystals in the unsaturated regime, i.e. when the coupling "constant" and absorption depend on the light intensity. A few

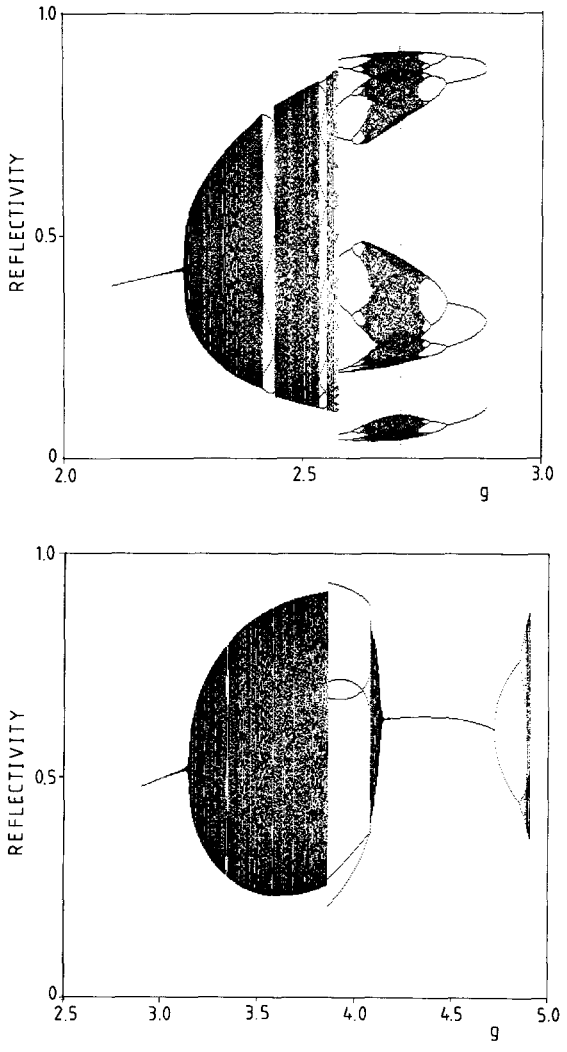

Fig. 3. (a) Bifurcation diagram of the intensity reflectivity as g_s is varied, for the saturated regime ($C=0$). To the left quasiperiodic behavior is observed. To the right quasiperiodic behavior has changed into (direct and inverse) period doubling. (b) The corresponding diagram for the saturable regime ($C=1$) reveals only quasiperiodic behaviour, with a terminal period doubling. Note that for the values of g_s where the saturated crystal displays complicated behavior, the unsaturated is simple.

models of saturable absorption in 2WM and 4WM are introduced, and the corresponding wave equations are solved exactly. The solutions are written in terms of quadratures. From the solutions it is evident that the energy transfer between the waves is less effective in the unsaturated stage of the process.

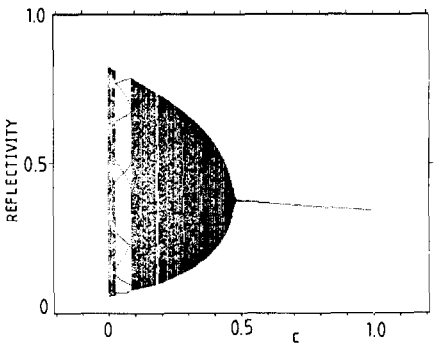

Fig. 4. The bifurcation diagram as the saturation parameter C is varied. The system is chaotic for $C=0$ (saturation), and then proceeds through a series of changes until a unique solution is obtained for $C=1$ (unsaturated system).

In 4WM stable and unstable (multiple) solutions are found, depending on the strength of the coupling. Different types of unstable behavior are observed: quasiperiodic motion on a torus, and period doubling to chaos. Chaotic behavior in this context means that the intensity reflectivity does not settle onto any particular value, but wanders on a strange attractor in the parameter space. We should note that an instability in this context does not mean a temporal instability. It means that the iterative solution procedure as a map in the plane has become unstable, allowing multiple (and sometimes infinitely many) reflectivity values. Each of these values represents an allowed solution to the boundary value problem at hand, however, not all of them could be dynamically accessible to the system. It is also found that (in this sense of instability) the unsaturated system is more stable than the saturated.

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